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# Statistical $\sigma$-convergence of positive linear operators

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ABSTRACT

Mursaleen and Edely [M. Mursaleen and O.H.H Edely, On invariant mean and statistical convergence, Appl. Math. Lett. (2009) doi:10.1016/j.aml.2009.06.005] have recently introduced the notion of statistical $\sigma$-convergence. In this paper, we study its use in the Korovkin-type approximation theorem. Then, we construct an example such that our new result works but its classical and statistical cases do not work. We also compute the rates of statistical $\sigma$-convergence of a sequence of positive linear operators.



## 1. Introduction

For a sequence $\{L_n\}$ of positive linear operators on $C(X)$, which is the space of real valued continuous functions on a compact subset $X$ of real numbers, Korovkin [1] first introduced the necessary and sufficient conditions for the uniform convergence of $L_n(f)$ to a function $f$ by using the test function $e_i$ defined by $f_i(x) = x^i$, $(i = 0, 1, 2)$ (see, for instance, [2]). Later many researchers investigated these conditions for various operators defined on different spaces. Using the concept of statistical convergence in the approximation theory provides us with many advantages. In particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of linear operators such as the interpolation operator of Hermite–Fejér [3], because these types of operators do not converge at points of simple discontinuity. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results have been proved [4–10]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem. Mursaleen and Edely [11] have recently introduced various kinds of statistical convergence which are stronger than the statistical convergence. We first recall these convergence methods.

Let $K$ be a subset of $\mathbb{N}$, the set of natural numbers, then the natural density of $K$, denoted by $\delta(K)$, is given by:

$$\delta(K) := \lim_n \frac{1}{n} |\{k \le n : k \in K\}|$$

whenever the limit exists, where $|B|$ denotes the cardinality of the set $B$. Then a sequence $x = \{x_k\}$ of numbers is statistically convergent to $L$ provided that, for every $\varepsilon > 0$, $\delta \{k : |x_k - L| \ge \varepsilon\} = 0$ holds. In this case we write $st - \lim x_k = L$.

Notice that every convergent sequence is statistically convergent to the same value, but its converse is not true. Such an example may be found in [12–14].

Let $\sigma$ be a mapping of the set of $\mathbb{N}$ into itself. A continuous linear functional $\varphi$ defined on the space $l_\infty$ of all bounded sequences is called an invariant mean (or $\sigma$-mean) [15] if it is nonnegative, normal and $\varphi(x) = \varphi((x_{\sigma(n)}))$.

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A sequence $x = \{x_k\}$ is said to be statistically $\sigma$-convergent to $L$ if for every $\varepsilon > 0$ the set $K_\varepsilon(\varphi) := \{k \in \mathbb{N} : \varphi(|x_k - L|) \geq \varepsilon\}$ has natural density zero, i.e. $\delta(K_\varepsilon(\varphi)) = 0$. In this case we write $\delta(\sigma) - \lim x_k = L$. That is,

$$\lim_n \frac{1}{n} \left|\left\{p \leq n : \left|t_{pm}(x_m) - L\right| \geq \varepsilon\right\}\right| = 0, \quad \text{uniformly in } m$$

where

$$t_{pm}(x_m) := \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^p(m)}}{p+1}, \quad t_{-1,m}(x_m) = 0$$

(see, for details, [11]). Using the above definitions, the next result follows immediately.

**Lemma 1.** *Statistical convergence implies statistical $\sigma$-convergence.*

However, one can construct an example which guarantees that the converse of Lemma 1 is not always true. Such an example was given in [11] as follows:

**Example 1.** Consider the case $\sigma(n) = n + 1$ and the sequence $u = \{u_m\}$ defined as

$$u_m = \begin{cases} 1, & \text{if } m \text{ is odd,} \\ -1, & \text{if } m \text{ is even,} \end{cases} \tag{1.1}$$

is statistically $\sigma$-convergence ($\delta(\sigma) - \lim u_m = 0$) but it is neither convergent nor statistically convergent.

With the above terminology, our primary interest in the present paper is to obtain a Korovkin-type approximation theorem by means of the concept of statistical $\sigma$-convergence. Also, by considering Lemma 1 and the above Example 1, we will construct a sequence of positive linear operators such that while our new results work, their classical and statistical cases do not work. We also compute the rates of statistical $\sigma$-convergence of a sequence of positive linear operators.

## 2. A Korovkin-type approximation theorem

We denote by $C_M[a, b]$ the space of all continuous functions on $[a, b]$ and bounded on an entire line, that is

$$|f(x)| \leq K_f, \quad -\infty < x < \infty,$$

where $K_f$ is a constant depending on $f$. Also by $B[a, b]$ we denote the space of all bounded functions on $[a, b]$. This space is equipped with the supremum norm

$$\|f\|_{B[a,b]} = \sup_{x \in [a,b]} |f(x)|, \quad (f \in B[a, b]).$$

Let $L$ be a linear operator from $C_M[a, b]$ into $B[a, b]$. Then, as usual, we say that $L$ is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in [a, b]$ by $L(f(u); x)$ or, briefly, $L(f; x)$.

Throughout the paper, we also use the following test functions

$$f_0(u) = 1, \qquad f_1(u) = u, \quad \text{and} \quad f_2(u) = u^2.$$

We now recall the classical and the statistical cases of the Korovkin-type results introduced in [1,9], respectively.

**Theorem 1.** *Let $\{L_m\}$ be a sequence of positive linear operators acting from $C_M[a, b]$ into $B[a, b]$. Then, for all $f \in C_M[a, b]$,*

$$\lim \|L_m(f) - f\|_{B[a,b]} = 0$$

*if and only if*

$$\lim \left\|L_m(f_j) - f_j\right\|_{B[a,b]} = 0, \quad (j = 0, 1, 2),$$

*where $f_j(u) = u^j$, $j = 0, 1, 2$.*

**Theorem 2.** *Let $\{L_m\}$ be a sequence of positive linear operators acting from $C_M[a, b]$ into $B[a, b]$. Then, for all $f \in C_M[a, b]$,*

$$st - \lim \|L_m(f) - f\|_{B[a,b]} = 0$$

*if and only if*

$$st - \lim \left\|L_m(f_j) - f_j\right\|_{B[a,b]} = 0, \quad (j = 0, 1, 2),$$

*where $f_j(u) = u^j$, $j = 0, 1, 2$.*

Now we have the following main result.

**Theorem 3.** *Let $\{L_m\}$ be a sequence of positive linear operators acting from $C_M[a,b]$ into $B[a,b]$. Assume that the following conditions hold:*

$$\delta(\sigma) - \lim \|L_m(f_i) - f_i\|_{B[a,b]} = 0, \quad i = 0, 1, 2. \tag{2.1}$$

*Then, for all $f \in C_M[a,b]$, we have*

$$\delta(\sigma) - \lim \|L_m(f) - f\|_{B[a,b]} = 0. \tag{2.2}$$

**Proof.** Let $f \in C_M[a,b]$ and $x \in [a,b]$ be fixed. Since $f$ is bounded on the entire line, we can write

$$|f(x)| \le K.$$

Also, since $f$ is continuous on $[a,b]$, we write that for every $\varepsilon > 0$, there exists a number $\delta_1 > 0$ such that $|f(u) - f(x)| < \varepsilon$ for all $x \in [a,b]$ satisfying $|u - x| < \delta_1$. Hence, we get

$$|f(u) - f(x)| < \varepsilon + \frac{2K}{\delta_1^2}(t - x)^2. \tag{2.3}$$

Using the positivity and monotonicity of $L_m$, we obtain, from (2.3),

$$\begin{aligned}
\left|t_{pm}(L_m(f;x)) - f(x)\right| &= \left| \frac{L_m(f;x) + L_{\sigma(m)}(f;x) + \cdots + L_{\sigma^p(m)}(f;x)}{p+1} - f(x) \right. \\
&\quad - \frac{\left[L_m(f_0;x) + L_{\sigma(m)}(f_0;x) + \cdots + L_{\sigma^p(m)}(f_0;x)\right]f(x)}{p+1} \\
&\quad \left. + \frac{\left[L_m(f_0;x) + L_{\sigma(m)}(f_0;x) + \cdots + L_{\sigma^p(m)}(f_0;x)\right]f(x)}{p+1} \right| \\
&\le \frac{L_m(|f(u) - f(x)|;x) + L_{\sigma(m)}(|f(u) - f(x)|;x) + \cdots + L_{\sigma^p(m)}(|f(u) - f(x)|;x)}{p+1} \\
&\quad + |f(x)| \left|t_{pm}(L_m(f_0;x)) - f_0(x)\right| \\
&\le \varepsilon + \left(\varepsilon + K + \frac{2K}{\delta_1^2}c^2\right)\left|t_{pm}(L_m(f_0;x)) - f_0(x)\right| \\
&\quad + \frac{4K}{\delta_1^2}c\left|t_{pm}(L_m(f_1;x)) - f_1(x)\right| + \frac{2K}{\delta_1^2}\left|t_{pm}(L_m(f_2;x)) - f_2(x)\right|
\end{aligned}$$

where $c := \max|x|$. Then taking the supremum we get

$$\begin{aligned}
\left\|t_{pm}(L_m(f)) - f\right\|_{B[a,b]} &\le \varepsilon + N\left\{\left\|t_{pm}(L_m(f_0)) - f_0\right\|_{B[a,b]} + \left\|t_{pm}(L_m(f_1)) - f_1\right\|_{B[a,b]}\right. \\
&\quad \left. + \left\|t_{pm}(L_m(f_2)) - f_2\right\|_{B[a,b]}\right\}
\end{aligned} \tag{2.4}$$

where $N := \max\left\{\varepsilon + K + \frac{2K}{\delta_1^2}c^2, \frac{4K}{\delta_1^2}c, \frac{2K}{\delta_1^2}\right\}$. Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Then, from (2.4), we obtain

$$\left|\left\{k : \left\|t_{pm}(L_m(f)) - f\right\| \ge r\right\}\right| \le \sum_{i=0}^{2}\left|\left\{k : \left\|t_{pm}(L_m(f_i)) - f_i\right\| \ge \frac{r - \varepsilon}{3N}\right\}\right|.$$

Using (2.1), we obtain (2.2). The proof is complete. □

**Remark 1.** We now show that our result Theorem 3 is stronger than its classical and statistical versions. To see this first consider the following Bernstein-type operators introduced by Bernstein:

$$B_m(f;x) = \sum_{k=0}^{m} f\left(\frac{k}{m}\right)\binom{m}{k}x^k(1-x)^{m-k}, \tag{2.5}$$

where $x \in [0,1]$; $f \in C_M[0,1]$. It is known that

$$\begin{aligned}
B_m(f_0;x) &= f_0(x), \\
B_m(f_1;x) &= f_1(x), \\
B_m(f_2;x) &= f_2(x) + \frac{x - x^2}{m}.
\end{aligned}$$

Now using (2.5) and (1.1), we define the following positive linear operators on $C_M[0, 1]$:

$$L_m(f; x) = (1 + u_m) B_m(f; x). \tag{2.6}$$

Then, observe that the sequence of positive linear operators $\{L_m\}$ defined by (2.6) satisfy all hypotheses of Theorem 3. So, by Example 1, we have

$$\delta(\sigma) - \lim \|L_m(f) - f\|_{B[0,1]} = 0.$$

However, since $(u_m)$ is not convergent and statistical convergent, we conclude that classical (Theorem 1) and statistical (Theorem 2) versions of our result do not work for the operators $L_m$ in (2.6) while our Theorem 3 still works.

## 3. Rate of $\sigma$-statistical convergence

In this section, we compute the corresponding rates in statistical $\sigma$-convergence in Theorem 3.

**Definition 1.** A sequence $\{x_m\}$ is statistically $\sigma$-convergent to a number $L$ with the rate of $\beta \in (0, 1)$ if for every $\varepsilon > 0$,

$$\lim_n \frac{\left|\left\{p \le n : \left|t_{pm}(x_m) - L\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} = 0, \quad \text{uniformly in } m.$$

In this case, it is denoted by

$$x_m - L = o(n^{-\beta}) (\delta(\sigma)).$$

Using this definition, we obtain the following auxiliary result.

**Lemma 2.** *Let $\{x_m\}$ and $\{y_m\}$ be sequences. Assume that $x_m - L_1 = o(n^{-\beta_1}) (\delta(\sigma))$ and $y_m - L_2 = o(n^{-\beta_2}) (\delta(\sigma))$. Then we have*

(i) $(x_m - L_1) \mp (y_m - L_2) = o(n^{-\beta}) (\delta(\sigma))$, *where* $\beta := \min\{\beta_1, \beta_2\}$,
(ii) $\lambda(x_m - L_1) = o(n^{-\beta_1}) (\delta(\sigma))$, *for any real number* $\lambda$.

**Proof.** (i) Assume that $x_m - L_1 = o(n^{-\beta_1}) (\delta(\sigma))$ and $y_m - L_2 = o(n^{-\beta_2}) (\delta(\sigma))$. Then, for $\varepsilon > 0$, observe that

$$\begin{aligned}
&\frac{\left|\left\{p \le n : \left|\left(t_{pm}(x_m) - L_1\right) \mp \left(t_{pm}(y_m) - L_2\right)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} \\
&\quad \le \frac{\left|\left\{p \le n : \left|t_{pm}(x_m) - L_1\right| \ge \frac{\varepsilon}{2}\right\}\right| + \left|\left\{p \le n : \left|t_{pm}(y_m) - L_2\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}} \\
&\quad \le \frac{\left|\left\{p \le n : \left|t_{pm}(x_m) - L_1\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_1}} + \frac{\left|\left\{p \le n : \left|t_{pm}(y_m) - L_2\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_2}}.
\end{aligned} \tag{3.1}$$

Now by taking the limit as $n \to \infty$ in (3.1) and using the hypotheses, we conclude that

$$\lim_n \frac{\left|\left\{p \le n : \left|\left(t_{pm}(x_m) - L_1\right) \mp \left(t_{pm}(y_m) - L_2\right)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} = 0, \quad \text{uniformly in } m,$$

which completes the proof of (i). Since the proof of (ii) is similar, we omit it. □

Now we recall the concept of the modulus of continuity. For $f \in C_M[a, b]$, the modulus of continuity of $f$, denoted by $\omega(f; \delta_1)$, is defined to be

$$\omega(f; \delta_1) := \sup\{|f(u) - f(x)| : u, x \in [a, b], |u - x| \le \delta_1\}$$

where $f \in C_M[a, b]$ and $\delta_1 > 0$. In order to obtain our result, we will make use of the elementary inequality, for all $f \in C_M[a, b]$ and for $\lambda, \delta_1 > 0$,

$$\omega(f; \lambda\delta_1) \le (1 + [\lambda]) \, \omega(f; \delta_1) \tag{3.2}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to $\lambda$.

Then we have the following result.

**Theorem 4.** *Let $\{L_m\}$ be a sequence of positive linear operators acting from $C_M[a, b]$ into $B[a, b]$. Assume that the following conditions holds:*

$$\|L_m(f_0) - f_0\|_{B[a,b]} = o(n^{-\beta_1}) (\delta(\sigma)) \quad \text{on } [a, b], \tag{3.3}$$

*and*

$$\omega\left(f;\alpha_{pm}\right)=o(n^{-\beta_2})\,(\delta(\sigma))\quad \text{on } [a,b], \tag{3.4}$$

*where* $\alpha_{pm} := \sqrt{\left\| t_{pm}\left(L_m(\varphi)\right)\right\|_{B[a,b]}}$ *with* $\varphi(u)=(u-x)^2$*.Then we have, for all* $f\in C_M[a,b]$*,*

$$\|L_m(f)-f\|_{B[a,b]}=o(n^{-\beta})\,(\delta(\sigma))\quad \text{on } [a,b],$$

*where* $\beta := \min\{\beta_1,\beta_2\}$*.*

**Proof.** Let $x\in[a,b]$ and $f\in C_M[a,b]$ be fixed. Using the properties of $\omega$ and the positivity and monotonicity of $L_m$, we get, for any $\delta_1>0$ and $m\in\mathbb{N}$, that

$$\begin{aligned}
&\left|t_{pm}\left(L_m\left(f;x\right)\right)-f(x)\right| \\
&\quad\le \frac{L_m\left(\left|f(u)-f(x)\right|;x\right)+L_{\sigma(m)}\left(\left|f(u)-f(x)\right|;x\right)+\cdots+L_{\sigma^p(m)}\left(\left|f(u)-f(x)\right|;x\right)}{p+1} \\
&\quad\quad+\left|f(x)\right|\left|t_{pm}\left(L_m\left(f_0;x\right)\right)-f_0(x)\right| \\
&\quad\le \frac{L_m\left(1+\frac{(u-x)^2}{\delta_1^2};x\right)+L_{\sigma(m)}\left(1+\frac{(u-x)^2}{\delta_1^2};x\right)+\cdots+L_{\sigma^p(m)}\left(1+\frac{(u-x)^2}{\delta_1^2};x\right)}{p+1}\,\omega\left(f;\delta_1\right) \\
&\quad\quad+\left|f(x)\right|\left|t_{pm}\left(L_m\left(f_0;x\right)\right)-f_0(x)\right| \\
&\quad= \left(t_{pm}\left(L_m\left(f_0;x\right)\right)+\frac{1}{\delta_1^2}\,\frac{L_m\left(\varphi;x\right)+L_{\sigma(m)}\left(\varphi;x\right)+\cdots+L_{\sigma^p(m)}\left(\varphi;x\right)}{p+1}\right)\omega\left(f;\delta_1\right) \\
&\quad\quad+\left|f(x)\right|\left|t_{pm}\left(L_m\left(f_0;x\right)\right)-f_0(x)\right|.
\end{aligned}$$

Then, taking the supremum we get

$$\begin{aligned}
\left\|t_{pm}\left(L_m(f)\right)-f\right\|_{B[a,b]} \le{}& \left\|t_{pm}\left(L_m\left(f_0\right)\right)-f_0\right\|_{B[a,b]}\omega\left(f;\delta_1\right)+2\omega\left(f;\delta_1\right) \\
&+K\left\|t_{pm}\left(L_m\left(f_0\right)\right)-f_0\right\|_{B[a,b]}
\end{aligned} \tag{3.5}$$

where $\delta_1 := \alpha_{pm} := \sqrt{\left\|t_{pm}\left(L_m(\varphi)\right)\right\|_{B[a,b]}}$ and $K := \|f\|_{B[a,b]}$. Hence, given $\varepsilon>0$, it follows from (3.5) that

$$\begin{aligned}
&\frac{\left|\left\{p\le n:\left\|t_{pm}\left(L_m(f)\right)-f\right\|_{B[a,b]}\ge\varepsilon\right\}\right|}{n^{1-\beta}} \\
&\quad\le \frac{\left|\left\{p\le n:\left\|t_{pm}\left(L_m\left(f_0\right)\right)-f_0\right\|_{B[a,b]}\ge\sqrt{\frac{\varepsilon}{3}}\right\}\right|}{n^{1-\beta_1}}+\frac{\left|\left\{p\le n:\omega\left(f;\delta_1\right)\ge\sqrt{\frac{\varepsilon}{3}}\right\}\right|}{n^{1-\beta_2}} \\
&\quad\quad+\frac{\left|\left\{p\le n:\omega\left(f;\delta_1\right)\ge\frac{\varepsilon}{6}\right\}\right|}{n^{1-\beta_2}}+\frac{\left|\left\{p\le n:\left\|t_{pm}\left(L_m\left(f_0\right)\right)-f_0\right\|_{B[a,b]}\ge\frac{\varepsilon}{3K}\right\}\right|}{n^{1-\beta_1}}
\end{aligned} \tag{3.6}$$

where $\beta=\min\{\beta_1,\beta_2\}$. Letting $n\to\infty$ in (3.6), we conclude from (3.3) and (3.4) that

$$\lim_n \frac{\left|\left\{p\le n:\left\|t_{pm}\left(L_m(f)\right)-f\right\|_{B[a,b]}\ge\varepsilon\right\}\right|}{n^{1-\beta}}=0,\quad \text{uniformly in } m,$$

which means

$$\|L_m(f)-f\|_{B[a,b]}=o(n^{-\beta})\,(\delta(\sigma))\quad \text{on } [a,b].$$

The proof is completed. □

## References


[1] P.P. Korovkin, Linear Operators and Approximation Theory, Hindustan Publ. Co., Delhi, 1960.
[2] F. Altomare, M. Campiti, Korovkin-Type Approximation Theory and its Applications, in: de Gruyter Stud. Math., vol. 17, Walter de Gruyter, Berlin, 1994.
[3] R. Bojanic, M.K. Khan, Summability of Hermite–Fejér interpolation for functions of bounded variation, J. Natur. Sci. Math. 32 (1992) 5–10.
[4] G.A. Anastassiou, O. Duman, A Baskakov type generalization of statistical Korovkin theory, J. Math. Anal. Appl. 340 (2008) 476–486.
[5] G.A. Anastassiou, O. Duman, Statistical fuzzy approximation by fuzzy positive linear operators, Comput. Math. Appl. 55 (2008) 573–580.
[6] O. Duman, M.K. Khan, C. Orhan, *A*-statistical convergence of approximating operators, Math. Inequal. Appl. 4 (2003) 689–699.
[7] O. Duman, E. Erkuş, V. Gupta, Statistical rates on the multivariate approximation theory, Math. Comput. Modelling 44 (2006) 763–770.

[8] E. Erkuş, O. Duman, H.M. Srivastava, Statistical approximation of certain positive linear operators constructed by means of the Chan–Chyan–Srivastava polynomials, Appl. Math. Comput. 182 (2006) 213–222.
[9] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32 (2002) 129–138.
[10] S. Karakuş, K. Demirci, O. Duman, Equi-statistical convergence of positive linear operators, J. Math. Anal. Appl. 339 (2008) 1065–1072.
[11] M. Mursaleen, O.H.H. Edely, On invariant mean and statistical convergence, Appl. Math. Lett. (2009) doi:10.1016/j.aml.2009.06.005.
[12] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[13] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.
[14] H. Steinhaus, Sur la convergence ordinaire et la convergence asymtotique, Colloq. Math. 2 (1951) 73–74.
[15] R.A. Raimi, Invariant means and invariant matrix methods of summability, Duke Math. J. 30 (1963) 81–94.