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Korovkin type approximation theorems in *B*-statistical sense

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ABSTRACT

In this paper, using the concept of $\mathfrak{B}$-statistical convergence for sequence of infinite matrices $\mathfrak{B} = (\mathfrak{B}_i)$ with $\mathfrak{B}_i = (b_{nk}(i))$ we investigate various approximation results concerning the classical Korovkin theorem. Then we present two examples of sequences of positive linear operators. The first one shows that the statistical Korovkin type theorem does not work but our approximation theorem works. The second one gives that our approximation theorem does not work but the statistical Korovkin type theorem works. Also, we study the rates of $\mathfrak{B}$-statistical convergence of approximating positive linear operators and give a Voronovskaya-type theorem.

1. Introduction

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, numerical solutions of differential and integral equations [1–3]. In recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved [4–6]. Recall that every convergent sequence (in the usual sense) is statistically convergent but its converse is not always true. Also, statistical convergent sequences do not need to be bounded. So, the usage of this method of convergence in the approximation theory provides us many advantages. Recently, Kolk [7] generalized the idea of statistical convergence to $\mathfrak{B}$-statistical convergence by using the idea of $\mathfrak{B}$-summability. Neither of the two methods, statistical convergence and $\mathfrak{B}$-statistical convergence, implies the other. But, for a constant sequence $\mathfrak{B} = (A)$, the method of $\mathfrak{B}$-statistical convergence reduces to the method of A-statistical convergence. The main goal of this paper is to investigate their use in approximation theory.

We now recall some basic definitions and notations used in the paper.

Let $A = (a_{nk})$, $n, k = 1, 2, \ldots$ be an infinite summability matrix. For a given sequence $x := \{x_k\}$, the A-transform of x, denoted by $Ax := \{(Ax)_n\}$ is given by

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k,$$

provided the series converges for each $n \in \mathbb{N}$. We say that A is regular (see [8]) if

$$\lim Ax = L \quad \text{whenever } \lim x = L.$$

The set of all regular matrices $A = (a_{nk})$ with $a_{nk} \geq 0$ for all n, k, we denote by $\mathcal{T}^+$. Assume that A is a nonnegative regular summability matrix. Then the sequence $x = \{x_k\}$ is called A-statistically convergent to L provided that, for every $\varepsilon > 0$,

$$\lim_n \sum_{k:|x_k - L| \geq \varepsilon} a_{nk} = 0. \tag{1}$$

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We denote this limit as follows (*cf*.[9]; see also [10,11])

$$st_A - \lim x = L.$$

Actually, this convergence method is based on the concept of A-density. Recall the A-density of a subset $K \subset \mathbb{N}$, denoted by $\delta_A\{K\}$, is given by

$$\delta_A\{K\} = \lim_n \sum_{k=1}^{\infty} a_{nk}\chi_K(k),$$

provided the limits exists, where χ_K is the characteristic function of K; or equivalently

$$\delta_A\{K\} = \lim_n \sum_{k \in K} a_{nk}.$$

Recently, Kolk [7] generalized the idea of A-statistical convergence to $\mathfrak{B}$-statistically convergence by using the idea of $\mathfrak{B}$-summability (or $F_{\mathfrak{B}}$-convergence) due to Steiglitz [12].

Let $\mathfrak{B} = (\mathfrak{B}_i)$ be a sequence of infinite matrices with $\mathfrak{B}_i = (b_{nk}(i))$. Then $x \in \ell_\infty$, which is the space of bounded sequences, is called to be $F_{\mathfrak{B}}$-convergent (or $\mathfrak{B}$-summable) to the value $\mathfrak{B} - \lim x$ if

$$\lim_n (\mathfrak{B}_i x)_n = \lim_n \sum_k b_{nk}(i)\, x_k = \mathfrak{B} - \lim x, \quad \text{uniformly in } i \geq 0.$$

The method of summability $\mathfrak{B}$ is called regular if every convergent sequence $x = \{x_k\}$ is $\mathfrak{B}$-summable and $\mathfrak{B}$- $\lim x = \lim x$. Method $\mathfrak{B} = (\mathfrak{B}_i)$ with $\mathfrak{B}_i = (b_{nk}(i))$ is regular [13,12] if and only if

(R1) for each $n, i = 1, 2, \ldots, \sum_k |b_{nk}(i)| < \infty$ and there exists integers N, M such that $\sum_k |b_{nk}(i)| < M$ for $n \geq N$ and all $i = 1, 2, \ldots$,
(R2) $\lim_n b_{nk}(i) = 0$ for all $k \in \mathbb{N}$, uniformly in i,
(R3) $\lim_n \sum_k b_{nk}(i) = 1$, uniformly in i.

The set of all regular methods $\mathfrak{B}$ with $b_{nk}(i) \geq 0$ for all n, k, i we denote by the symbol $\mathfrak{R}^+$. Since for a constant sequence $\mathfrak{B} = (A)$ the method $\mathfrak{B}$ reduces the method A, we may write $\mathcal{T}^+ \subset \mathfrak{R}^+$.

The $\mathfrak{B}$-density of a subset $K = \{k_i\} \subset \mathbb{N}$, where $k_i \leq k_{i+1}$ for all i, denoted by $\delta_{\mathfrak{B}}\{K\}$, is given by

$$\lim_n \sum_{k \in K} b_{nk}(i) = d \quad \text{uniformly in } i.$$

Let $\mathfrak{B} \in \mathfrak{R}^+$. A sequence $x = \{x_k\}$ is said to be $\mathfrak{B}$-statistically convergent to the number L if,for every $\varepsilon > 0$,

$$\delta_{\mathfrak{B}}\{k : |x_k - L| \geq \varepsilon\} = 0.$$

i.e.

$$\lim_n \sum_{k:|x_k - L| \geq \varepsilon} b_{nk}(i) = 0 \quad \text{uniformly in } i.$$

In this case we write $st_{\mathfrak{B}} - \lim x = L$.

With the following example, we can see that neither of the two methods, statistical convergence and $\mathfrak{B}$-statistically convergence, implies the other.

Example 1. Let us consider the sequence of infinite matrices $\mathfrak{B} = (\mathfrak{B}_i)$ with $\mathfrak{B}_i = (b_{nk}(i))$ is given by

$$b_{nk}(i) = \begin{cases} \dfrac{n^2+1}{n^2 i}, & \text{if } k = n^2, \\ 1 - \dfrac{n^2}{\left(n^2+1\right) i}, & \text{if } k = n^2+1, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

It is observe that $\mathfrak{B} \in \mathfrak{R}^+$. Also let us define the sequence $x = \{x_k\}$ and $y = \{y_k\}$ by

$$x_k = \begin{cases} 1, & \text{if } k = n^2, \text{ for some } n \in \mathbb{N}, \\ 0, & \text{if } k = n^2+1, \text{ for some } n \in \mathbb{N}, \\ k, & \text{otherwise,} \end{cases} \tag{3}$$

and

$$y_k = \begin{cases} \sqrt{k}, & \text{if } k = n^2, \text{for some } n \in \mathbb{N}, \\ k, & \text{if } k = n^2+1, \text{for some } n \in \mathbb{N}, \\ \dfrac{1}{k}, & \text{otherwise.} \end{cases} \tag{4}$$

It is clear that x is not statistically convergent to zero but x is $\mathfrak{B}$-statistically convergent to zero. On the other hand y is statistically convergent to zero but y is not $\mathfrak{B}$-statistically convergent to zero.

2. A Korovkin-type approximation theorem

We denote by $C_M[a,b]$ the space of all continuous functions on $[a,b]$ and bounded on entire line, that is

$$|f(x)| \leq K_f, \quad -\infty < x < \infty,$$

where K_f is a constant depending on f. Also by $B[a,b]$ we denote the space of all bounded functions on $[a,b]$. This space is equipped with the supremum norm

$$\|f\|_{B[a,b]} = \sup_{x\in[a,b]} |f(x)|, \quad (f \in B[a,b]).$$

Theorem 2. *Let $\mathfrak{B} \in \mathfrak{R}^+$. Let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a,b]$ into $B[a,b]$. Then, for all $f \in C_M[a,b]$,*

$$st_{\mathfrak{B}} - \lim \|L_k(f) - f\|_{B[a,b]} = 0 \tag{5}$$

if and only if

$$st_{\mathfrak{B}} - \lim \left\|L_k(f_j) - f_j\right\|_{B[a,b]} = 0, \ (j = 0, 1, 2), \tag{6}$$

where $f_j(t) = t^j$, $j = 0, 1, 2$.

Proof. Since each $f_j \in C_M[a,b]$, $(j = 0, 1, 2, 3)$, the implication (5) $\Rightarrow$ (6) is obvious. Suppose now that (6) holds. Since f is bounded on the entire line, we can write

$$|f(x)| \leq K.$$

Also, since f is continuous on $[a,b]$, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for all $x \in [a,b]$ satisfying $|t - x| < \delta$. Hence, we get

$$|f(t) - f(x)| < \varepsilon + \frac{2K}{\delta^2}(t - x)^2. \tag{7}$$

Since L_k is linear and positive, we obtain

$$\begin{aligned}
|L_k(f;x) - f(x)| &= |L_k(f(t) - f(x); x) + f(x)(L_k(f_0;x) - f_0(x))| \\
&\leq L_k(|f(t) - f(x)|; x) + K|L_k(f_0;x) - f_0(x)| \\
&\leq \left|L_k\left(\varepsilon + \frac{2K}{\delta^2}(t-x)^2; x\right)\right| + K|L_k(f_0;x) - f_0(x)| \\
&\leq \left(\varepsilon + K + \frac{2K}{\delta^2}c^2\right)|L_k(f_0;x) - f_0(x)| + \frac{4K}{\delta^2}c\,|L_k(f_1;x) - f_1(x)| + \frac{2K}{\delta^2}|L_k(f_2;x) - f_2(x)| + \varepsilon
\end{aligned}$$

where $c := \max\{|a|, |b|\}$. Then taking supremum we get

$$\|L_k(f) - f\|_{B[a,b]} \leq N\left\{\|L_k(f_0) - f_0\|_{B[a,b]} + \|L_k(f_1) - f_1\|_{B[a,b]} + \|L_k(f_2) - f_2\|_{B[a,b]}\right\} + \varepsilon \tag{8}$$

where $N := \max\left\{\varepsilon + K + \frac{2K}{\delta^2}c^2, \frac{4K}{\delta^2}c, \frac{2K}{\delta^2}\right\}$. Now, for a given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$. Then, setting

$$D := \left\{k : \|L_k(f) - f\|_{B[a,b]} \geq \varepsilon'\right\},$$

$$D_j := \left\{k : \left\|L_k(f_j) - f_j\right\|_{B[a,b]} \geq \frac{\varepsilon' - \varepsilon}{3N}\right\}, \quad j = 0, 1, 2.$$

Then it is easy to see that

$$D \subseteq \bigcup_{j=0}^{2} D_j$$

which gives, for all $n \in \mathbb{N}$,

$$\sum_{k\in D} b_{nk}(i) \leq \sum_{j=0}^{2}\sum_{k\in D_j} b_{nk}(i).$$

Letting $n \to \infty$ and using (6), we obtain (5). The proof is complete. $\square$

Remark 3. If we take $\mathfrak{B} = \{\mathcal{I}\}$, the identity matrix, in Theorem 2, then we immediately get the following classical result which was introduced by Korovkin [3].

Corollary 4. *Let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a, b]$ into $B[a, b]$. Then, for all $f \in C_M[a, b]$,*

$$\lim_k \|L_k(f) - f\|_{B[a,b]} = 0 \tag{9}$$

if and only if

$$\lim_k \left\|L_k\left(f_j\right) - f_j\right\|_{B[a,b]} = 0, \ (j = 0, 1, 2), \tag{10}$$

where $f_j(t) = t^j$, $j = 0, 1, 2$.

Remark 5. If we take $\mathfrak{B} = \{C_1\}$, the Cesáro matrix, in Theorem 2, then we immediately get the following statistical result which was introduced by Gadjiev and Orhan [14].

Corollary 6. *Let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a, b]$ into $B[a, b]$. Then, for all $f \in C_M[a, b]$,*

$$st - \lim \|L_k(f) - f\|_{B[a,b]} = 0 \tag{11}$$

if and only if

$$st - \lim \left\|L_k\left(f_j\right) - f_j\right\|_{B[a,b]} = 0, \ (j = 0, 1, 2), \tag{12}$$

where $f_j(t) = t^j$, $j = 0, 1, 2$.

Remark 7. Now we present two examples of sequences of positive linear operators. The first one shows that Corollary 6 does not work but our approximation theorem works. The second one gives that our approximation theorem does not work but Corollary 6 works. We note that the sequence of classical Bernstein polynomials on $C_M[0, 1]$ (see [3])

$$B_k(f; x) = \sum_{t=0}^{k} f\left(\frac{t}{k}\right)\binom{k}{t} x^t (1 - x)^{k-t} \tag{13}$$

satisfies the conditions of the classical Korovkin theorem. It is known that

$$B_k(f_0; x) = f_0(x), \ B_k(f_1; x) = f_1(x) \text{ and } B_k(f_2; x) = f_2(x) + \frac{x - x^2}{k}.$$

(a) Now we define the following positive linear operators on $C_M[0, 1]$ into $B[0, 1]$ as follows:

$$L_k(f; x) = (1 + x_k) B_k(f; x) \tag{14}$$

where $x = \{x_k\}$ is defined as in Example 1. Also, $\mathfrak{B} = (\mathfrak{B}_i)$ is the same as in Example 1. Since $st_{\mathfrak{B}} - \lim x = 0$, we conclude that

$$st_{\mathfrak{B}} - \lim \left\|L_k\left(f_j\right) - f_j\right\|_{B[0,1]} = 0, j = 0, 1, 2.$$

Then, by Theorem 2, we obtain, for all $f \in C_M[0, 1]$,

$$st_{\mathfrak{B}} - \lim \|L_k(f) - f\|_{B[0,1]} = 0.$$

However, since x is not statistically convergent, the sequence $\{L_k(f; x)\}$ given by (14) does not satisfy the Corollary 6 for all $f \in C_M[0, 1]$.
(b) Now we consider the following positive linear operators on $C_M[0, 1]$ into $B[0, 1]$ as follows:

$$L_k(f; x) = (1 + y_k) B_k(f; x) \tag{15}$$

where $y = \{y_k\}$ is defined as in Example 1. Also, $\mathfrak{B} = (\mathfrak{B}_i)$ is the same as in Example 1. Since $st - \lim y = 0$, we conclude that

$$st - \lim \left\|L_k\left(f_j\right) - f_j\right\|_{B[0,1]} = 0, j = 0, 1, 2.$$

Then, by Corollary 6, we obtain, for all $f \in C_M[0, 1]$,

$$st - \lim \|L_k(f) - f\|_{B[0,1]} = 0.$$

However, since x is not $\mathfrak{B}$-statistically convergent, the sequence $\{L_k(f; x)\}$ given by (15) does not satisfy the Theorem 2 for all $f \in C_M[0, 1]$.

Remark 8. For a constant $\mathfrak{B} = \{A\}$, the A is a non-negative regular summability matrix, then $\mathfrak{B}$-statistically convergence is reduced to the A-statistical convergence [15,9]. For a constant $\mathfrak{B} = (\mathfrak{B}_1)$, it is reduced to uniform statistical convergence [16], where $\mathfrak{B}_1 = \left(b^1_{nk}(i)\right)$ with

$$b^1_{nk}(i) = \begin{cases} \frac{1}{n}, & \text{if } 1+i \le k \le n+i, \\ 0, & \text{otherwise.} \end{cases}$$

3. Rates of $\mathfrak{B}$-statistical convergence of Theorem 2

In this section we study the rates of $\mathfrak{B}$-statistical convergence of a sequence of positive linear operators defined $C_M[a,b]$ into $B[a,b]$ with the help modulus of continuity.

We now present the following definition.

Definition 9. Let $\mathfrak{B} \in \mathfrak{R}^+$ and let $\{a_k\}$ be a positive non-increasing sequence. A sequence $x = \{x_k\}$ is $\mathfrak{B}$-statistical convergent to a number P with the rate $o(a_k)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k:|x_k-P|\ge\varepsilon} b_{nk}(i) = 0, \quad \text{uniformly in } i.$$

In this case, it is denoted by

$$x_k - P = st_{\mathfrak{B}} - o(a_k).$$

Definition 10. Let $\mathfrak{B} \in \mathfrak{R}^+$ and let $\{a_k\}$ be a positive non-increasing sequence. A sequence $x = \{x_k\}$ is $\mathfrak{B}$-statistical convergent to a number P with the rate $o_m(a_k)$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k:|x_k-P|\ge\varepsilon a_k} b_{nk}(i) = 0, \quad \text{uniformly in } i.$$

In this case, it is denoted by

$$x_k - P = st_{\mathfrak{B}} - o_m(a_k).$$

In Definition 9 the "rate" is more controlled by the entries of the summability method rather than the terms of the sequence $\{x_k\}$. For instance, when one takes $\mathfrak{B} = \{\mathcal{I}\}$, the identity matrix if $b_{nn} = o(a_k)$ then $x_k - P = st_{\mathfrak{B}} - o(a_k)$,as $k \to \infty$, for any convergent sequence $\{x_k - P\}$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation one may borrow the concept of convergence in measure from measure theory to define the rate of convergences as in Definition 10.

Now we first need the following lemma to get the rates of $\mathfrak{B}$-statistical convergence in Theorem 2 by using Definition 9.

Lemma 11. *Let $\{x_k\}$ and $\{y_k\}$ be sequences. Assume that let $\mathfrak{B} \in \mathfrak{R}^+$ and let $\{a_k\}$, $\{b_k\}$ be positive non-increasing sequences. If $x_k - P_1 = st_{\mathfrak{B}} - o(a_k)$ and $y_k - P_2 = st_{\mathfrak{B}} - o(b_k)$, then the following hold:*

(1) $(x_k - P_1) \mp (y_k - P_2) = st_{\mathfrak{B}} - o(c_k)$,
(2) $(x_k - P_1)(y_k - P_2) = st_{\mathfrak{B}} - o(c_k)$,
(3) $\lambda(x_k - P_1) = st_{\mathfrak{B}} - o(a_k)$ *for any real number* λ,
(4) $\sqrt{|x_k - P_1|} = st_{\mathfrak{B}} - o(a_k)$

where $c_k = \max\{a_k, b_k\}$.

Proof. (1) Assume that $x_k - P_1 = st_{\mathfrak{B}} - o(a_k)$ and $y_k - P_2 = st_{\mathfrak{B}} - o(b_k)$. Also, for $\varepsilon > 0$, define

$$D := \{k : |(x_k - P_1) \mp (y_k - P_2)| \ge \varepsilon\},$$
$$D_1 := \left\{k : |x_k - P_1| \ge \frac{\varepsilon}{2}\right\},$$
$$D_2 := \left\{k : |y_k - P_2| \ge \frac{\varepsilon}{2}\right\}.$$

Then observe that

$$D \subset D_1 \cup D_2$$

which gives for $n \in \mathbb{N}$,

$$\sum_{k\in D} b_{nk}(i) \le \sum_{k\in D_1} b_{nk}(i) + \sum_{k\in D_2} b_{nk}(i). \tag{16}$$

Since $c_n = \max\{a_n, b_n\}$, by (16), we get

$$\frac{1}{c_n}\sum_{k\in D} b_{nk}(i) \le \frac{1}{a_n}\sum_{k\in D_1} b_{nk}(i) + \frac{1}{b_n}\sum_{k\in D_2} b_{nk}(i)\,. \tag{17}$$

Now by taking limit as $n\to\infty$ in (17) and using the hypotheses, we conclude that

$$\lim_{n\to\infty}\frac{1}{c_n}\sum_{k\in D} b_{nk}(i) = 0, \text{ uniformly in } i.$$

which completes the proof of (1). Since the proof of (2), (3) and (4) are similar, we omit them. □

On the other hand, we recall that the modulus of continuity of a function $f\in C_M[a,b]$ is defined by

$$w(f,\delta) = \sup_{|t-x|\le\delta,\, x,t\in[a,b]} |f(t)-f(x)|\,, (\delta>0).$$

Then it is clear that for any $\delta>0$ and each $x,t\in[a,b]$

$$|f(t)-f(x)| \le w(f,\delta)\left(\frac{|t-x|}{\delta}+1\right), \quad (f\in C_M[a,b])\,. \tag{18}$$

Now we have the following result.

Theorem 12. *Let $\mathfrak{B}\in\mathfrak{R}^+$. Let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a,b]$ into $B[a,b]$. Let $\{a_k\}$, $\{b_k\}$ be a positive non-increasing sequences. Assume that the following conditions holds:*

(a) $\|L_k(f_0)-f_0\|_{B[a,b]} = st_{\mathfrak{B}} - o(a_k)$ *on* $[a,b]$, *where* $f_0(t)=1$,
(b) $w(f,\alpha_k) = st_{\mathfrak{B}} - o(b_k)$ *on* $[a,b]$, *where* $\alpha_k := \sqrt{\|L_k(\varphi;x)\|_{B[a,b]}}$ *with* $\varphi(t)=(t-x)^2$.

Then we have, for all $f\in C_M[a,b]$,

$$\|L_k(f)-f\|_{B[a,b]} = st_{\mathfrak{B}} - o(c_k) on\ [a,b]\,,$$

where $c_k = \max\{a_k,b_k\}$.

Proof. Let $f\in C_M[a,b]$ and $x\in[a,b]$. Then, since $f_0(t)=1$, by (18) and using linearity and monotonicity of the operators L_k, we get, for any $\delta>0$ and $k\in\mathbb{N}$, that

$$\begin{aligned}
|L_k(f;x)-f(x)| &\le L_k(|f(t)-f(x)|\,;x) + |f(x)|\,|L_k(f_0;x)-f_0(x)| \\
&\le L_k(|f(t)-f(x)|\,;x) + K\,|L_k(f_0;x)-f_0(x)| \\
&\le w(f,\delta)L_k(\frac{|t-x|}{\delta}+f_0(t);x) + K\,|L_k(f_0;x)-f_0(x)| \\
&\le w(f,\delta)L_k(\frac{\varphi(t)}{\delta^2}+f_0(t);x) + K\,|L_k(f_0;x)-f_0(x)| \\
&= w(f,\delta)\left\{\frac{1}{\delta^2}L_k(\varphi(t);x)+L_k(f_0;x)\right\} + K\,|L_k(f_0;x)-f_0(x)|
\end{aligned}$$

where $K = ||f||_{B[a,b]}$. Then we obtain

$$\|L_k(f)-f\|_{B[a,b]} \le w(f,\delta)\left\{\frac{1}{\delta^2}\,\|L_k(\varphi;x)\|_{B[a,b]} + \|L_k(f_0)-f_0\|_{B[a,b]}+1\right\} + K\,\|L_k(f_0)-f_0\|_{B[a,b]}\,.$$

Put $\delta=\alpha_k := \sqrt{\|L_k(\varphi;x)\|_{B[a,b]}}$, so we get

$$\begin{aligned}
\|L_k(f)-f\|_{B[a,b]} &\le w(f,\alpha_k)\left\{\|L_k(f_0)-f_0\|_{B[a,b]}+2\right\} + K\,\|L_k(f_0)-f_0\|_{B[a,b]} \\
&= 2w(f,\alpha_k) + w(f,\alpha_k)\,\|L_k(f_0)-f_0\|_{B[a,b]} + K\,\|L_k(f_0)-f_0\|_{B[a,b]}\,.
\end{aligned} \tag{19}$$

Given $\varepsilon>0$, by (19) we have

$$\frac{1}{c_n}\sum_{k:\|L_k(f)-f\|_{B[a,b]}\ge\varepsilon} b_{nk}(i) \le \frac{1}{b_n}\sum_{k:w(f,\alpha_k)\ge\frac{\varepsilon}{6}} b_{nk}(i) + \frac{1}{c_n}\sum_{k:w(f,\alpha_k)\|L_k(f_0)-f_0\|_{B[a,b]}\ge\frac{\varepsilon}{3}} b_{nk}(i) + \frac{1}{a_n}\sum_{k:\|L_k(f_0)-f_0\|_{B[a,b]}\ge\frac{\varepsilon}{3K}} b_{nk}(i)\,.$$

Letting $n\to\infty$ and considering the above inequality, the hypotheses (a), (b), and Lemma 11, the proof is completed at once. □

4. A Voronovskaya-type theorem

In this section, we obtain a Voronovskaya-type theorem $\mathfrak{B}$-statistical case for the positive linear operators $\{L_k\}$ given by (14).

Lemma 13. *Let $\mathfrak{B} \in \mathfrak{R}^{+}$. Let $x \in [0, 1]$. Then, we get*

$$st_{\mathfrak{B}} - \lim k^2 L_k\left((t-x)^4 ; x\right) = 3f_2(x)\left[f_2(x) - 2f_1(x) + f_0(x)\right]. \tag{20}$$

Proof. After some simple calculations, we can write form (20) that

$$k^2 L_k\left((t-x)^4 ; x\right) = (1+x_k)\left[3x^4 - 6x^3 + 3x^2 + \frac{-6x^4 + 12x^3 - 7x^2 + x}{k}\right].$$

Hence, we obtain

$$\left|k^2 L_k\left((t-x)^4 ; x\right) - 3f_2(x)\left[f_2(x) - 2f_1(x) + f_0(x)\right]\right| \leq 12x_k + (1+x_k)\frac{26}{k} \tag{21}$$

for every $x \in [0, 1]$. Since $st_{\mathfrak{B}} - \lim x = 0$, it is easy see that

$$st_{\mathfrak{B}} - \lim\left[12x_k + (1+x_k)\frac{26}{k}\right] = 0. \tag{22}$$

Combining (21) and (22), the proof is complete. $\square$

Theorem 14. *Let $\mathfrak{B} \in \mathfrak{R}^{+}$. For every $f \in C_M[0, 1]$ such that $f^{'}, f^{''} \in C_M[0, 1]$, we have*

$$st_{\mathfrak{B}} - \lim k\left\{L_k(f; x) - f(x)\right\} = \frac{1}{2}\left(f_1(x) - f_2(x)\right) f^{''}(x).$$

Proof. Let $x \in [0, 1]$ and $f, f^{'}, f^{''} \in C_M[0, 1]$. Define the function ϕ by

$$\phi_x(t) = \begin{cases} \dfrac{f(t) - f(x) - f^{'}(x)(t-x) - \frac{1}{2}f^{''}(x)(t-x)^2}{(t-x)^2}, & t \neq x, \\ 0, & t = x. \end{cases}$$

By the Taylor formula for $f \in C_M[0, 1]$, we have

$$f(t) = f(x) + f^{'}(x)(t-x) + \frac{1}{2}f^{''}(x)(t-x)^2 + \phi_x(t)(t-x)^2$$

where $\phi_x(x) = 0$ and $\phi_x(.) \in C_M[0, 1]$. Since the operator L_k is linear, we obtain

$$L_k(f; x) = f(x)(1+x_k) + f^{'}(x) L_k(t-x; x) + \frac{1}{2}f^{''}(x) L_k\left((t-x)^2 ; x\right) + L_k\left(\phi_x(t)(t-x)^2 ; x\right)$$

Now we obtain

$$L_k(f; x) = f(x)(1+x_k) + \frac{1+x_k}{2k}\left(f_1(x) - f_2(x)\right) f^{''}(x) + L_k\left(\phi_x(t)(t-x)^2 ; x\right)$$

which yields

$$k\left\{L_k(f; x) - f(x)\right\} = kx_k f(x) + \frac{1+x_k}{2}\left(f_1(x) - f_2(x)\right) f^{''}(x) + kL_k\left(\phi_x(t)(t-x)^2 ; x\right). \tag{23}$$

Then we obtain

$$\left|k\left\{L_k(f; x) - f(x)\right\} - \frac{1}{2}\left(f_1(x) - f_2(x)\right) f^{''}(x)\right| \leq M_1 k x_k + \frac{M_2}{8}x_k + k\left|L_k\left(\phi_x(t)(t-x)^2 ; x\right)\right| \tag{24}$$

where $M_1 = \|f\|_{B[0,1]}$ and $M_2 = \left\|f^{''}\right\|_{B[0,1]}$. Applying the Cauchy–Schwarz inequality for the third term on the right-hand side of (24), we get

$$k\left|L_k\left(\phi_x(t)(t-x)^2 ; x\right)\right| \leq \left(L_k\left(\phi_x^2(t) ; x\right)\right)^{1/2} \cdot \left(k^2 L_k\left((t-x)^4 ; x\right)\right)^{1/2}. \tag{25}$$

Let $\eta_x(t) = \phi_x^2(t)$. In this case, we show that $\eta_x(x) = 0$ and $\eta_x(.) \in C_M[0, 1]$. From Theorem 2,

$$st_{\mathfrak{B}} - \lim L_k\left(\phi_x^2(t); x\right) = st_{\mathfrak{B}} - \lim L_k(\eta_x(t); x) = \eta_x(x) = 0. \tag{26}$$

Using (26) and Lemma 13, we obtain

$$st_{\mathfrak{B}} - \lim kL_k\left(\phi_x(t)(t-x)^2; x\right) = 0. \tag{27}$$

Using (27) in (24) and $st_{\mathfrak{B}} - \lim x = 0$, the proof is complete. □

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