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On the Surface the Fermi–Walker Derivative in Minkowski 3-Space

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Abstract. In this paper Fermi–Walker derivative and Fermi–Walker parallelism and non-rotating frame concepts are given along the curve lying on the spacelike surface and the timelike surface in Minkowski 3-space. First, we consider a curve lying on the spacelike surface and investigate the Fermi–Walker derivative along the curve. The concepts which Fermi–Walker derivative and its theorems are analyzed along the curve lying on the spacelike surface in Minkowski 3-space. And then we consider a curve lying on the timelike surface and investigate the Fermi–Walker derivative along the curve.

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1. Introduction

In Special Relativity, the term "fix" is quite clear. On arbitrary spacetimes, the respective notion must be covariant; if γ is freely falling, its restspaces are transported through Levi-Civita parallelism, so a fix spacelike direction has, by definition, a null covariant derivative [10,11]. For accelerated observers, the restspace is not transported by the Levi-Civita parallelism, anymore. Therefore, in order to define "constant" directions, one uses the Fermi–Walker parallelism which is an isometry between the tangent space along the curve [2,3,12,16–18]. They enlarged the context by defining a class of generalized Fermi–Walker connections. In [1], Fermi–Walker derivative along a space curve in R^3 was identified and was given any properties. In [5], we have shown Fermi–Walker derivative according to the frames and non-rotating frame being conditions are analyzed along the curve in Euclid space. In [6], we have shown Fermi–Walker derivative and non-rotating frame being conditions are analyzed in Minkowski space E_1^3.

In this paper we investigate that Fermi–Walker derivative along any curve lying on the spacelike surface and the timelike surface in Minkowski 3-space E_1^3. By using the definition of Fermi–Walker derivative, Fermi–Walker parallelism, non-rotating frame and Darboux vector concepts are defined again for the curve lying on the spacelike surface and the timelike surface in Minkowski 3-space. Non-rotating frame being conditions are analyzed along the curve lying on the spacelike surface and the timelike surface.

2. Preliminaries

The Minkowski 3-space E_1^3 is an affine 3-space endowed with an indefinite inner product given as

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is natural coordinates of E_1^3. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are arbitrary vectors in E_1^3, we define the (Lorentzian) vector product of u and v as the following:

$$u \times v = \begin{vmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Since $\langle\ ,\ \rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian casual characters: it can be spacelike if $\langle v\ ,\ v\rangle > 0$ or $v = 0$, timelike if $\langle v\ ,\ v\rangle < 0$ and null (lightlike) if $\langle v\ ,\ v\rangle = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{\langle v, v\rangle}$ and two vectors v and w are said to be orthogonal if $\langle v, w\rangle = 0$. An arbitrary curve $\alpha : I \subset IR \longrightarrow E_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha^{'}(s)$ are respectively spacelike, timelike or null (lightlike) for each $s \in I \subset IR$. A spacelike or timelike curve α has unit speed, if $\left\langle \alpha^{'}(s), \alpha^{'}(s) \right\rangle = \pm 1$. A null curve α is parameterized by pseudo-arc s if $\left\langle \alpha^{''}(s), \alpha^{''}(s) \right\rangle = 1$ [7,9,14].

Let S be an oriented surface in three-dimensional Minkowski space E_1^3 and let consider a non-null curve $\alpha(s)$ lying on S fully. Since the curve $\alpha(s)$ is also in space, three exists Frenet frame $\{T, N, B\}$ at each points of the curve where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively.

Since the curve $\alpha(s)$ lies on the surface S there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{T, \overrightarrow{g}, \overrightarrow{n}\}$. In this frame T is the unit tangent vector of the curve, $\overrightarrow{n}$ is the unit normal vector of the surface S and $\overrightarrow{g}$ is a unit vector given by $\overrightarrow{g} = \overrightarrow{n} \wedge T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors $N, B, \overrightarrow{g}$ and $\overrightarrow{n}$ lie on the same plane. If the surface S is an oriented spacelike surface, then the curve $\alpha = \alpha(s)$ lying on S is a spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{g} \\ \overrightarrow{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}$$

φ is the angle between the vectors $\overrightarrow{g}$ and N [13,14].

Then, if the surface S is an oriented timelike surface, the relations between the frames can be given as follows.

If the curve $\alpha(s)$ is timelike,

$$\begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{g} \\ \overrightarrow{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}$$

If the curve $\alpha(s)$ is spacelike,

$$\begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{g} \\ \overrightarrow{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix}$$

In all cases, φ is the angle between the vectors $\overrightarrow{g}$ and N[13–15].

According to the Lorentzian causal characters of the surface S and the curve $\alpha(s)$ lying on S, the derivative formulae of the Darboux frame can be changed as follows:

If the surface S is a spacelike surface, then the curve $\alpha(s)$ lying on S is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} \nabla_T \overrightarrow{T} \\ \nabla_T \overrightarrow{g} \\ \nabla_T \overrightarrow{n} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ \overrightarrow{g} \\ \overrightarrow{n} \end{bmatrix}$$

where $\langle T , T\rangle = 1, \langle \overrightarrow{g} , \overrightarrow{g}\rangle = 1, \langle \overrightarrow{n} , \overrightarrow{n}\rangle = -1$ [13,14].

If the surface S is a timelike surface, then the curve $\alpha(s)$ lying on S can be a spacelike or timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} \nabla_T \overrightarrow{T} \\ \nabla_T \overrightarrow{g} \\ \nabla_T \overrightarrow{n} \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon k_n \\ k_g & 0 & \varepsilon\tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ \overrightarrow{g} \\ \overrightarrow{n} \end{bmatrix}$$

where $\langle T , T\rangle = \varepsilon = \pm 1, \langle \overrightarrow{g} , \overrightarrow{g}\rangle = -\varepsilon, \langle \overrightarrow{n} , \overrightarrow{n}\rangle = 1$ [13–15].

In these formulae k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows:

If both S and $\alpha(s)$ are spacelike or timelike,

$$k_g = \kappa \cos\varphi$$
$$k_n = \kappa \sin\varphi$$
$$\tau_g = \tau - \frac{d\varphi}{ds}$$

If S is timelike and $\alpha(s)$ is spacelike,

$$k_g = \kappa \cosh\varphi$$
$$k_n = \kappa \sinh\varphi$$
$$\tau_g = \tau + \frac{d\varphi}{ds}$$

[13–15].

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of the curve $\alpha(s)$ can be calculated as follows

$$k_g = \left\langle \frac{d\alpha}{ds}, \frac{d^2\alpha}{ds^2} \wedge \overrightarrow{n} \right\rangle, \quad \tau_g = \left\langle \frac{d\alpha}{ds}, \overrightarrow{n} \wedge \frac{d\overrightarrow{n}}{ds} \right\rangle$$

[13–15].

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on surface S the followings are well-known

$$\alpha \text{ is } \begin{cases} \text{an asymptotic line iff } k_n = 0, \\ \text{a geodesic curve iff } k_g = 0, \quad [8]. \\ \text{a principal line iff } \tau_g = 0 \end{cases}$$

Definition 2.1. A surface in the Minkowski 3-space E_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [4,7,9].

Lemma 2.2. *In the Minkowski 3-space E_1^3, the following properties are satisfied:*

(i) *Two timelike vectors are never orthogonal.*
(ii) *Two null vectors are orthogonal if and only if they are linearly dependent.*
(iii) *A timelike vector is never orthogonal to a null vector* [4,7,9].

Definition 2.3. Let $\alpha : I \subset IR \longrightarrow E_1^3$ be any curve in Minkowski 3-space E_1^3 and X be any vector field along the curve α. Then, the Fermi–Walker derivative of X is defined as

$$\widetilde{\nabla}_T X = \nabla_T X - \varepsilon \langle T, X \rangle A + \varepsilon \langle A, X \rangle T$$

Here T is the tangent vector field of the curve α, $\langle T\,, T \rangle = \varepsilon = \pm 1$ and $A = \nabla_T T$ [1–3,6].

Definition 2.4. In Minkowski 3-space E_1^3, let $\alpha : I \subset IR \longrightarrow E_1^3$ be a curve and X be any vector field along the curve α. If the Fermi–Walker derivative of the vector field X vanishes, i.e., if $\widetilde{\nabla}_T X = 0$, then X is called Fermi–Walker parallel along the curve [1–3,6].

Definition 2.5. Let a unit speed curve $\alpha : I \subset IR \longrightarrow E_1^3$ together with orthogonal vector field U, V, W along α be given. If the Fermi–Walker derivative of the vector fields vanish, then $\{U, V, W\}$ is called non-rotating frame [1–3,6].

3. The Fermi–Walker Derivative On Spacelike Surfaces

In this section, we investigate the Fermi–Walker derivative along the curve lying on the spacelike surface in Minkowski 3-space. Let $\alpha : I \subset IR \longrightarrow E_1^3$ be unit speed curve lying on the surface with the Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$, S be a spacelike surface and X any vector field along α with the corresponding connection ∇.

Lemma 3.1. *The Fermi–Walker derivative along the curve α which is spacelike, can be expressed as*

$$\widetilde{\nabla}_T X = \nabla_T X - (k_g \overrightarrow{n} + k_n \overrightarrow{g}) \wedge X$$

where $\{T, \overrightarrow{g}, \overrightarrow{n}\}$ is Darboux frame of the spacelike surface and $\langle T , T\rangle = 1$.

Proof. By definition 2.3,

$$\widetilde{\nabla}_T X = \nabla_T X - \langle T, X\rangle A + \langle A, X\rangle T$$

$$\widetilde{\nabla}_T X = \nabla_T X - \langle T, X\rangle \nabla_T T + \langle \nabla_T T, X\rangle T$$

using the Darboux equations and Lorentzian vector product

$$\widetilde{\nabla}_T X = \nabla_T X - (k_g \overrightarrow{n} + k_n \overrightarrow{g}) \wedge X$$

which gives the result. □

Corollary 3.2. *If the curve $\alpha(s)$ is a principal line lying on a spacelike surface then the Fermi–Walker derivative*

$$\widetilde{\nabla}_T X = \nabla_T X - w \wedge X$$

where $w = -\tau_g T + k_n \overrightarrow{g} + k_g \overrightarrow{n}$ is the Darboux instantaneous rotation vector of the Darboux trihedron.

Corollary 3.3. *Let α be any principal line lying on a spacelike surface. Fermi–Walker derivative along the curve of the vector field X coincides with derivative of the vector field X if and only if*

$$X = \lambda w$$

where $w = -\tau_g T + k_n \overrightarrow{g} + k_g \overrightarrow{n}$ is the Darboux instantaneous rotation vector of the Darboux trihedron and λ is constant.

Theorem 3.4. *Let α be a curve lying on a spacelike surface and $X = \lambda_1 T + \lambda_2 \overrightarrow{g} + \lambda_3 \overrightarrow{n}$ be any vector field along α. The vector field X is Fermi–Walker parallel along α if and only if*

$$\begin{aligned} \lambda_1(s) &= const., \\ \lambda_2(s) &= c_1 \cosh\left(\int_1^s \tau_g(s)\,ds\right) + c_2 \sinh\left(\int_1^s \tau_g(s)\,ds\right), \\ \lambda_3(s) &= -c_1 \sinh\left(\int_1^s \tau_g(s)\,ds\right) - c_2 \cosh\left(\int_1^s \tau_g(s)\,ds\right). \end{aligned}$$

Here $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of a real parameter s.

Proof. Using Lemma 2.2 and Lemma 3.1, X be Fermi–Walker parallel along α,

$$\begin{aligned} \widetilde{\nabla}_T X &= \nabla_T X - (k_g \overrightarrow{n} + k_n \overrightarrow{g}) \wedge X \\ \widetilde{\nabla}_T X &= \left(\frac{d\lambda_1}{ds}\right) T + \left(\frac{d\lambda_2}{ds} + \tau_g(s)\lambda_3\right) \overrightarrow{g} + \left(\frac{d\lambda_3}{ds} + \tau_g(s)\lambda_2\right) \overrightarrow{n} \end{aligned}$$

is obtained. X is Fermi–Walker parallel along the curve iff

$$\begin{aligned} \frac{d\lambda_1}{ds} &= 0, \\ \frac{d\lambda_2}{ds} + \tau_g(s)\lambda_3 &= 0, \\ \frac{d\lambda_3}{ds} + \tau_g(s)\lambda_2 &= 0. \end{aligned}$$

This is equivalent to,

$$\begin{aligned} \lambda_1(s) &= const., \\ \lambda_2(s) &= c_1 \cosh\left(\int_1^s \tau_g(s)\,ds\right) + c_2 \sinh\left(\int_1^s \tau_g(s)\,ds\right), \\ \lambda_3(s) &= -c_1 \sinh\left(\int_1^s \tau_g(s)\,ds\right) - c_2 \cosh\left(\int_1^s \tau_g(s)\,ds\right). \end{aligned}$$

The rest is obvious. □

Theorem 3.5. *If the curve α is a principal line and the vector field $X = \lambda_1 T + \lambda_2 \overrightarrow{g} + \lambda_3 \overrightarrow{n}$ is Fermi–Walker parallel along the curve α then the parameters λ_i are constant.*

Proof. Using Lemma 3.1, X be Fermi–Walker parallel along α,

$$\widetilde{\nabla}_T X = \nabla_T X - (k_g \overrightarrow{n} + k_n \overrightarrow{g}) \wedge X$$

$\forall \lambda_i = const.$,

$$\widetilde{\nabla}_T X = (\tau_g(s)\lambda_3)\,\overrightarrow{g} + (\tau_g(s)\lambda_2)\,\overrightarrow{n}$$

then

$$\widetilde{\nabla}_T X = \tau_g(s)(\lambda_3 \overrightarrow{g} + \lambda_2 \overrightarrow{n})$$

is obtained. α is a principal line so,

$$\widetilde{\nabla}_T X = 0$$

□

Corollary 3.6. *Let α be a curve lying on a spacelike surface with the Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$. If the curve α is a principal line then the $\{T, \overrightarrow{g}, \overrightarrow{n}\}$ is non-rotating frame along α.*

Definition 3.7. Let α be a curve lying on a spacelike surface with the Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$. The Fermi–Walker termed Darboux vector is defined by

$$w_* = -\tau_g(s)\, T$$

and where $\widetilde{\nabla}_T T = w_* \wedge T, \widetilde{\nabla}_T \overrightarrow{g} = w_* \wedge \overrightarrow{g}, \widetilde{\nabla}_T \overrightarrow{n} = w_* \wedge \overrightarrow{n}$ are provided.

Theorem 3.8. *Let α be a curve lying on a spacelike surface with the Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$. The Fermi–Walker termed Darboux vector is Fermi–Walker parallel along α if and only if $\tau_g(s) = const.$*

Proof. By using Lemma 3.1,

$$\widetilde{\nabla}_T w_* = -\frac{d\tau_g}{ds} T$$

is obtained.

If $w_*(s)$ is Fermi–Walker parallel then,

$$\tau_g(s) = const.$$

The rest is obvious □

4. The Fermi–Walker Derivative On Timelike Surfaces

In this section, we investigate the Fermi–Walker derivative along the curve lying on the timelike surface in Minkowski 3-space. Let $\alpha : I \subset IR \longrightarrow E_1^3$ be unit speed curve lying on the surface with the Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$, S be a timelike surface and X any vector field along α with the corresponding connection ∇.

Lemma 4.1. *The Fermi–Walker derivative along the curve α which is timelike or spacelike, can be expressed as*

$$\widetilde{\nabla}_T X = \nabla_T X - \varepsilon\,(k_n \overrightarrow{g} - k_g \overrightarrow{n}) \wedge X$$

where $\{T, \overrightarrow{g}, \overrightarrow{n}\}$ is Darboux frame of the timelike surface and $\langle T\ , T\rangle = \varepsilon = \pm 1$.

Proof. By definition 2.3,

$$\widetilde{\nabla}_T X = \nabla_T X + \langle T, X\rangle A - \langle A, X\rangle T$$

$$\widetilde{\nabla}_T X = \nabla_T X + \langle T, X\rangle \nabla_T T - \langle \nabla_T T, X\rangle T$$

using the Darboux equations and Lorentzian vector product

$$\widetilde{\nabla}_T X = \nabla_T X - \varepsilon \left(k_n \overrightarrow{g} - k_g \overrightarrow{n} \right) \wedge X$$

which gives the result. □

Corollary 4.2. *If the curve $\alpha(s)$ is a principal line lying on a timelike surface then the Fermi–Walker derivative*

$$\widetilde{\nabla}_T X = \nabla_T X - \varepsilon \omega \wedge X$$

where $\omega = \tau_g T + k_n \overrightarrow{g} - k_g \overrightarrow{n}$ is the Darboux instantaneous rotation vector of the Darboux trihedron.

Corollary 4.3. *Let α be any principal line lying on a timelike surface. Fermi–Walker derivative along the curve of the vector field X coincides with derivative of the vector field X if and only if*

$$X = \lambda \varepsilon \omega$$

where $\omega = \tau_g T + k_n \overrightarrow{g} - k_g \overrightarrow{n}$ is the Darboux instantaneous rotation vector of the Darboux trihedron and λ is constant.

Definition 4.4. *Let α be a curve lying on a timelike surface with the Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$ .The Fermi–Walker termed Darboux vector is defined by*

$$\overrightarrow{w}^* = \varepsilon \tau_g(s) T$$

where $\widetilde{\nabla}_T T = \overrightarrow{w}^* \wedge T$, $\widetilde{\nabla}_T \overrightarrow{g} = \overrightarrow{w}^* \wedge \overrightarrow{g}$, $\widetilde{\nabla}_T \overrightarrow{n} = \overrightarrow{w}^* \wedge \overrightarrow{n}$ and $\langle T , T \rangle = \varepsilon = \pm 1$.

Theorem 4.5. *Let α be a curve lying on a timelike surface with the Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$ .The Fermi–Walker termed Darboux vector is Fermi–Walker parallel along α if and only if $\tau_g(s) = const.$*

Proof. By using Lemma 4.1,

$$\widetilde{\nabla}_T \overrightarrow{w}^* = \varepsilon \frac{d\tau_g}{ds} T$$

is obtained.

If $\overrightarrow{w}^*$is Fermi–Walker parallel then,

$$\tau_g(s) = const.$$

The rest is obvious. □

Theorem 4.6. *Let α be a timelike curve lying on a timelike surface and $X = \lambda_1 T + \lambda_2 \overrightarrow{g} + \lambda_3 \overrightarrow{n}$ be any vector field along α. The vector field X is Fermi–Walker parallel along α if and only if*

$$\lambda_1(s) = const.,$$

$$\lambda_2(s) = -c_1 \sin \left(\int_1^s \tau_g(s)\, ds \right) + c_2 \cos \left(\int_1^s \tau_g(s)\, ds \right),$$

$$\lambda_3(s) = c_1 \cos \left(\int_1^s \tau_g(s)\, ds \right) + c_2 \sin \left(\int_1^s \tau_g(s)\, ds \right).$$

Here $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of a real parameter s.

Proof. Using Lemma 4.1, X be Fermi–Walker parallel along α,

$$\widetilde{\nabla}_T X = \nabla_T X + (k_n \overrightarrow{g} - k_g \overrightarrow{n}) \wedge X$$

$$\widetilde{\nabla}_T X = \left(\frac{d\lambda_1}{ds}\right) T + \left(\frac{d\lambda_2}{ds} + \tau_g(s)\lambda_3\right) \overrightarrow{g} + \left(\frac{d\lambda_3}{ds} - \tau_g(s)\lambda_2\right) \overrightarrow{n}$$

is obtained. X is Fermi–Walker parallel along the timelike curve iff

$$\frac{d\lambda_1}{ds} = 0,$$
$$\frac{d\lambda_2}{ds} + \tau_g(s)\lambda_3 = 0,$$
$$\frac{d\lambda_3}{ds} - \tau_g(s)\lambda_2 = 0.$$

This equivalent to,

$$\lambda_1(s) = const.,$$
$$\lambda_2(s) = -c_1 \sin\left(\int_1^s \tau_g(s)\, ds\right) + c_2 \cos\left(\int_1^s \tau_g(s)\, ds\right),$$
$$\lambda_3(s) = c_1 \cos\left(\int_1^s \tau_g(s)\, ds\right) + c_2 \sin\left(\int_1^s \tau_g(s)\, ds\right).$$

The rest is obvious. □

Theorem 4.7. *If the curve α is a timelike principal line and the vector field $X = \lambda_1 T + \lambda_2 \overrightarrow{g} + \lambda_3 \overrightarrow{n}$ is Fermi–Walker parallel along the curve α then the parameters λ_i are constant.*

Proof. Using Lemma 4.1, X be Fermi–Walker parallel along α,

$$\widetilde{\nabla}_T X = \nabla_T X + (k_n \overrightarrow{g} - k_g \overrightarrow{n}) \wedge X$$

$\forall \lambda_i = const.$,

$$\widetilde{\nabla}_T X = (\tau_g(s)\lambda_3)\, \overrightarrow{g} - (\tau_g(s)\lambda_2)\, \overrightarrow{n}$$

then

$$\widetilde{\nabla}_T X = \tau_g(s)\,(\lambda_3 \overrightarrow{g} - \lambda_2 \overrightarrow{n})$$

is obtained. α is a timelike principal line so,

$$\widetilde{\nabla}_T X = 0$$

□

Corollary 4.8. *Let α be a timelike curve lying on a timelike surface with Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$. If the curve α is a principal line then the $\{T, \overrightarrow{g}, \overrightarrow{n}\}$ is non-rotating frame along α.*

Theorem 4.9. *Let α be a spacelike curve lying on a timelike surface and $X = \lambda_1 T + \lambda_2 \overrightarrow{g} + \lambda_3 \overrightarrow{n}$ be any vector field along α. The vector field X is Fermi–Walker parallel along α. The vector field X is Fermi–Walker parallel along α if and only if*

$$\begin{aligned}
\lambda_1(s) &= const.,\\
\lambda_2(s) &= c_1 \cosh\left(\int_1^s \tau_g(s)\,ds\right) + c_2 \sinh\left(\int_1^s \tau_g(s)\,ds\right),\\
\lambda_3(s) &= -c_1 \sinh\left(\int_1^s \tau_g(s)\,ds\right) - c_2 \cosh\left(\int_1^s \tau_g(s)\,ds\right).
\end{aligned}$$

Here $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of a real parameter s.

Proof. Using Lemma 4.1, X be Fermi–Walker parallel along α,

$$\widetilde{\nabla}_T X = \nabla_T X - (k_n \overrightarrow{g} - k_g \overrightarrow{n}) \wedge X$$

$$\widetilde{\nabla}_T X = \left(\frac{d\lambda_1}{ds}\right) T + \left(\frac{d\lambda_2}{ds} + \tau_g(s)\lambda_3\right) \overrightarrow{g} + \left(\frac{d\lambda_3}{ds} + \tau_g(s)\lambda_2\right) \overrightarrow{n}$$

is obtained. X is Fermi–Walker parallel along the curve iff

$$\begin{aligned}
\frac{d\lambda_1}{ds} &= 0,\\
\frac{d\lambda_2}{ds} + \tau_g(s)\lambda_3 &= 0,\\
\frac{d\lambda_3}{ds} + \tau_g(s)\lambda_2 &= 0.
\end{aligned}$$

This is equivalent to,

$$\begin{aligned}
\lambda_1(s) &= const.,\\
\lambda_2(s) &= c_1 \cosh\left(\int_1^s \tau_g(s)\,ds\right) + c_2 \sinh\left(\int_1^s \tau_g(s)\,ds\right),\\
\lambda_3(s) &= -c_1 \sinh\left(\int_1^s \tau_g(s)\,ds\right) - c_2 \cosh\left(\int_1^s \tau_g(s)\,ds\right).
\end{aligned}$$

The rest is obvious. □

Theorem 4.10. *If the curve α is a spacelike principal line and the vector field $X = \lambda_1 T + \lambda_2 \overrightarrow{g} + \lambda_3 \overrightarrow{n}$ is Fermi–Walker parallel along the curve α then the parameters λ_i are constant.*

Proof. Using Lemma 4.1, X be Fermi–Walker parallel along α,

$$\widetilde{\nabla}_T X = \nabla_T X - (k_n \overrightarrow{g} - k_g \overrightarrow{n}) \wedge X$$

$\forall \lambda_i = const.$,

$$\widetilde{\nabla}_T X = (\tau_g(s)\lambda_3)\, \overrightarrow{g} + (\tau_g(s)\lambda_2)\, \overrightarrow{n}$$

then

$$\widetilde{\nabla}_T X = \tau_g(s)\left(\lambda_3 \overrightarrow{g} + \lambda_2 \overrightarrow{n}\right)$$

is obtained. α is a spacelike principal line so,

$$\widetilde{\nabla}_T X = 0$$

□

Corollary 4.11. *Let α be a spacelike curve lying on a timelike surface with Darboux frame $\{T, \overrightarrow{g}, \overrightarrow{n}\}$. If the curve α is a principal line then the$\{T, \overrightarrow{g}, \overrightarrow{n}\}$ is non-rotating frame along α.*

5. Conclusions

In this paper we explained Fermi–Walker derivative along a curve lying on a spacelike surface and timelike surface in Minkowski 3-space. The concepts of Fermi–Walker parallelism, non-rotating frame and Fermi–Walker termed Darboux vector are analyzed along the curve lying on the spacelike surface and timelike surface.

Initially we get any curve lying on a surface which is an oriented spacelike surface in Minkowski 3-space. According to the curve Fermi–Walker derivative, Fermi–Walker parallelism, non-rotating frame and Fermi–Walker termed Darboux vector concepts are given. We proved while α is a spacelike principal line Darboux frame is a non-rotating frame.Then we proved when the geodesic torsion is constant Fermi–Walker termed Darboux vector is Fermi–Walker parallel. And then we get any curve lying on a surface which is oriented timelike surface in Minkowski 3-space.According to the curve Fermi–Walker derivative, Fermi–Walker parallelism, non-rotating frame and Fermi–Walker termed Darboux vector concepts are given for timelike surface. We proved while α is a timelike principal line Darboux frame is a non-rotating frame.

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