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CHAPTER I

On Bicomplex Lorentz Sequence Space

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1.Introduction

The widely known appendage of the complex number field to the four-dimensional field is by a work titled "Algebra on Quaternions or a new imaginary system" by W.R. Hamilton in 1844. As an idea, Quaternions emerged by take into account three imaginary units (i, j, k) , where $k = ij$, which are non-commutative. The importance of the speculation of quaternions is that it creates a field in which entire known operations can be carried out. Loss of commutativity can be considered as a deficiency. Although from a

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fully algebraic point of view the deficiency of commutativity is not such a horrible problem, in some studies it poses many hardship, for example, when we try to expand the fertile speculation of holomorphic functions of a complex variable to quaternions.

Therefore, it is not illogical to think whether a four-dimensional algebra that includes $\mathbb{C}$ as the lower algebra could be included in a way that conserves the commutative property. It is not surprising that this is done by taking into account two imaginary units i, j by publicity $k = ij$ (even as in the quaternionic case), but now majestic $ij = ji$. This converts k to the hyperbolic imaginary unit, i.e. something like $k^2 = 1$. As far as is known, J. Cockle wrote a series of articles in 1848 regarding these new algebra studies. Cockle's work was absolutely encouraged by Hamilton's and he was the first to use tessarini-algebra to insulate the hyperbolic trigonometric series as components of the exponential series. Not surprisingly, Cockle instantly noticed that there was a price to be paid for commutativity in four dimensions, and the compensation was the asset of zero-divisors.

Only in 1892, inspired by the study of Hamilton and Clifford, did the mathematician Corrado Segre introduce what he named the algebra of bicomplex numbers, the equivalent of tessarine algebra. In his individual papers, Segre recognized that the elements $((1+ij))^2$ and $((1-ij))^2$ are idempotent and play a head role in even complex number speculation. After Segre, several other mathematicians, notably (Spampinato, 1935, 1936) and (Scorza Dragoni, 1934), advanced the first basics of a speculation of functions on even complex numbers.

The subsequent grand breakthrough in the work of bicomplex analysis was the work of J.D. Riley in 1953, in which he further advanced the speculation of functions of bicomplex variables. However, the most significant contribution was undoubtedly the work of G. B. Price (Price, 1991); here the speculation of holomorphic functions of two complex variables (as well as very complex variables) was fully developed. Until this monograph, the

work of G. B. Price is accepted the foundational study of this speculation.

However, recently there has been a significant revival of interest in the work of holomorphic functions on one and several pairs of complex variables, as well as in improvement functional analysis on spaces having a build consisting of modules on two rings of complex numbers. B. Sağır one of those who carried out these studies has carried out many studies on $\mathbb{BC}$.

We now explain the content of this study. First of all, the bicomplex number $\mathbb{BC}$ is introduced and its features is explained with examples. Hyperbolic numbers, which are a subset of the bicomplex set, will be mentioned. A hyperbolic-valued norm is defined. The measure space is introduced and accordingly the Distribution, Decreasing rearrangement and Average function definitions of a function is made. Without changing the basis of these definitions, by using the set of Hyperbolic numbers and the Hyperbolic valued norm, the $\mathbb{D}$ -Distribution, $\mathbb{D}$ -Decreasing rearrangement and $\mathbb{D}$ -Average functions of a sequence with Bicomplex terms are created and the properties of these functions are examined.

Definition of Bicomplex Numbers

The members of set represented by $\mathbb{BC} = \{z_1 + jz_2: z_1, z_2 \in \mathbb{C}\}$ is called bicomplex numbers, where $\mathbb{C}$ is the set of complex numbers with the virtual unit i , and also where i and j are commuting virtual units, i.e., $ij = ji = k$, $i^2 = j^2 = -1$ and

$$k^2 = (ij)^2 = (ij)(ij) = i(ji)j = i(ij)j = (ii)(jj) = i^2j^2 = 1.$$

So bicomplex numbers are “complex numbers with complex coefficients”, which express the name of bicomplex, and in what follows we will attempt to emphasize the similarities between the feature of complex and bicomplex numbers.

Any two elements in the set of bicomplex numbers can be added and multiplied. Let $u = u_1 + ju_2$ and $v = v_1 + jv_2$ are two

bicomplex numbers, the formulas for the summation and the product of two bicomplex numbers are given by

$$u + v = (u_1 + v_1) + j(u_2 + v_2) \quad (1.1)$$

and

$$u \cdot v = (u_1 v_1 - u_2 v_2) + j(u_1 v_2 + u_2 v_1) \quad (1.2)$$

respectively. According to operations (1.1) and (1.2) described above, the set $\mathbb{BC}$ is a commutative and unitary ring with $1_{\mathbb{BC}} = 1 + j \cdot 0$.

Example 1. Investigate whether the bicomplex number $u = (2 - i) + j(1 + 2i)$ is invertible in the $\mathbb{BC}$ space.

Solution. Assume that

$$u^{-1} = v = v_1 + jv_2 = (x_1 + iy_1) + j(x_2 + iy_2)$$

with $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$. Then

$$\begin{aligned} u \cdot v &= ((2 - i) + j(1 + 2i)) \cdot ((x_1 + iy_1) + j(x_2 + iy_2)) \\ &= (2x_1 + y_1 - x_2 + 2y_2) + i(-x_1 + 2y_1 - 2x_2 - y_2) \\ &\quad + j(x_1 - 2y_1 + 2x_2 + y_2) + ij(2x_1 + y_1 - x_2 + 2y_2) \\ &= 1 + j \cdot 0 \end{aligned}$$

and hence a system of equations

$$\begin{aligned} 2x_1 + y_1 - x_2 + 2y_2 &= 1 \\ -x_1 + 2y_1 - 2x_2 - y_2 &= 0 \\ x_1 - 2y_1 + 2x_2 + y_2 &= 0 \\ 2x_1 + y_1 - x_2 + 2y_2 &= 0 \end{aligned} \quad (1.3)$$

is obtained, if the coefficients of the mutual imaginary components in this equation are equalized. Since the system of equations (1.3) has no solution, $u = (2 - i) + j(1 + 2i)$ has no inverse in $\mathbb{BC}$. Then, since every element of $\mathbb{BC}$ is not invertible with respect to multiplication, we can say that $\mathbb{BC}$ is not a field.

In addition, when $z_2 = 0$ in $z = z_1 + jz_2$, that is, $z = z_1$, the set of these numbers is represented by $\mathbb{C}(i)$. If the z_1 and z_2 coefficients are real numbers, that is, $z = x_1 + jx_2$ with $x_1, x_2 \in \mathbb{R}$, the set of those numbers is represented by $\mathbb{C}(j)$. $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are isomorphic fields but coexisting internal of $\mathbb{BC}$ they are distinct. We will see many times in what follows that there is a specific asymmetry in their attitude.

Definition 1. The set of hyperbolic numbers is described by

$$\mathbb{D} = \{x + ky : x, y \in \mathbb{R}, k = ij\},$$

where k is a hyperbolic virtual unit, i.e., $k^2 = 1$. In the studies conducted in the current literature, hyperbolic numbers are sometimes called duplex, double or bireal numbers. The following subsets $\mathbb{D}^+$ and $\mathbb{D}^+ \setminus \{0\}$ of $\mathbb{D}$ are called non-negative and positive hyperbolic numbers, respectively:

$$\mathbb{D}^+ = \{x + ky : x^2 - y^2 \geq 0, x^2 \geq 0\} \quad (1.4)$$

$$\mathbb{D}^+ \setminus \{0\} = \{x + ky : x^2 - y^2 \geq 0, x^2 > 0\}. \quad (1.5)$$

Similarly, non-negative and negative hyperbolic numbers are defined as follows:

$$\mathbb{D}^- = \{x + ky : x^2 - y^2 \geq 0, x^2 \leq 0\} \quad (1.6)$$

$$\mathbb{D}^- \setminus \{0\} = \{x + ky : x^2 - y^2 \geq 0, x^2 < 0\}. \quad (1.7)$$

Idempotent Representations of Bicomplex Numbers

Consider the bicomplex numbers $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$. It can be easily seen that $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$. There are also equations $(e_1)^n = e_1$, $(e_2)^n = e_2$ with $n \geq 2$. For any bicomplex number $u = u_1 + ju_2 \in \mathbb{BC}$, we have

$$\begin{aligned} u &= \frac{u_1 + iu_2 + u_1 - iu_2}{2} + j \frac{u_2 + iu_1 + u_2 - iu_1}{2} \\ &= \frac{u_1 - iu_2}{2} + \frac{u_1 + iu_2}{2} + j \left(i \frac{u_1 - iu_2}{2} - i \frac{u_1 + iu_2}{2} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{u_1 - iu_2}{2}(1 + ij) + \frac{u_1 + iu_2}{2}(1 + ij) \\
&= (u_1 - iu_2)e_1 + (u_1 + iu_2)e_2 = \delta_1 e_1 + \delta_2 e_2 \quad (1.8)
\end{aligned}$$

with $\delta_1 = (u_1 - iu_2)$ and $\delta_2 = (u_1 + iu_2)$ in $\mathbb{C}(i)$. This equality (1.8) is named the $\mathbb{C}(i)$ -idempotent representation of the bicomplex number u .

Similarly, along with the coefficients in $\mathbb{C}(j)$, there is also a representation of the bicomplex number u with respect to e_1 and e_2 .

As a result, any bicomplex number has an idempotent representation with its coefficients in any of $\mathbb{C}(i)$ or $\mathbb{C}(j)$, that is, $u = \delta_1 e_1 + \delta_2 e_2 = \rho_1 e_1 + \rho_2 e_2$ where $\delta_1, \delta_2 \in \mathbb{C}(i)$ and $\rho_1, \rho_2 \in \mathbb{C}(j)$.

A Partial Order on The Set of Hyperbolic Numbers

Let u and v be two elements of the set $\mathbb{D}^+$. If $u - v \in \mathbb{D}^+$, that is, $u - v$ is a non-negative hyperbolic number, then we write $u \succcurlyeq v$ or $v \preccurlyeq u$, and also u is said to be $\mathbb{D}$ -greater or $\mathbb{D}$ -equal than v , or that v is $\mathbb{D}$ -less or $\mathbb{D}$ -equal than u .

If $u = u_1 e_1 + u_2 e_2$ and $v = v_1 e_1 + v_2 e_2$, with real numbers u_1, u_2, v_1 and v_2 , we have

$$v \preccurlyeq u \Leftrightarrow v_1 \leq u_1 \wedge v_2 \leq u_2. \quad (1.9)$$

Also u is a (absolutely) positive hyperbolic number, then it is reversed and its inverse is also positive. Additionally, if $u > 0$ and $u < v$, then $v^{-1} > 0$ and $v^{-1} < u^{-1}$ (Luna-Elizarrarás, M.E. & et al., 2015).

The Hyperbolic-Valued $\mathbb{D}$ -Norm on $\mathbb{BC}$

A function $|\cdot|_k$ defined from $\mathbb{BC}$ to $\mathbb{D}^+$ with $|u|_k = |u_1|e_1 + |u_2|e_2$ for each $u = u_1 e_1 + u_2 e_2 \in \mathbb{BC}$ provides the following properties.

N_1) Since $|u_1| \geq 0$ and $|u_2| \geq 0$ for a $u = (u_1 e_1 + u_2 e_2)$ in $\mathbb{BC}$, we have $|u|_k = |u_1|e_1 + |u_2|e_2 \succcurlyeq 0e_1 + 0e_2 = 0$.

$N_2)$ Let $|u|_k = |u_1|e_1 + |u_2|e_2 = 0 = 0e_1 + 0e_2$ for any $u = (u_1e_1 + u_2e_2) \in \mathbb{B}\mathbb{C}$. Hence $|u_1| = 0$ ve $|u_2| = 0$, namely,

$u = 0e_1 + 0e_2 = 0$. It is clear that $|u|_k = 0$ when $u = 0$.

$N_3)$ Let $u = (u_1e_1 + u_2e_2)$ in $\mathbb{B}\mathbb{C}$ and λ in $\mathbb{D}$. Then

$$\begin{aligned} |\lambda u|_k &= |(\lambda_1e_1 + \lambda_2e_2)(u_1e_1 + u_2e_2)|_k \\ &= |(\lambda_1u_1)e_1 + (\lambda_2u_2)e_2|_k \\ &= |\lambda_1u_1|e_1 + |\lambda_2u_2|e_2 \\ &= |\lambda_1||u_1|e_1 + |\lambda_2||u_2|e_2 \\ &= (|\lambda_1|e_1 + |\lambda_2|e_2)(|u_1|e_1 + |u_2|e_2) \\ &= |\lambda|_k|u|_k. \end{aligned}$$

$N_4)$ Let u and v with $u = (u_1e_1 + u_2e_2)$ and $v = (v_1e_1 + v_2e_2)$ be two elements in $\mathbb{B}\mathbb{C}$. Then

$$\begin{aligned} |u + v|_k &= |(u_1e_1 + u_2e_2) + (v_1e_1 + v_2e_2)|_k \\ &= |(u_1 + v_1)e_1 + (u_2 + v_2)e_2|_k \\ &= |u_1 + v_1|e_1 + |u_2 + v_2|e_2 \\ &\leq (|u_1| + |v_1|)e_1 + (|u_2| + |v_2|)e_2 \\ &= (|u_1|e_1 + |u_2|e_2) + (|v_1|e_1 + |v_2|e_2) \\ &= |u|_k + |v|_k. \end{aligned}$$

As a result of the conditions $N_1), N_2), N_3)$ and $N_4)$, it is seen that $|\cdot|_k$ is a $\mathbb{D}$ -norm. This norm is named the hyperbolic-valued norm.

Distribution Functions

A collection $\mathcal{G}$ of subsets of a set G is named a σ – algebra if it supply the following circumstances:

- i. $G \in \mathcal{G}$.
- ii. $S \in \mathcal{G}$ then $G \setminus S \in \mathcal{G}$.

iii. If $(I_k)_k \subseteq \mathcal{G}$ is a sequence, then $\bigcup_{k=1}^{\infty} I_k \in \mathcal{G}$.

The couple $(G, \mathcal{G})$ is named a measurable space and each element of $\mathcal{G}$ is named a measurable set (Halmos, 1978).

Let $\mathcal{G}$ be a σ -algebra. A function $\mu: \mathcal{G} \rightarrow [0, \infty)$ is named a measure, if it supply the following circumstances:

i. $\mu(\emptyset) = 0$.

ii. $\mu(\bigcup_{k=1}^{\infty} I_k) = \sum_{k=1}^{\infty} \mu(I_k)$ for any sequence $(I_k)_k$ of pairwise disjoint sets from $\mathcal{G}$, that is, $I_k \cap I_j = \emptyset$ for $j \neq k$.

Additionally, the trinity $(G, \mathcal{G}, \mu)$ is said a measure space.

Definition 2. Let $(G, \mathcal{G}, \mu)$ be a measure space and $\mathfrak{M}(G, \mathcal{G})$ be the set of all measurable complex-valued functions on G . The distribution function D_g of a function g in $\mathfrak{M}(G, \mathcal{G})$ is given by

$$D_g(\lambda) = \mu\{x \in G: |g(x)| > \lambda \geq 0\} \quad (1.10)$$

(Castillo & Rafeiro, 2015).

Decreasing Rearrangement

Definition 3. Let a $g \in \mathfrak{M}(G, \mathcal{G})$ be given. The function determined as

$$g^*(t) = \inf\{\lambda \geq 0: D_g(\lambda) \leq t\} = \sup\{\lambda \geq 0: D_g(\lambda) > t\} \quad (1.11)$$

is described the decreasing rearrangement function of g . According to this definition, it is clear that the function g^* is defined from $[0, \infty]$ to $[0, \infty]$ (Eryılmaz& Işık, 2019).

Definition 4. Let a $g \in \mathfrak{M}(G, \mathcal{G})$ be given. The average function of g^* , which we will denote by g^{**} , is defined by

$$g^{**}(m) = \frac{1}{m} \int_0^m g^*(t) dt, m > 0 \quad (1.12)$$

(Eryılmaz& Işık, 2019).

2. $\mathbb{D}$ -Distribution and $\mathbb{D}$ -Decreasing Rearrangement Functions of Sequences with Bicomplex Terms

Let $\mathcal{W}_{\mathbb{BC}}$ be the set of sequences with all bicomplex terms and $\mathcal{G}$ be the power set of $\mathbb{N}$, namely $\mathcal{G} = 2^{\mathbb{N}}$, and μ be counting measure on $\mathcal{G}$.

$\mathbb{D}$ -Distribution function

In the definition of the $\mathbb{D}$ -distribution function, a sequence with bicomplex terms instead of the measurable function and the non-negative hyperbolic number instead of the non-negative real number in definition of general distribution function, given in (1.10), is used.

Definition 5. Let $u = (u(n))_n$ with $u(n) = u_1(n)e_1 + u_2(n)e_2$ be a arbitrary sequence in $\mathbb{BC}$ and a number $\lambda_1e_1 + \lambda_2e_2 = \lambda \in \mathbb{D}^+$ be given. The $\mathbb{D}$ -distribution function $\mathcal{D}_u$ of u is defined by

$$\mathcal{D}_u(\lambda) = D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2. \quad (2.1)$$

Definition 6. A function $\hbar$ defined from $\mathbb{D}^+$ to $\mathbb{D}^+$, is called a $\mathbb{D}$ -decreasing function, if there is a $\mathbb{D}$ -inequality $\hbar(\beta) \preceq \hbar(\alpha)$, whenever $\alpha \prec \beta$.

Lemma 1. Let λ be an element in $\mathbb{D}$ with $\lambda = \lambda_1e_1 + \lambda_2e_2$ and δ be an element in $\mathbb{D}$ with $\delta = \delta_1e_1 + \delta_2e_2$, $\delta_1 \neq 0$ and $\delta_2 \neq 0$. There exist the equality

$$\frac{\lambda}{|\delta|_k} = \frac{\lambda_1}{|\delta_1|}e_1 + \frac{\lambda_2}{|\delta_2|}e_2.$$

Proof. We have

$$\begin{aligned} \frac{\lambda}{|\delta|_k} &= \frac{\lambda_1e_1 + \lambda_2e_2}{|\delta_1|e_1 + |\delta_2|e_2} = \frac{\lambda_1e_1}{|\delta_1|e_1 + |\delta_2|e_2} + \frac{\lambda_2e_2}{|\delta_1|e_1 + |\delta_2|e_2} \\ &= \frac{\lambda_1e_1e_1}{(|\delta_1|e_1 + |\delta_2|e_2)e_1} + \frac{\lambda_2e_2e_2}{(|\delta_1|e_1 + |\delta_2|e_2)e_2} \end{aligned}$$

$$= \frac{\lambda_1 e_1 e_1}{|\delta_1| e_1} + \frac{\lambda_2 e_2 e_2}{|\delta_2| e_2} = \frac{\lambda_1}{|\delta_1|} e_1 + \frac{\lambda_2}{|\delta_2|} e_2.$$

Theorem 1. Let $u = (u(n))$ and $v = (v(n))$ be two sequences in $\mathcal{W}_{\mathbb{B}\mathbb{C}}$ and λ, δ and c be in $\mathbb{D}^+$. Then, the following features are satisfied:

- a) $\mathbb{D}$ -distribution function is $\mathbb{D}$ -decreasing.
- b) If $|v(n)|_k \leq |u(n)|_k$ for all $n \geq 1$, then $\mathcal{D}_v(\lambda) \leq \mathcal{D}_u(\lambda)$.
- c) $\mathcal{D}_{cu}(\lambda) = \mathcal{D}_u\left(\frac{\lambda}{|c|_k}\right)$ for all $c \in \mathbb{D}^+$, where $c = c_1 e_1 + c_2 e_2$ and $c_1 \neq 0 \neq c_2$.
- d) $\mathcal{D}_{u+v}(\lambda + \delta) \leq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta)$.
- e) $\mathcal{D}_{u \cdot v}(\lambda \cdot \delta) \leq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta)$.

Proof. a) Assume that $\lambda = \lambda_1 e_1 + \lambda_2 e_2$, $\delta = \delta_1 e_1 + \delta_2 e_2$ and $\delta < \lambda$. Since distribution functions D_{u_1} and D_{u_2} are decreasing (Castillo & Rafeiro, 2015), obviously

$$\begin{aligned} \mathcal{D}_u(\lambda) &= D_{u_1}(\lambda_1) e_1 + D_{u_2}(\lambda_2) e_2 \\ &\leq D_{u_1}(\delta_1) e_1 + D_{u_2}(\delta_2) e_2 = \mathcal{D}_u(\delta). \end{aligned}$$

Hence $\mathcal{D}_u$ is $\mathbb{D}$ -decreasing function.

b) Assume that $|v(n)|_k \leq |u(n)|_k$ for all $n \geq 1$. Then we can write

$$\begin{aligned} |v_1(n)| e_1 + |v_2(n)| e_2 &\leq |u_1(n)| e_1 + |u_2(n)| e_2 \\ \Leftrightarrow |v_1(n)| &\leq |u_1(n)| \text{ and } |v_2(n)| \leq |u_2(n)|. \end{aligned}$$

Thus

$\{n \in \mathbb{N} : |v_m(n)| > \lambda_1\} \subseteq \{n \in \mathbb{N} : |u_m(n)| > \lambda_1\}, m = 1, 2$, and therefore $D_{v_1}(\lambda_1) \leq D_{u_1}(\lambda_1)$ and $D_{v_2}(\lambda_2) \leq D_{u_2}(\lambda_2)$. This shows that $\mathcal{D}_v(\lambda) \leq \mathcal{D}_u(\lambda)$.

- c) According to the definition of $\mathbb{D}$ -distribution function,

$$\mathcal{D}_{cu}(\lambda) = D_{c_1u_1}(\lambda_1)e_1 + D_{c_2u_2}(\lambda_2)e_2$$

is written. We can easily see that

$$\begin{aligned} \mu\{n \in \mathbb{N}: |c_m u_m(n)| > \lambda_1\} &= \mu\left\{n \in \mathbb{N}: |u_m(n)| > \frac{\lambda_1}{|c_m|}\right\} \\ &= D_{u_m}\left(\frac{\lambda_1}{|c_m|}\right), m = 1, 2. \end{aligned}$$

Then

$$D_{c_1u_1}(\lambda_1)e_1 + D_{c_2u_2}(\lambda_2)e_2 = D_{u_1}\left(\frac{\lambda_1}{|c_1|}\right)e_1 + D_{u_2}\left(\frac{\lambda_2}{|c_2|}\right)e_2$$

and hence

$$\mathcal{D}_{cu}(\lambda) = \mathcal{D}_u\left(\frac{\lambda}{|c|_k}\right).$$

d) Considering the distribution functions $D_{(u_m+v_m)}(\lambda_m + \delta_m)$ with $m = 1, 2$; we have

$$\begin{aligned} &\{n \in \mathbb{N}: \lambda_m + \delta_m < |u_m(n) + v_m(n)|\} \subset \\ &\{n \in \mathbb{N}: \lambda_m < |u_m(n)|\} \cup \{n \in \mathbb{N}: \delta_m < |v_m(n)|\} \end{aligned}$$

and hence

$$D_{(u_m+v_m)}(\lambda_m + \delta_m) \leq D_{u_m}(\lambda_m) + D_{v_m}(\delta_m).$$

Using these inequalities, we obtain $\mathbb{D}$ -inequality

$$\begin{aligned} \mathcal{D}_{u+v}(\lambda + \delta) &= D_{(u_1+v_1)}(\lambda_1 + \delta_1)e_1 + D_{(u_2+v_2)}(\lambda_2 + \delta_2)e_2 \\ &\leq \{D_{u_1}(\lambda_1) + D_{v_1}(\delta_1)\}e_1 + \{D_{u_2}(\lambda_2) + D_{v_2}(\delta_2)\}e_2 \\ &= \{D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2\} + \{D_{v_1}(\delta_1)e_1 + D_{v_2}(\delta_2)e_2\} \end{aligned}$$

finally

$$\mathcal{D}_{u+v}(\lambda + \delta) \leq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta). \quad (2.2)$$

e) Using the definitions of the distribution functions

$D_{(u_m \cdot v_m)}(\lambda_m \cdot \delta_m)$ with $m = 1, 2$; we have

$$\begin{aligned} & \{n \in \mathbb{N}: \lambda_m \cdot \delta_m < |u_m(n) \cdot v_m(n)|\} \\ & \subset \{n \in \mathbb{N}: \lambda_m < |u_m(n)|\} \cup \{n \in \mathbb{N}: \delta_m < |v_m(n)|\} \end{aligned}$$

and hence

$$D_{(u_m \cdot v_m)}(\lambda_m \cdot \delta_m) \leq D_{u_m}(\lambda_m) + D_{v_m}(\delta_m)$$

Also the inequalities above allow us to write

$$\begin{aligned} \mathcal{D}_{u \cdot v}(\lambda \cdot \delta) &= D_{(u_1 \cdot v_1)}(\lambda_1 \cdot \delta_1)e_1 + D_{(u_2 \cdot v_2)}(\lambda_2 \cdot \delta_2)e_2 \\ &\leq \{D_{u_1}(\lambda_1) + D_{v_1}(\delta_1)\}e_1 + \{D_{u_2}(\lambda_2) + D_{v_2}(\delta_2)\}e_2 \\ &= \{D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2\} + \{D_{v_1}(\delta_1)e_1 + D_{v_2}(\delta_2)e_2\} \end{aligned}$$

and so

$$\mathcal{D}_{u \cdot v}(\lambda \cdot \delta) \leq \mathcal{D}_u(\lambda) + \mathcal{D}_v(\delta). \quad (2.3)$$

$\mathbb{D}$ -Decreasing Rearrangement Function

Definition 7. If there is $\rho \in \mathbb{D}$ such that $g \leq \rho$ ($\rho \leq g$) for all $g \in G$, then it is said that a subset $G \subset \mathbb{D}$ is $\mathbb{D}$ -frontiered from above(lower). This number $\rho \in \mathbb{D}$ is called a $\mathbb{D}$ -top($\mathbb{D}$ -inferior) frontier of G .

If $G \subset \mathbb{D}$ is a $\mathbb{D}$ -frontiered set from upstairs, then we describe the its $\mathbb{D}$ -supremum showed by $\sup_{\mathbb{D}} G$, the smallest top frontier of G , and its $\mathbb{D}$ -infimum showed by $\inf_{\mathbb{D}} G$, the hugest inferior frontier of G . The “least” top frontier here means that $\sup_{\mathbb{D}} G \leq \rho$ for any $\mathbb{D}$ -top frontier ρ even if not all of the $\mathbb{D}$ -top frontiers are comparable. Likewise, the meaning of the “hugest” inferior frontier can be evaluate. Surely, every non-empty set of hyperbolic numbers which is $\mathbb{D}$ -frontier from upstairs has its $\mathbb{D}$ -supremum, and if it is $\mathbb{D}$ -frontier from inferior, then it has a $\mathbb{D}$ -infimum. Given a set $G \subset \mathbb{D}$, think the sets

$$G_1 = \{g_1: g_1 e_1 + g_2 e_2 \in G\} \text{ and } G_2 = \{g_2: g_1 e_1 + g_2 e_2 \in G\}.$$

If G is a $\mathbb{D}$ -frontiered set from upstairs, then the $\sup_{\mathbb{D}} G$ can be

calculated by the formula

$$\sup_{\mathbb{D}} G = \sup G_1 e_1 + \sup G_2 e_2. \quad (2.4)$$

If G is a $\mathbb{D}$ -frontiered set from below, then the $\inf_{\mathbb{D}} G$ can be calculated by the formula

$$\inf_{\mathbb{D}} G = \inf G_1 e_1 + \inf G_2 e_2. \quad (2.5)$$

If G and H are two $\mathbb{D}$ -frontiered set from inferior, then so is $G + H$ and

$$\inf_{\mathbb{D}} (G + H) = \inf_{\mathbb{D}} G + \inf_{\mathbb{D}} H \quad (2.6)$$

holds.

If two subsets $G \subset \mathbb{D}^+$ and $H \subset \mathbb{D}^+$ are $\mathbb{D}$ -frontiered from inferior, then so is $G \cdot H$ and

$$\inf_{\mathbb{D}} (G \cdot H) = \inf_{\mathbb{D}} G \cdot \inf_{\mathbb{D}} H \quad (2.7)$$

holds.

For the top $\mathbb{D}$ -frontiered subsets of $\mathbb{D}$, the equations (2.6) to (2.7) are still true when $\sup_{\mathbb{D}}$ is written instead of $\inf_{\mathbb{D}}$ (Luna-Elizarrarás, M.E. & et al., 2015).

Definition 8. The $\mathbb{D}$ -decreasing rearrangement of a sequence $u = (u(n))$ in $\mathcal{W}_{\mathbb{B}\mathbb{C}}$ is a function u^* defined by

$$u^*(t) = \inf_{\mathbb{D}} \{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq t\}$$

from $\mathbb{D}^+$ to $\mathbb{D}^+$. Since D_{u_i} with $i = 1, 2$ is decreasing,

$$\inf \{\lambda_1 \geq 0 : D_{u_1}(\lambda_1) \leq D_{u_2}(\beta_1)\} = \beta_1$$

and

$$\inf \{\lambda_2 \geq 0 : D_{u_2}(\lambda_2) \leq D_{u_2}(\beta_2)\} = \beta_2$$

holds. Therefore,

$$\begin{aligned} u^*(\mathcal{D}_u(\beta)) &= \inf_{\mathbb{D}} \{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq \mathcal{D}_u(\beta)\} \\ &= \inf \{\lambda_1 \geq 0 : D_{u_1}(\lambda_1) \leq D_{u_2}(\beta_1)\} e_1 \end{aligned}$$

$$+ \inf\{\lambda_2 \geq 0: D_{u_2}(\lambda_2) \leq D_{u_2}(\beta_2)\}e_2 = \beta,$$

u^* is the left inverse of $\mathcal{D}_u$.

Moreover, let $u(n) = u_1(n)e_1 + u_2(n)e_2$, $\lambda = (\lambda_1e_1 + \lambda_2e_2)$ and $t = (t_1e_1 + t_2e_2)$. Then

$$\begin{aligned} u^*(t) &= \inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+: \mathcal{D}_u(\lambda) \leq t\} \\ &= \inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+: D_{u_1}(\lambda_1)e_1 + D_{u_2}(\lambda_2)e_2 \leq t_1e_1 + t_2e_2\} \\ &= \inf\{\lambda_1: D_{u_1}(\lambda_1) \leq t_1\}e_1 + \inf\{\lambda_2: D_{u_2}(\lambda_2) \leq t_2\}e_2 \end{aligned}$$

and so

$$u^*(t) = u_1^*(t_1)e_1 + u_2^*(t_2)e_2. \quad (2.8)$$

Theorem 2. Let $u = (u(n))$ be a sequence in $\mathcal{W}_{\mathbb{BC}}$ and $\lambda = (\lambda_1e_1 + \lambda_2e_2)$, $t = (t_1e_1 + t_2e_2)$ be two elements in $\mathbb{D}^+$. The $\mathbb{D}$ -decreasing rearrangement function of u has the following properties:

- a) u^* is $\mathbb{D}$ -decreasing.
- b) $u^*(t) > \lambda$ necessary and sufficient condition $\mathcal{D}_u(\lambda) > t$.
- c) $(\kappa u)^*(t) = |\kappa|_k u^*(t)$, $\kappa \in \mathbb{D}^+$.
- d) Let $(u(n))$ and $(v(n))$ be two sequences in $\mathcal{W}_{\mathbb{BC}}$. If $|u(n)|_k < |v(n)|_k$ for every $n = 1, 2, \dots$ then $u^*(t) \leq v^*(t)$.
- e) $[(|u(n)|_k)^p]^* = (u^*(n))^p$, $p \geq 1$.

Proof.

- a) Let $0 \leq t < k$, then $t_1 \leq k_1 \wedge t_2 \leq k_2$.

This shows that

$$\{\lambda_1 \geq 0: D_{u_1}(\lambda_1) \leq t_1\} \subset \{\lambda_1 \geq 0: D_{u_1}(\lambda_1) \leq k_1\}$$

and

$$\{\lambda_2 \geq 0: D_{u_2}(\lambda_2) \leq t_2\} \subset \{\lambda_2 \geq 0: D_{u_2}(\lambda_2) \leq k_2\}$$

therefore,

$$\{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq t\} \subset \{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq k\}$$

obviously

$$\inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq k\} \leq \inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq t\}$$

that is,

$$u^*(k) \leq u^*(t).$$

b) Let $\lambda < u^*(t)$. Then

$$\lambda < \inf_{\mathbb{D}}\{\alpha \in \mathbb{D}^+ : \mathcal{D}_u(\alpha) \leq t\}$$

namely

$$\lambda_1 e_1 + \lambda_2 e_2$$

$$\begin{aligned} &< \inf_{\mathbb{D}}\{\alpha \in \mathbb{D}^+ : D_{u_1}(\alpha_1)e_1 + D_{u_2}(\alpha_2)e_2 \leq t_1 e_1 + t_2 e_2\} \\ &= \inf\{\alpha_1 : D_{u_1}(\alpha_1) \leq t_1\}e_1 + \inf\{\alpha_2 : D_{u_2}(\alpha_2) \leq t_2\}e_2. \end{aligned}$$

Thus

$$\lambda_1 < \inf\{\alpha_1 : D_{u_1}(\alpha_1) \leq t_1\} \text{ and } \lambda_2 < \inf\{\alpha_2 : D_{u_2}(\alpha_2) \leq t_2\}$$

and so $D_{u_1}(\lambda_1) > t_1$, $D_{u_2}(\lambda_2) > t_2$. Obviously $\mathcal{D}_u(\lambda) > t$.

Conversely, let $t < \mathcal{D}_u(\lambda)$. Suppose that $u^*(t) \leq \lambda$. Since $\mathcal{D}_u$ is $\mathbb{D}$ -decreasing, we have $\mathcal{D}_u(\lambda) \leq \mathcal{D}_u(u^*(t)) \leq t$ contradicting our hypothesis.

c) By the Theorem 1, we write

$$\begin{aligned} (\kappa u)^*(t) &= \inf_{\mathbb{D}}\{\alpha \in \mathbb{D}^+ : \mathcal{D}_{\kappa u}(\alpha) \leq t\} \\ &= \inf_{\mathbb{D}}\left\{\alpha \in \mathbb{D}^+ : \mathcal{D}_u\left(\frac{\alpha}{|\kappa|_k}\right) \leq t\right\}. \end{aligned}$$

If it is said $\frac{\alpha}{|\kappa|_k} = \beta \in \mathbb{D}^+$, then we have $\alpha = |\kappa|_k \cdot \beta$ and so

$$\inf_{\mathbb{D}}\{|\kappa|_k \cdot \beta \in \mathbb{D}^+ : \mathcal{D}_u(\beta) \leq t\}$$

$$\begin{aligned}
&= \inf_{\mathbb{D}} \{ (|\kappa_1|e_1 + |\kappa_2|e_2) \cdot (\beta_1e_1 + \beta_2e_2) \in \mathbb{D}^+ : \mathcal{D}_u(\beta) \leq t \} \\
&= \inf_{\mathbb{D}} \{ (|\kappa_1|\beta_1)e_1 + (|\kappa_2|\beta_2)e_2 \in \mathbb{D}^+ : \mathcal{D}_u(\beta) \leq t \} \\
&= \inf \{ |\kappa_1|\beta_1 : D_{u_1}(\beta_1) \leq t_1 \} e_1 + \inf \{ |\kappa_2|\beta_2 : D_{u_2}(\beta_2) \leq t_2 \} e_2 \\
&= (|\kappa_1| \inf \{ \beta_1 : D_{u_1}(\beta_1) \leq t_1 \}) e_1 \\
&\quad + (|\kappa_2| \inf \{ \beta_2 : D_{u_2}(\beta_2) \leq t_2 \}) e_2 \\
&= (|\kappa_1|e_1 + |\kappa_2|e_2) (\inf \{ \beta_1 : D_{u_1}(\beta_1) \leq t_1 \} e_1 \\
&\quad + \inf \{ \beta_2 : D_{u_2}(\beta_2) \leq t_2 \} e_2) \\
&= |\kappa|_k \cdot \inf_{\mathbb{D}} \{ \beta \in \mathbb{D}^+ : \mathcal{D}_u(\beta) \leq t \} \\
&= |\kappa|_k \cdot u^*(t).
\end{aligned}$$

d) If $|u|_k < |v|_k$, then the inequality $\mathcal{D}_u(\lambda) \leq \mathcal{D}_v(\lambda)$ is obtained from the Theorem 1.

Since $\mathcal{D}_u(\lambda) \leq \mathcal{D}_v(\lambda)$, we have

$$D_{u_1}(\lambda_1) \leq D_{v_1}(\lambda_1) \wedge D_{u_2}(\lambda_2) \leq D_{v_2}(\lambda_2).$$

This shows that

$$\{\lambda_1 \geq 0 : D_{v_1}(\lambda_1) \leq t_1\} \subset \{\lambda_1 \geq 0 : D_{u_1}(\lambda_1) \leq t_1\}$$

and

$$\{\lambda_2 \geq 0 : D_{v_2}(\lambda_2) \leq t_2\} \subset \{\lambda_2 \geq 0 : D_{u_2}(\lambda_2) \leq t_2\}$$

as a result

$$\{\lambda \in \mathbb{D}^+ : \mathcal{D}_v(\lambda) \leq t\} \subset \{\lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq t\}$$

and so

$$\inf_{\mathbb{D}} \{ \lambda \in \mathbb{D}^+ : \mathcal{D}_u(\lambda) \leq t \} \leq \inf_{\mathbb{D}} \{ \lambda \in \mathbb{D}^+ : \mathcal{D}_v(\lambda) \leq t \},$$

namely

$$u^*(t) \leq v^*(t).$$

e) Let $p \in \mathbb{R}$ and $\lambda \in \mathbb{BC}$ then $(\lambda_1e_1 + \lambda_2e_2)^p = \lambda_1^p e_1 + \lambda_2^p e_2$ (Sağır & Değirmen, 2022).

Also in case of using the properties of $\mathcal{D}_u(\lambda)$

$$\begin{aligned}\mathcal{D}_{(|u|_k)^p}(\lambda) &= \mathcal{D}_{(|u_1|e_1+|u_2|e_2)^p}(\lambda) = \mathcal{D}_{(|u_1|^pe_1+|u_2|^pe_2)}(\lambda) \\ &= D_{|u_1|^p}(\lambda_1)e_1 + D_{|u_2|^p}(\lambda_2)e_2.\end{aligned}\quad (2.9)$$

Using the definitions of the distribution functions $D_{u_i}(\lambda_i)$ with $i = 1, 2$ we have

$$\begin{aligned}D_{|u_i|^p}(\lambda_i) &= \mu\{n \in \mathbb{N}: \lambda_i < ||u_i(n)|^p|\} \\ &= \mu\left\{n \in \mathbb{N}: (\lambda_i)^{\frac{1}{p}} < |u_i(n)|\right\} \\ &= D_{u_i}\left((\lambda_i)^{\frac{1}{p}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{D}_{(|u|_k)^p}(\lambda) &= D_{u_1}\left((\lambda_1)^{\frac{1}{p}}\right)e_1 + D_{u_2}\left((\lambda_2)^{\frac{1}{p}}\right)e_2 \\ &= \mathcal{D}_u\left((\lambda)^{\frac{1}{p}}\right)\end{aligned}$$

and hence

$$\begin{aligned}((|u|_k)^p)^*(t) &= \inf_{\mathbb{D}}\{\lambda \in \mathbb{D}^+: \mathcal{D}_{(|u|_k)^p}(\lambda) \leq t\} \\ &= \inf_{\mathbb{D}}\left\{\lambda \in \mathbb{D}^+: \mathcal{D}_u\left(\lambda^{\frac{1}{p}}\right) \leq t\right\} \\ &= \inf_{\mathbb{D}}\{\alpha^p \in \mathbb{D}^+: \mathcal{D}_u(\alpha) \leq t\} \\ &= \inf_{\mathbb{D}}\{(\alpha_1e_1 + \alpha_2e_2)^p \in \mathbb{D}^+: \mathcal{D}_u(\alpha) \leq t\} \\ &= \inf_{\mathbb{D}}\{\alpha_1^pe_1 + \alpha_2^pe_2 \in \mathbb{D}^+: \mathcal{D}_u(\alpha_1e_1 + \alpha_2e_2) \leq t\}.\end{aligned}$$

Now if (2.5) is used,

$$\begin{aligned}((|u|_k)^p)^*(t) &= \inf\{\alpha_1^p: \alpha^p \in \mathbb{D}^+ \wedge \mathcal{D}_u(\alpha) \leq t\}e_1 \\ &\quad + \inf\{\alpha_2^p: \alpha^p \in \mathbb{D}^+ \wedge \mathcal{D}_u(\alpha) \leq t\}e_2\end{aligned}$$

$$\begin{aligned}
&= \left(\inf \{ \alpha_1 : \alpha^p \in \mathbb{D}^+ \wedge D_{u_1}(\alpha_1) \leq t_1 \} \right)^p e_1 \\
&\quad + \left(\inf \{ \alpha_2 : \alpha^p \in \mathbb{D}^+ \wedge D_{u_2}(\alpha_2) \leq t_2 \} \right)^p e_2 \\
&= \left(\left(\inf \{ \alpha_1 : \alpha^p \in \mathbb{D}^+ \wedge D_{u_1}(\alpha_1) \leq t_1 \} \right) e_1 \right. \\
&\quad \left. + \left(\inf \{ \alpha_2 : \alpha^p \in \mathbb{D}^+ \wedge D_{u_2}(\alpha_2) \leq t_2 \} \right) e_2 \right)^p \\
&= (\inf_{\mathbb{D}} \{ \alpha = \alpha_1 e_1 + \alpha_2 e_2 \in \mathbb{D}^+ : \mathcal{D}_u(\alpha_1 e_1 + \alpha_2 e_2) \leq t \})^p,
\end{aligned}$$

and so

$$((|u|_k)^p)^*(t) = (u^*)^p(t). \quad (2.10)$$

Definition 9. For a $H \in \mathcal{G} = 2^{\mathbb{N}}$, the function defined by $\chi_H(n) = \begin{cases} 1, n \in H \\ 0, n \notin H \end{cases}$ is the known characteristic function of the set H , therefore the function defined from $\mathbb{N}$ to $\mathbb{D}^+$ as

$$\chi_H^{\mathbb{D}}(n) = \chi_H(n)e_1 + \chi_H(n)e_2$$

is called $\mathbb{D}$ -characteristic function of the set H . At the time, it can be written by

$$\chi_H^{\mathbb{D}}(n) = \begin{cases} 1e_1 + 1e_2, n \in H \\ 0e_1 + 0e_2, n \notin H \end{cases} = \begin{cases} 1, n \in H \\ 0, n \notin H \end{cases}$$

Now let us construct the $\mathbb{D}$ -distribution and $\mathbb{D}$ -decreasing rearrangement functions of this function χ_H . Let $\lambda = (\lambda_1 e_1 + \lambda_2 e_2)$ be an element in $\mathbb{D}^+$, we have

$$\begin{aligned}
\mathcal{D}_{(\chi_H^{\mathbb{D}}(n))}(\lambda) &= \mathcal{D}_{(\chi_H(n)e_1 + \chi_H(n)e_2)}(\lambda_1 e_1 + \lambda_2 e_2) \\
&= D_{\chi_H(n)}(\lambda_1)e_1 + D_{\chi_H(n)}(\lambda_2)e_2.
\end{aligned}$$

Using the definitions of the distribution functions $D_{u_i}(\lambda_i)$ with $i = 1, 2$ we have

$$D_{\chi_H(n)}(\lambda_i) = \begin{cases} \mu(H), & 0 \leq \lambda_i < 1 \\ 0, & \lambda_i \geq 1 \end{cases}.$$

Finally,

$$\begin{aligned}
\mathcal{D}_{(\chi_H^{\mathbb{D}}(n))}(\lambda) &= \begin{cases} \mu(H)e_1 + \mu(H)e_2, & 0 \leq \lambda_1 < 1 \wedge 0 \leq \lambda_2 < 1 \\ \mu(H)e_1 + 0e_2, & 0 \leq \lambda_1 < 1 \wedge \lambda_2 \geq 1 \\ 0e_1 + \mu(H)e_2, & \lambda_1 \geq 1 \wedge 0 \leq \lambda_2 < 1 \\ 0e_1 + 0e_2, & \lambda_1 \geq 1 \wedge \lambda_2 \geq 1 \end{cases} \\
\mathcal{D}_{(\chi_H^{\mathbb{D}}(n))}(\lambda) &= \begin{cases} \mu(H) & , & 0 \leq \lambda_1 < 1 \wedge 0 \leq \lambda_2 < 1 \\ \mu(H)e_1 & , & 0 \leq \lambda_1 < 1 \wedge \lambda_2 \geq 1 \\ \mu(H)e_2 & , & \lambda_1 \geq 1 \wedge 0 \leq \lambda_2 < 1 \\ 0 & , & \lambda_1 \geq 1 \wedge \lambda_2 \geq 1 \end{cases} \quad (2.11)
\end{aligned}$$

is obtained.

Now, if the $\mathbb{D}$ -decreasing rearrangement function of the $\mathbb{D}$ -characteristic function $\chi_H^{\mathbb{D}}$ is written according to (2.11), it is clearly seen that

$$\begin{aligned}
(\chi_H^{\mathbb{D}})^*(t) &= \inf_{\mathbb{D}} \left\{ \lambda \in \mathbb{D}^+ : \mathcal{D}_{(\chi_H^{\mathbb{D}}(n))}(\lambda) \leq t \right\} \\
&= \inf_{\mathbb{D}} \left\{ \lambda_1 e_1 + \lambda_2 e_2 \in \mathbb{D}^+ : \mathcal{D}_{(\chi_H^{\mathbb{D}}(n))}(\lambda) \leq t_1 e_1 + t_2 e_2 \right\} \\
&= \inf \left\{ \lambda_1 \geq 0 : D_{\chi_H(n)}(\lambda_1) \leq t_1 \right\} e_1 \\
&\quad + \inf \left\{ \lambda_2 \geq 0 : D_{\chi_H(n)}(\lambda_2) \leq t_2 \right\} e_2 \\
&= (\chi_H)^*(t_1) e_1 + (\chi_H)^*(t_2) e_2
\end{aligned}$$

and so

$$(\chi_H^{\mathbb{D}})^*(t_1 e_1 + t_2 e_2) = \begin{cases} 1e_1 + 1e_2 & , & t_1 < \mu(H) \wedge t_2 < \mu(H) \\ 1e_1 + 0e_2 & , & t_1 < \mu(H) \wedge t_2 \geq \mu(H) \\ 0e_1 + 1e_2 & , & t_1 \geq \mu(H) \wedge t_2 < \mu(H) \\ 0e_1 + 0e_2 & , & t_1 \geq \mu(H) \wedge t_2 \geq \mu(H) \end{cases}$$

$$= \begin{cases} 1 & , \quad t_1 < \mu(H) \wedge t_2 < \mu(H) \\ e_1 & , \quad t_1 < \mu(H) \wedge t_2 \geq \mu(H) \\ e_2 & , \quad t_1 \geq \mu(H) \wedge t_2 < \mu(H) \\ 0 & , \quad t_1 \geq \mu(H) \wedge t_2 \geq \mu(H) \end{cases}$$

also

$$(\chi_H^{\mathbb{D}})^*(t) = \chi_{[1, \mu(H))}(t_1)e_1 + \chi_{[1, \mu(H))}(t_2)e_2. \quad (2.12)$$

Theorem 3. Let $H \in \mathcal{G} = 2^{\mathbb{N}}$ and $u = (u(n))$ is a sequence in $\mathcal{W}_{\mathbb{B}\mathbb{C}}$. For $t \in \mathbb{D}^+$,

$$(u \cdot \chi_H)^*(t) \leq u^*(t).$$

Proof. Since $|u \cdot \chi_H|_k \leq |u|_k$, by Theorem 1.(d), we have

$$(u \cdot \chi_H)^*(t) \leq u^*(t).$$

Lemma 2. Let $(G, \mathcal{G}, \mu)$ is be a measurement space further $\mathcal{g}$ and $\mathcal{h}$ are be two measurable functions. Then the inequalities

$$(\mathcal{g} + \mathcal{h})^*(\alpha + \beta) \leq \mathcal{g}^*(\alpha) + \mathcal{h}^*(\beta)$$

and

$$(\mathcal{g} \cdot \mathcal{h})^*(\alpha + \beta) \leq \mathcal{g}^*(\alpha) \cdot \mathcal{h}^*(\beta)$$

hold for all $\alpha, \beta \geq 0$ (Castillo and Rafeiro, 2015).

Theorem 4. Let $(\mathbb{N}, \mathcal{G}, \mu)$ be a measure space and $u = (u(n))$, $v = (v(n))$ be two sequence in $\mathcal{W}_{\mathbb{B}\mathbb{C}}$. Then, the $\mathbb{D}$ -inequalities

$$(u + v)^*(\lambda + \delta) \leq u^*(\lambda) + v^*(\delta) \quad (2.13)$$

and

$$(u \cdot v)^*(\lambda + \delta) \leq u^*(\lambda) \cdot v^*(\delta) \quad (2.14)$$

hold for all $\lambda, \delta \in \mathbb{D}^+$ with $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ and $\delta = \delta_1 e_1 + \delta_2 e_2$.

Proof. Since $u^*(\lambda) = u_1^*(\lambda_1)e_1 + u_2^*(\lambda_2)e_2$ and

$v^*(\delta) = v_1^*(\delta_1)e_1 + v_2^*(\delta_2)e_2$ from (2.8) and also if use Lemma 2, we have

$$\begin{aligned}
(u + v)^*(\lambda + \delta) &= ((u_1e_1 + u_2e_2) + (v_1e_1 + v_2e_2))^*(\lambda + \delta) \\
&= ((u_1 + v_1)e_1 + (u_2 + v_2)e_2)^*((\lambda_1 + \delta_1)e_1 + (\lambda_2 + \delta_2)e_2) \\
&= (u_1 + v_1)^*(\lambda_1 + \delta_1)e_1 + (u_2 + v_2)^*(\lambda_2 + \delta_2)e_2 \\
&\leq (u_1^*(\lambda_1) + v_1^*(\delta_1))e_1 + (u_2^*(\lambda_2) + v_2^*(\delta_2))e_2 \\
&= (u_1^*(\lambda_1)e_1 + u_2^*(\lambda_2)e_2) + (v_1^*(\delta_1)e_1 + v_2^*(\delta_2))e_2 \\
&= u^*(\lambda) + v^*(\delta).
\end{aligned}$$

Again, we have

$$\begin{aligned}
(u \cdot v)^*(\lambda + \delta) &= ((u_1e_1 + u_2e_2) \cdot (v_1e_1 + v_2e_2))^*(\lambda + \delta) \\
&= ((u_1 \cdot v_1)e_1 + (u_2 \cdot v_2)e_2)^*((\lambda_1 + \delta_1)e_1 + (\lambda_2 + \delta_2)e_2) \\
&= (u_1 \cdot v_1)^*(\lambda_1 + \delta_1)e_1 + (u_2 \cdot v_2)^*(\lambda_2 + \delta_2)e_2 \\
&\leq (u_1^*(\lambda_1) \cdot v_1^*(\delta_1))e_1 + (u_2^*(\lambda_2) \cdot v_2^*(\delta_2))e_2 \\
&= (u_1^*(\lambda_1)e_1 + u_2^*(\lambda_2)e_2) \cdot (v_1^*(\delta_1)e_1 + v_2^*(\delta_2))e_2 \\
&= u^*(\lambda) \cdot v^*(\delta).
\end{aligned}$$

If $\lambda = \delta = \frac{\alpha}{2}$ is chosen in (2.13) and (2.14), then

$$(u + v)^*(\alpha) \leq u^*\left(\frac{\alpha}{2}\right) + v^*\left(\frac{\alpha}{2}\right)$$

and

$$(u \cdot v)^*(\alpha) \leq u^*\left(\frac{\alpha}{2}\right) \cdot v^*\left(\frac{\alpha}{2}\right)$$

is obtained.

Definition 10. $u = (u(n))$ be a sequence in $\mathcal{W}_{\mathbb{BC}}$. The function defined by

$$u^{**}(n) = \frac{1}{n} \sum_{x=1}^n u^*(x)$$

is called the $\mathbb{D}$ -average function of u^* .

Theorem 5. Let $u = (u(n))$ be a sequence in $\mathcal{W}_{\mathbb{BC}}$. The $\mathbb{D}$ -average function of u^* has the following features:

- a) For every $n = 1, 2, 3, \dots$, $u^*(n) \leq u^{**}(n)$.
- b) u^{**} is $\mathbb{D}$ -decreasing.
- c) If n is an even natural number,

$$(u + v)^{**}(n) \leq u^{**}(n/2) + v^{**}(n/2).$$

d) If n is an odd natural number and $\llbracket \cdot \rrbracket$ is an greatest integer function, then

$$(u + v)^{**}(n) \leq \frac{n+1}{n} (u^{**}(\llbracket n/2 \rrbracket + 1) + v^{**}(\llbracket n/2 \rrbracket + 1)).$$

Proof. a) Since u^* a $\mathbb{D}$ -decreasing function and $n = ne_1 + ne_2$ for all $n \geq 1$,

$$\begin{aligned} u^{**}(n) &= \frac{1}{n} \sum_{x=1}^n u^*(x) = \frac{1}{n} ((u^*(1) + u^*(2) + \dots + u^*(n))) \\ &\geq \frac{1}{n} (n \cdot u^*(n)) = u^*(n) \end{aligned}$$

is obtained.

- b) For $n \geq 1$, we have

$$\begin{aligned} u^{**}(n+1) &= \frac{1}{n+1} \sum_{x=1}^{n+1} u^*(x) \\ &= \frac{1}{n+1} \left(\sum_{x=1}^n u^*(x) + u^*(n+1) \right) \end{aligned}$$

$$= \frac{1}{n+1} \sum_{x=1}^n u^*(x) + \frac{1}{n+1} u^*(n+1). \quad (2.15)$$

Since $u^*(n+1) \leq u^*(i)$ for $1 \leq i \leq n$, there exist

$$u^*(n+1) \leq \frac{1}{n} (u^*(1) + u^*(2) + \cdots + u^*(n)).$$

If this inequality is written in (2.15), we have

$$\begin{aligned} u^{**}(n+1) &\leq \frac{1}{n+1} \sum_{x=1}^n u^*(x) \\ &\quad + \frac{1}{n(n+1)} (u^*(1) + \cdots + u^*(n)) = u^{**}(n). \end{aligned}$$

As a result, u^{**} is $\mathbb{D}$ -decreasing.

c) If the $\mathbb{D}$ -average function definition and (2.13) are used,

$$\begin{aligned} (u+v)^{**}(n) &= \frac{1}{n} \sum_{x=1}^n (u+v)^*(x) \\ &\leq \frac{1}{n} \sum_{x=1}^n \left(u^*\left(\frac{x}{2}\right) + v^*\left(\frac{x}{2}\right) \right) \\ &= \frac{1}{n} \left\{ \sum_{x=1}^n u^*\left(\frac{x}{2}\right) + \sum_{x=1}^n v^*\left(\frac{x}{2}\right) \right\}. \end{aligned}$$

Since $(n-1) \leq t_i < n$, we have $u_i^*(t_i) = u_i^*(n)$ with $i = 1, 2$ (Castillo & Rafeiro, 2015).

For $(n-1) \leq t < n$, we have

$$u^*(t) = u_1^*(t_1)e_1 + u_2^*(t_2)e_2 = u_1^*(n)e_1 + u_2^*(n)e_2 = u^*(n).$$

Therefore,

$$\sum_{x=1}^n u^*\left(\frac{x}{2}\right)$$

$$\begin{aligned}
&= u^*\left(\frac{1}{2}\right) + u^*\left(\frac{2}{2}\right) + u^*\left(\frac{3}{2}\right) + u^*\left(\frac{4}{2}\right) + \cdots + u^*\left(\frac{n}{2}\right) \\
&= u^*(1) + u^*(1) + u^*(2) + u^*(2) + \cdots + u^*\left(\frac{n}{2}\right) \\
&= 2 \left\{ u^*(1) + u^*(2) + \cdots + u^*\left(\frac{n}{2}\right) \right\} \\
&= 2 \sum_{x=1}^{n/2} u^*(x)
\end{aligned}$$

similarly,

$$\sum_{x=1}^n v^*\left(\frac{x}{2}\right) = 2 \sum_{x=1}^{n/2} v^*(x).$$

As a result,

$$\begin{aligned}
(u + v)^{**}(n) &\leq \frac{1}{n} \left\{ 2 \sum_{x=1}^{n/2} u^*(x) + 2 \sum_{x=1}^{n/2} v^*(x) \right\} \\
&= \frac{1}{n/2} \left\{ \sum_{x=1}^{n/2} u^*(x) + \sum_{x=1}^{n/2} v^*(x) \right\} \\
&= u^{**}(n/2) + v^{**}(n/2).
\end{aligned}$$

d) Assume that n is the odd natural number. When using the proof method in c,

$$\begin{aligned}
(u + v)^{**}(n) &= \frac{1}{n} \sum_{x=1}^n (u + v)^*(x) \\
&\leq \frac{1}{n} \sum_{x=1}^n \left(u^*\left(\frac{x}{2}\right) + v^*\left(\frac{x}{2}\right) \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{x=1}^n \left(u^* \left(\frac{x}{2} \right) \right) + \frac{1}{n} \sum_{x=1}^n \left(v^* \left(\frac{x}{2} \right) \right) \\
&= \frac{1}{n} \left\{ u^* \left(\frac{1}{2} \right) + u^* \left(\frac{2}{2} \right) + u^* \left(\frac{3}{2} \right) + \cdots + u^* \left(\frac{n-1}{2} \right) + u^* \left(\frac{n}{2} \right) \right\} \\
&\quad + \frac{1}{n} \left\{ v^* \left(\frac{1}{2} \right) + v^* \left(\frac{2}{2} \right) + v^* \left(\frac{3}{2} \right) + \cdots + v^* \left(\frac{n-1}{2} \right) + v^* \left(\frac{n}{2} \right) \right\} \\
&\leq \frac{1}{n} \left\{ u^* \left(\frac{1}{2} \right) + u^* \left(\frac{2}{2} \right) + \cdots + u^* \left(\frac{n-1}{2} \right) + u^* \left(\frac{n}{2} \right) + u^* \left(\frac{n}{2} \right) \right\} \\
&\quad + \frac{1}{n} \left\{ v^* \left(\frac{1}{2} \right) + v^* \left(\frac{2}{2} \right) + \cdots + v^* \left(\frac{n-1}{2} \right) + v^* \left(\frac{n}{2} \right) + v^* \left(\frac{n}{2} \right) \right\} \\
&= \frac{2}{n} \sum_{x=1}^{\llbracket n/2 \rrbracket + 1} u^*(x) + \frac{2}{n} \sum_{x=1}^{\llbracket n/2 \rrbracket + 1} v^*(x) \\
&= \frac{2 \{ \llbracket n/2 \rrbracket + 1 \}}{n \{ \llbracket n/2 \rrbracket + 1 \}} \sum_{x=1}^{\llbracket n/2 \rrbracket + 1} (u^*(x) + v^*(x)) \\
&= \frac{2 \left\{ \left\lfloor \frac{n+1}{2} - \frac{1}{2} \right\rfloor + 1 \right\}}{n} \left\{ \frac{1}{(\llbracket n/2 \rrbracket + 1)} \sum_{k=1}^{\llbracket n/2 \rrbracket + 1} u^*(x) \right. \\
&\quad \left. + \frac{1}{(\llbracket n/2 \rrbracket + 1)} \sum_{x=1}^{\llbracket n/2 \rrbracket + 1} v^*(x) \right\} \\
&= \frac{2 \left\{ \left(\frac{n+1}{2} - 1 \right) + 1 \right\}}{n} (u^{**}(\llbracket n/2 \rrbracket + 1) + v^{**}(\llbracket n/2 \rrbracket + 1)) \\
&= \frac{n+1}{n} (u^{**}(\llbracket n/2 \rrbracket + 1) + v^{**}(\llbracket n/2 \rrbracket + 1))
\end{aligned}$$

is obtained.

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CHAPTER II

Q Tensor on Para-Sasakian Manifolds

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Introduction

One of the most comprehensive and active theories of modern differential geometry is Riemannian manifolds. Therefore, such manifolds have been examined and studied frequently by several mathematicians in literature in recent years. Over time, different structures have been defined on Riemannian manifolds and thanks

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to these structures, some special different manifolds have been introduced to the literature. One of these manifolds is the almost paracontact manifolds and their subclasses defined by Sato (Sato, 1976). Since then, many authors have made significant contributions to these manifolds.

As a special subclass of almost paracontact manifolds, para-Sasakian manifolds represent a captivating and significant area of study in the realm of differential geometry and mathematics. Such manifolds are a natural extension of Sasakian manifolds, and they possess remarkable geometric properties that have far-reaching implications across various mathematical disciplines.

Para-Sasakian manifolds are closely related to the broader family of Sasakian manifolds. A Sasakian manifold is a Riemannian manifold equipped with a special contact structure known as the Reeb foliation. What distinguishes para-Sasakian manifolds is their compatibility with a pseudo-Riemannian metric alongside the Reeb foliation, resulting in a perfect blend of Riemannian and Lorentzian geometries. This combination of geometric structures provides a unique setting for the exploration of mathematical concepts.

On the other hand, using the Riemannian metric on manifolds plays an important role in examining and characterizing manifolds and their submanifolds. While examining Riemannian manifolds and their submanifolds, curvatures are generally used, classifications are made with the help of these curvatures, and characterization theorems and results are given depending on the curvatures. While investigating Riemannian manifolds and their submanifolds, one of the new methods is to make use of certain curvature tensor conditions on the manifolds. There are many interesting paper about various type of manifolds satisfying curvature conditions.

In their work (Adati & Matsumoto, 1977), T. Adati and K. Matsumoto defined the concepts of para-Sasakian and special para-Sasakian manifolds, which are regarded as specific instances of almost paracontact manifolds initially established by I. Sato (Sato, 1976). Within the same study, the authors conducted an investigation

into conformally symmetric para-Sasakian manifolds, establishing the result that a para-Sasakian manifold of dimension n (where $n > 3$) that is conformally symmetric is both conformally flat and special para-Sasakian. Subsequently, in their study (De & Guha, 1992), the authors examined para-Sasakian manifolds satisfying the condition Weyl-semisymmetric, and their findings demonstrated that a Weyl-semisymmetric manifold of dimension n that is also para-Sasakian exhibits conformal flatness. This research is devoted to achieving classifications of para-Sasakian manifolds with Weyl-pseudosymmetric condition, which represent an expanded class of Weyl-semisymmetric manifolds that have para-Sasakian structure, as well as additional characterizations of para-Sasakian manifolds that satisfy the condition curvature $C \cdot S = 0$.

In his research, Özgür (Özgür, 2005) explored Weyl-pseudosymmetric manifolds that have para-Sasakian structure. Also, in same research he investigate para-Sasakian manifolds under the curvature condition $C \cdot S = 0$. Para-Sasakian manifolds have been the subject of study by various authors, including Deshmukh and Ahmed (Deshmukh & Ahmed, 1980), De et al. (De & et al., 2008), Matsumoto, Ianus, and Mihai (Matsumoto, Ianus & Mihai, 1986), Sharfuddin, Deshmukh, and Husain (Sharfuddin, Deshmukh & Husain, 1980), De et al. (De, Han & Mandal, 2017), Adati and Miyazawa (Adati & Miyazawa, 1979), Acet et al. (Acet, Kılıç & Yüksel Perktas, 2012), Ozgür and Tripathi (Özgür & Tripathi, 2007) among several others.

Motivated and inspired by the above studies, in this work we deal with Q curvature tensor on para-Sasakian manifolds, which is a fruitful topic of paracontact manifolds, and we obtain some significant characterization theorems depending on this tensor. Our work is structured as follows. First section is dedicated to the introduction, which contains a background of manifolds. In the next section, in preliminaries, we give some requirement notions and formulas that we will make use of in the proof of our main results. The final section, as main results, we give some characterizations on

para-Sasakian manifolds which are equipped with the curvature tensor Q .

Preliminaries

In this section, we collect some necessary notions and formulas related to almost paracontact metric manifolds and para-Sasakian manifolds which will be used later in the next section.

Let M be a contact manifold of dimension $2n + 1$ together with a contact form η such that $\eta \wedge (d\eta)^n \neq 0$. It is widely recognized that a contact manifold admits a special vector field ξ , which is said to be characteristic vector field of the contact manifold, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \tau_1) = 0$ for all τ_1 in $\Gamma(TM)$. Furthermore, M possesses a Riemannian metric g and a $(1,1)$ -type tensor field φ satisfying $\varphi^2 = I - \eta \otimes \xi$, $g(\tau_1, \xi) = \eta(\tau_1)$, and $g(\tau_1, \varphi \tau_2) = d\eta(\tau_1, \tau_2)$. In this context, we refer to (φ, ξ, η, g) as a contact metric structure. A manifold which is endowed with a such structure is termed Sasakian if

$$(\nabla_{\tau_1} \varphi) \tau_2 = g(\tau_1, \tau_2) \xi - \eta(\tau_2) \tau_1,$$

in which case

$$\nabla_{\tau_1} \xi = -\varphi \tau_1$$

and

$$R(\tau_1, \tau_2) \xi = \eta(\tau_2) \tau_1 - \eta(\tau_1) \tau_2$$

for all vector fields $\tau_1, \tau_2 \in \Gamma(TM)$. Here, ∇ denotes the connection of the manifold, which is called the Levi-Civita connection (Blair, 1976)

An almost paracontact manifold M having dimension $2n + 1$ is a differentiable manifold consisting of the structure (φ, ξ, η, g) , where φ represents a $(1,1)$ -tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M . This structure is characterized by the following conditions (Sato, 1976)

$$\varphi^2 \tau_1 = \tau_1 - \eta(\tau_1)\xi, \quad (1)$$

$$\eta(\xi) = 1, \quad (2)$$

$$\varphi\xi = 0, \quad (3)$$

$$\eta\varphi = 0, \quad (4)$$

and

$$g(\xi, \tau_1) = \eta(\tau_1), \quad (5)$$

$$g(\varphi\tau_1, \varphi\tau_2) = g(\tau_1, \tau_2) - \eta(\tau_1)\eta(\tau_2) \quad (6)$$

for all vector fields $\tau_1, \tau_2 \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of all vector field on the manifold M .

It is to be noted that as a direct consequence of the equations (1), (3), (4) and (6) we also have

$$g(\varphi\tau_1, \tau_2) + g(\tau_1, \varphi\tau_2) = 0. \quad (7)$$

If the structure (φ, ξ, η, g) on the manifold M satisfies

$$d\eta = 0, \quad (8)$$

$$\nabla_{\tau_1}\xi = \varphi\tau_1, \quad (9)$$

$$(\nabla_{\tau_1}\varphi)\tau_2 = -g(\tau_1, \tau_2)\xi - \eta(\tau_2)\tau_1 + 2\eta(\tau_1)\eta(\tau_2)\xi, \quad (10)$$

then M is said to define a para-Sasakian manifold, or shortly, a P -Sasakian manifold (Adati & Matsumoto, 1977). We denote a para-Sasakian manifold by $(M, \varphi, \xi, \eta, g)$.

Also we want to remark that in a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ the following properties are satisfied (Zamkovoy, 2009):

$$Ric \xi = -2n. \quad (11)$$

$$S(\tau_1, \xi) = -2n\eta(\tau_1), \quad (12)$$

$$R(\tau_1, \tau_2)\xi = \eta(\tau_1)\tau_2 - \eta(\tau_2)\tau_1, \quad (13)$$

$$R(\xi, \tau_1)\tau_2 = \eta(\tau_2)\tau_1 - g(\tau_1, \tau_2)\xi \quad (14)$$

$$R(\xi, \tau_1)\xi = \tau_1 - \eta(\tau_1)\xi, \quad (15)$$

$$\eta(R(\tau_1, \tau_2)\tau_3) = g(\tau_1, \tau_3)\eta(\tau_2) - g(\tau_2, \tau_3)\eta(\tau_1) \quad (16)$$

for all vector fields $\tau_1, \tau_2, \tau_3 \in \Gamma(TM)$. Here, R is the Riemannian curvature tensor of the manifold, which is given by the formula

$$R(\tau_1, \tau_2)\tau_3 = \nabla_{\tau_1}\nabla_{\tau_2}\tau_3 - \nabla_{\tau_2}\nabla_{\tau_1}\tau_3 - \nabla_{[\tau_1, \tau_2]}\tau_3$$

and S stands for the Ricci tensor defined by $S(\tau_1, \tau_2)g(Ric\tau_1, \tau_2)$, where Ric is the Ricci operator.

Additionally if the Ricci tensor S of a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ satisfies

$$S(\tau_1, \tau_2) = \rho g(\tau_1, \tau_2) + \delta \eta(\tau_1)\eta(\tau_2)$$

then the manifold is named as η – Einstein, where ρ and δ are smooth functions on the manifold M . An η – Einstein manifold becomes Einstein if $\delta = 0$ (Adati & Miyazawa, 1979).

On the other hand Mantica and Suh defined a new curvature tensor, which is named as Q curvature tensor, given by (Mantica & Suh, 2013)

$$Q(\tau_1, \tau_2)\tau_3 = R(\tau_1, \tau_2)\tau_3 - \frac{\psi}{2n}[g(\tau_2, \tau_3)\tau_1 - g(\tau_1, \tau_3)\tau_2]. \quad (17)$$

Here, ψ denotes an arbitrary scalar function. When $\psi = \frac{r}{(2n+1)}$, then this curvature tensor reduces to concircular curvature tensor.

Also it follows from the equations (13), (14), (15) and (17) one immediately has

$$Q(\tau_1, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)R(\tau_1, \tau_2)\xi \quad (18)$$

$$Q(\xi, \tau_2)\tau_3 = \left(1 + \frac{\psi}{2n}\right)R(\xi, \tau_2)\tau_3 \quad (19)$$

$$Q(\xi, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)R(\xi, \tau_2)\xi \quad (20)$$

for any vector fields τ_1 and τ_2 tangent to M . The curvature tensor Q has been studied by various researchers, including Yadav and Yıldız (Yadav & Yıldız, 2022), De and Majhi (De and Majhi, 2019), Yıldırım (Yıldırım, 2022), Yılmaz (Bağdatlı Yılmaz, 2020) as well as numerous other scholars.

Main Results

In this section we give our main results that are obtained in this work. We obtain that under some curvature conditions a para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ becomes a constant negative curvature manifold.

The first result of this section is in the following.

Theorem 1: Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold admitting curvature condition $Q(\xi, \tau_4).Q = 0$. Then, we have that either $\psi = -2n$ or the manifold M has constant negative curvature -1 .

Proof: Let us suppose that the structure (φ, ξ, η, g) of the manifold M satisfies the condition $Q(\xi, \tau_4).Q = 0$, that is

$$(Q(\xi, \tau_4)Q).(\tau_1, \tau_2)\tau_5 = 0$$

for all vector fields $\tau_1, \tau_2, \tau_4, \tau_5 \in \Gamma(TM)$. This means that

$$\begin{aligned} &Q(\xi, \tau_4).Q(\tau_1, \tau_2)\tau_5 - Q(Q(\xi, \tau_4)\tau_1, \tau_2)\tau_5 \\ &Q(\tau_1, Q(\xi, \tau_4)\tau_2)\tau_5 - Q(\tau_1, \tau_2)Q(\xi, \tau_4)\tau_5 = 0. \end{aligned} \quad (21)$$

Setting $\tau_5 = \xi$ in (21), one can write

$$\begin{aligned} &Q(\xi, \tau_4)Q(\tau_1, \tau_2)\xi - Q(Q(\xi, \tau_4)\tau_1, \tau_2)\xi \\ &Q(\tau_1, Q(\xi, \tau_4)\tau_2)\xi - Q(\tau_1, \tau_2)Q(\xi, \tau_4)\xi = 0. \end{aligned} \quad (22)$$

For the first term of (22), using (18) and (19), we get

$$Q(\xi, \tau_4)Q(\tau_1, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)^2 R(\xi, \tau_4)R(\tau_1, \tau_2)\xi \quad (23)$$

Also, making use of (19) in (23) we find that

$$Q(\xi, \tau_4)Q(\tau_1, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)^2 (\eta(\tau_2)g(\tau_4, \tau_1)\xi - \eta(\tau_1)g(\tau_4, \tau_2)\xi). \quad (24)$$

For the second term of (22), by means of (19) we achieve

$$Q(Q(\xi, \tau_4)\tau_1, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)^2 R(R(\xi, \tau_4)\tau_1, \tau_2)\xi,$$

which together with the equations (13) and (15) takes the form

$$Q(Q(\xi, \tau_4)\tau_1, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)^2 \left[\frac{\eta(\tau_1)\eta(\tau_4)\tau_2 - \eta(\tau_1)\eta(\tau_2)\tau_4}{+g(\tau_4, \tau_1)\eta(\tau_2)\xi - g(\tau_4, \tau_1)\tau_2} \right] \quad (25)$$

For the third term of the equation (22), with the help of (18) and (19), we arrive at

$$Q(Q(\xi, \tau_4)\tau_1, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)^2 R(\tau_1, R(\xi, \tau_4)\tau_2)\xi. \quad (26)$$

It follow from (13) and (15), the equation (25) becomes

$$Q(Q(\xi, \tau_4)\tau_1, \tau_2)\xi = \left(1 + \frac{\psi}{2n}\right)^2 \left[\frac{\eta(\tau_2)\eta(\tau_1)\tau_4 - \eta(\tau_2)\eta(\tau_4)\tau_1}{+g(\tau_4, \tau_2)\tau_1 - g(\tau_4, \tau_2)\eta(\tau_1)\xi} \right]. \quad (27)$$

For the fourth term of the equation (22), using (20) we write

$$Q(\tau_1, \tau_2)Q(\xi, \tau_4)\xi = \left(1 + \frac{\psi}{2n}\right) Q(\tau_1, \tau_2)R(\xi, \tau_4)\xi \quad (28)$$

In wiew of (15), we provide

$$Q(\tau_1, \tau_2)Q(\xi, \tau_4)\xi = \left(1 + \frac{\psi}{2n}\right) Q(\tau_1, \tau_2)(\tau_4 - \eta(\tau_4)\xi) \quad (29)$$

Using (18) in (29), we obtain that

$$Q(\tau_1, \tau_2)Q(\xi, \tau_4)\xi = \left(1 + \frac{\psi}{2n}\right) Q(\tau_1, \tau_2)\tau_4 \left(1 + \frac{\psi}{2n}\right)^2 [\eta(\tau_4)\eta(\tau_1)\tau_2 - \eta(\tau_4)\eta(\tau_2)\tau_1]. \quad (30)$$

Now, substituting the equations (24), (25), (27) and (30) into (22), we have

$$\left(1 + \frac{\psi}{2n}\right) [R(\tau_1, \tau_2)\tau_4 + g(\tau_2, \tau_4)\tau_1 - g(\tau_1, \tau_4)\tau_2] = 0,$$

which implies that either

$$\frac{\psi}{2n} + 1 = 0$$

or

$$R(\tau_1, \tau_2)\tau_4 = -(g(\tau_2, \tau_4)\tau_1 - g(\tau_1, \tau_4)\tau_2).$$

This result completes the proof.

Next, we have another important result.

Theorem 2: Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold admitting curvature condition $Q(\tau_1, \tau_2).Ric = 0$. Then, we have that either $\psi = -2n$ or the manifold M is an Einstein manifold with the scalar curvature $r = -2n(2n + 1)$.

Proof: Let us suppose that the structure (φ, ξ, η, g) of the manifold M satisfies the condition $Q(\tau_1, \tau_2).Ric = 0$, namely

$$(Q(\tau_1, \tau_4).Ric)\tau_3 = 0$$

and hence

$$Q(\tau_1, \tau_2).Ric \tau_3 - Ric (Q(\tau_1, \tau_2)\tau_3) = 0 \quad (31)$$

for all vector fields $\tau_1, \tau_2, \tau_3 \in \Gamma(TM)$. If we take $\tau_1 = \xi$ in (31) and also benefit from (19), we obtain

$$\left(1 + \frac{\psi}{2n}\right) [R(\xi, \tau_2)Ric \tau_3 - Ric (R(\xi, \tau_2)\tau_3)] = 0,$$

which together with (14) gives

$$\left(1 + \frac{\psi}{2n}\right) [\eta(Ric \tau_3)\tau_2 - g(Ric \tau_3, \tau_2)\xi - Ric (\eta(\tau_3)\tau_2 - g(\tau_2, \tau_3)\xi)] = 0. \quad (32)$$

Due to the equations (11) and (12), the equation (32) reduces to

$$\left(1 + \frac{\psi}{2n}\right)[-2n\eta(\tau_3)\tau_2 - S(\tau_2, \tau_3)\xi - \eta(\tau_3)Ric \tau_2 - 2ng(\tau_2, \tau_3)\xi] = 0. \quad (33)$$

Taking inner product of (33) with ξ yields

$$\left(1 + \frac{\psi}{2n}\right)[-2ng(\tau_2, \tau_3) - S(\tau_2, \tau_3)] = 0.$$

Thus, we have that either $\psi = -2n$ or $S(\tau_2, \tau_3) = -2ng(\tau_2, \tau_3)$. Taking the trace of the ricci tensor, we obtain $r = -2n(2n + 1)$. Therefore, we get the requested result.

The next result provides a important characterization regarding para-Sasakian manifolds.

Theorem 3: Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold admitting curvature condition $Q(\tau_1, \tau_2).S = 0$. Then, we have that either $\psi = -2n$ or the manifold is an Einstein manifold.

Proof: Under our hypothesis, we have

$$(Q(\tau_1, \tau_2).S)(\tau_3, \tau_5) = 0$$

for all vector fields $\tau_1, \tau_2, \tau_3, \tau_5 \in \Gamma(TM)$. This is equivalent to

$$S(Q(\tau_1, \tau_2)\tau_3, \tau_5) + S(\tau_3, Q(\tau_1, \tau_2)\tau_5) = 0. \quad (34)$$

Putting $\tau_1 = \xi$ in (34) and from (19) we get

$$\left(1 + \frac{\psi}{2n}\right)(S(R(\xi, \tau_2)\tau_3, \tau_5) + S(\tau_3, R(\xi, \tau_2)\tau_5)) = 0. \quad (35)$$

Implementing (14) in (35), we find that

$$\left(1 + \frac{\psi}{2n}\right) \left[\eta(\tau_3)S(\tau_2, \tau_5) + 2ng(\tau_2, \tau_3)\eta(\tau_5) + \eta(\tau_5)S(\tau_2, \tau_3) \right] + 2ng(\tau_2, \tau_5)\eta(\tau_3) \quad (36)$$

Moreover setting $\tau_3 = \xi$ in (36) provides

$$\left(1 + \frac{\psi}{2n}\right)[S(\tau_2, \tau_5) + 2ng(\tau_2, \tau_5)] = 0,$$

where we have used the equations (5) and (12). Therefore we have the desired result. The proof is completed.

Theorem 4: Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold admitting curvature condition $Q.\varphi = 0$. Then we have that $\psi = -2n$.

Proof: Let us consider that the structure (φ, ξ, η, g) of the manifold M satisfies the curvature condition $Q.\varphi = 0$, that is

$$(Q(\tau_1, \tau_2).\varphi)\tau_3 = 0,$$

which is equivalent to

$$Q(\tau_1, \tau_2).\varphi\tau_3 - \varphi(Q(\tau_1, \tau_2)\tau_3) = 0 \quad (37)$$

for all vector fields $\tau_1, \tau_2, \tau_3 \in \Gamma(TM)$. Putting ξ instead of τ_1 in (37), we write

$$Q(\xi, \tau_2)\varphi\tau_3 - \varphi(Q(\xi, \tau_2)\tau_3) = 0. \quad (38)$$

Also, making use of (19) in (38) we get

$$\left(1 + \frac{\psi}{2n}\right)[R(\xi, \tau_2)\varphi\tau_3 - \varphi(R(\xi, \tau_2)\tau_3)] = 0. \quad (39)$$

From (2), (3), (14) and (39), we arrive at

$$\left(1 + \frac{\psi}{2n}\right)[-g(\tau_2, \varphi\tau_3)\xi - \eta(\tau_3)\varphi\tau_2] = 0. \quad (40)$$

Replacing τ_3 by $\varphi\tau_3$ in (40) and after a straightforward computation, we have

$$\left(1 + \frac{\psi}{2n}\right)[g(\varphi\tau_2, \varphi\tau_3)\xi] = 0. \quad (41)$$

Taking inner product of (41) with ξ gives

$$\left(1 + \frac{\psi}{2n}\right)[g(\varphi\tau_2, \varphi\tau_3)] = 0.$$

Taking the orthonormal basis of the above equation, we obtain that $\psi = -2n$. Therefore, the proof is completed.

The final result of our work is as follows:

Theorem 5: Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold admitting curvature condition $\varphi.Q = 0$. Then we have that $\psi = -2n$.

Proof: Let us consider that the structure (φ, ξ, η, g) of the manifold M satisfies the curvature condition $\varphi.Q = 0$, which means

$$(\varphi.Q)(\tau_1, \tau_2)\tau_3$$

namely,

$$\varphi(Q(\tau_1, \tau_2)\tau_3 - Q(\varphi\tau_1, \tau_2)\tau_3 - Q(\tau_1, \varphi\tau_2)\tau_3 - Q(\tau_1, \tau_2)\varphi\tau_3) = 0 \quad (42)$$

for all vector fields $\tau_1, \tau_2, \tau_3 \in \Gamma(TM)$. Taking ξ in place of τ_3 in (42) and it follows from (3) that we have

$$\varphi(Q(\tau_1, \tau_2)\xi - Q(\varphi\tau_1, \tau_2)\xi - Q(\tau_1, \varphi\tau_2)\xi) = 0. \quad (43)$$

Using (13) and (18) in (43) one immediately has

$$\left(1 + \frac{\psi}{2n}\right)(\eta(\tau_1)\tau_2 - \eta(\tau_2)\tau_1 + \eta(\tau_2)\varphi\tau_1 - \eta(\tau_1)\varphi\tau_2) = 0. \quad (44)$$

Taking inner product of (44) with arbitrary vector field τ_3 gives

$$\left(1 + \frac{\psi}{2n}\right) \left[\begin{aligned} &\eta(\tau_1)g(\tau_2, \tau_3) - \eta(\tau_2)g(\tau_1, \tau_3) \\ &+ \eta(\tau_2)g(\varphi\tau_1, \tau_3) - \eta(\tau_1)g(\varphi\tau_2, \tau_3) \end{aligned} \right] = 0. \quad (45)$$

Setting $\tau_1 = \tau_3 = G_i$ in (45) and from (2), (3) we get

$$\left(1 + \frac{\psi}{2n}\right) \eta(\tau_2) = 0,$$

where $\{G_i\}_{i=1}^{2n+1}$ is a local orthonormal frame of the manifold M .

Taking $\tau_2 = \xi$ yields $\psi = -2n$. Therefore, the proof is completed.

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CHAPTER III

Exploring Bicomplex Lebesgue Spaces: Properties and Significance

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Introduction

Lebesgue spaces are essential structures in functional analysis and measure theory, offering a formal framework for understanding function convergence and integrability. The study of Lebesgue spaces has been helpful in increasing our understanding of diverse mathematical phenomena, providing a flexible platform for examining the behavior of functions in various circumstances. As we explore deeper into the complexities of Lebesgue spaces, this essay focuses on the concepts of sums and intersection within this mathematical realm. We want to get insights into the convergence

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features and interrelationships that emerge when combining or intersecting functions in Lebesgue spaces by investigating the interaction of these operations. This examination not only advances theoretical comprehension of these spaces, but also reveals their practical consequences in a variety of mathematical applications. Join us on a voyage through the intricacies of sums and crossings in Lebesgue spaces, where the convergence of mathematical concepts leads to a better knowledge of function structure and behavior $\mathbb{BC}$ -valued functions arise naturally in various mathematical fields, including probability theory, mathematical analysis, and functional analysis, and understanding their properties is crucial for advancing these areas of study. Indeed, the study of modules with bicomplex scalars in the context of functional analysis has gained significant attention in recent years. One influential work that has contributed to this area is the book (Alpay et al., 2014). The book likely presents groundbreaking results and insights related to this topic. Functional analysis traditionally deals with vector spaces over a field, such as the complex numbers or the real numbers. However, by considering modules with bicomplex scalars, where the scalars are elements of the bicomplex numbers, a broader framework is introduced. This extension allows for the exploration of new mathematical structures and the investigation of properties beyond the classical setting. The book by Alpay et al. is likely a valuable resource for researchers and enthusiasts interested in this area. It likely presents notable results, techniques, and applications pertaining to the study of modules with bicomplex scalars in the context of functional analysis. These results may encompass various aspects of functional analysis, such as operator theory, function spaces, and spectral theory, among others. They may shed light on the behavior of modules with bicomplex scalars, reveal connections to other areas of mathematics, and potentially find applications in physics, engineering, or other disciplines.

The series of articles mentioned in the references highlight the systematic study of topological bicomplex modules and various

fundamental theorems related to them. Here is a breakdown of the articles and their contributions:

In (Kumar & Saini, 2016), the authors studied topological bicomplex modules, likely exploring their topological properties and investigating concepts such as convergence, continuity, and compactness in this context.

The authors in (Kumar, Kumar & Rochon, 2011) presented fundamental theorems, including Banach-Steinhaus theorem, open mapping theorem, closed graph theorem and interior mapping theorem for bicomplex modules.

The papers (Saini, Sharma & Kumar, 2020), in collaboration with (Kumar, Kumar & Rochon, 2011), likely extends the study of fundamental theorems to the setting of topological bicomplex modules. The focus may be on generalizing classical results from functional analysis to the bicomplex module framework, providing a deeper understanding of their properties. Also, the authors likely delve further into the study of topological hyperbolic modules, topological bicomplex modules, exploring the properties of linear operators, continuity, and related topological concepts specific to these settings.

The authors in (Luna- Elizarrarás, Perez-Regalado & Shapiro, 2014) studied on bicomplex modules and hyperbolic modules and wrote the Hahn-Banach theorem for these modules.

The book (Luna-Elizarrarás et al., 2015) likely provides an in-depth exploration of bicomplex analysis and geometry. It may cover a wide range of topics, including holomorphic functions, integration, differential equations, and geometric properties specific to the bicomplex domain.

In (Colombo, Sabatini & Struppa, 2014), the authors focused on $\mathbb{BC}$ bounded linear operators and bicomplex functional calculus. It may provide a detailed study of operators acting on bicomplex modules and explore the construction and properties of functional calculi specific to the bicomplex framework.

In (Sağır, Değirmen & Duyar, 2023), properties of bicomplex matrix transformations between sequences' spaces c_0 and c are examined.

These references collectively represent significant contributions to the study of bicomplex modules, functional analysis, and related areas. They showcase the exploration of properties, the development of new theorems, and the application of functional analysis techniques in the context of bicomplex numbers. Researchers and readers interested in these topics can refer to these articles and the books for detailed insights into the respective areas of study. Now, we will give a small summary of bicomplex numbers with some basic properties.

Preliminaries on $\mathbb{BC}$ and $\mathbb{BC}$ -Lebesgue spaces

The set bicomplex numbers $\mathbb{BC}$ which is a four-dimensional extension of the complex numbers is defined as

$$\mathbb{BC} := \{W = w_1 + jw_2 \mid w_1, w_2 \in \mathbb{C}(i)\}.$$

Here i and j are imaginary units satisfying $ij = ji$ and $i^2 = -1 = j^2$. Here $\mathbb{C}(i)$ with the imaginary unit i , stands for the field of complex numbers. According to ring structure, for any $Z = z_1 + jz_2$, $W = w_1 + jw_2$ in $\mathbb{BC}$ usual addition and multiplication are defined as

$$Z + W = (z_1 + w_1) + j(z_2 + w_2)$$

$$ZW = (z_1w_1 - z_2w_2) + j(z_2w_1 + z_1w_2).$$

Under the ordinary addition and multiplication of bicomplex numbers, the set $\mathbb{BC}$ forms a commutative ring. The bicomplex numbers have a unit element denoted as $1_{\mathbb{BC}} := 1$ and this acts as the identity for multiplication, such that for any bicomplex number W , $1 \cdot W = W \cdot 1 = W$. In the sense of module structure, the set $\mathbb{BC}$ is

a module over itself. This means that $\mathbb{BC}$ satisfies the properties of a module, including scalar multiplication and distributivity. If one multiplies the imaginary units i and j , then a new hyperbolic unit k can be obtained such that $k^2 = 1$. This implies that k is a square root of 1 and is distinct from i and j . The product operation of all units i, j and k in the bicomplex numbers is commutative. Specifically, the following relations hold:

$$ij = k, \quad jk = -i \text{ and } ik = -j.$$

These properties summarize the basic characteristics of bicomplex numbers and their algebraic structure.

Hyperbolic numbers $\mathbb{D}$ are a two-dimensional extension of the real numbers that form a number system known as the hyperbolic plane or hyperbolic plane algebra. They can be represented in the form $\alpha = \beta_1 + k\beta_2$, where β_1 and β_2 are real numbers, and k is the hyperbolic unit. In the hyperbolic number system, for any two hyperbolic numbers $\alpha = \beta_1 + k\beta_2$ and $\gamma = \delta_1 + k\delta_2$, addition and multiplication are defined as follows:

$$\alpha + \gamma = (\beta_1 + \delta_1) + k(\beta_2 + \delta_2)$$

$$\alpha\gamma = (\beta_1\delta_1 + \beta_2\delta_2) + k(\beta_1\delta_2 + \beta_2\delta_1).$$

The hyperbolic numbers form a ring, however, unlike the complex numbers, the hyperbolic numbers do not have a multiplicative inverse for all nonzero elements. The nonzero hyperbolic numbers that have multiplicative inverses are called units. The bicomplex numbers contain two imaginary units i and j , and the hyperbolic numbers can be taken as a subset of the bicomplex numbers by restricting the imaginary part of j to be zero.

Let $W = w_1 + jw_2 \in \mathbb{BC}$ where $w_1, w_2 \in \mathbb{C}(i)$. By the notation of W with imaginary units i and j , three conjugations are brought out for bicomplex numbers in (Alpay et al., 2014) and (Luna-Elizarrarás et al., 2015) as $\bar{W}_1 = \bar{w}_1 + j\bar{w}_2$, $\bar{W}_2 = w_1 - jw_2$

and $\bar{W}_3 = \bar{w}_1 - j\bar{w}_2$ where $\bar{w}_1$ and $\bar{w}_2$ are the usual complex conjugates of $w_1, w_2 \in \mathbb{C}(i)$. For any bicomplex number W , they also wrote the following three moduli in (Alpay et al., 2014), (Luna-Elizarrarás et al., 2015) and (Price, 2018) as:

- i. $|W|_i^2 = W \cdot \bar{W}_2 = w_1^2 + w_2^2 \in \mathbb{C}(i),$
- ii. $|W|_j^2 = W \cdot \bar{W}_1 = (|w_1|^2 - |w_2|^2) + j(2\text{Re}(w_1\bar{w}_2)) \in \mathbb{C}(j)$
- iii. $|W|_k^2 = W \cdot \bar{W}_3 = (|w_1|^2 + |w_2|^2) + k(-2\text{Im}(w_1\bar{w}_2)) \in \mathbb{D}.$

Furthermore, $\mathbb{BC}$ is a normed space with the norm

$$\|W\|_{\mathbb{BC}} = \sqrt{|w_1|^2 + |w_2|^2}$$

for any $W = w_1 + jw_2$ in $\mathbb{BC}$ (Alpay et al., 2014). According to this,

$$\|W_1 W_2\|_{\mathbb{BC}} \leq \sqrt{2} \|W_1\|_{\mathbb{BC}} \|W_2\|_{\mathbb{BC}}$$

for every $W_1, W_2 \in \mathbb{BC}$, and finally $\mathbb{BC}$ is a quasi-Banach algebra (Alpay et al., 2014). If the hyperbolic numbers e_1 and e_2 defined as

$$e_1 = \frac{1+k}{2} \quad \text{and} \quad e_2 = \frac{1-k}{2},$$

then it is easy to see that the set $\{e_1, e_2\}$ is a fundamental set in $\mathbb{C}(i)$ -vector space $\mathbb{BC}$ and linearly independent. The set $\{e_1, e_2\}$ also satisfies the following properties:

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad (\bar{e}_1)_3 = e_1, \quad (\bar{e}_2)_3 = e_2$$

$$e_1 + e_2 = 1, \quad e_1 \cdot e_2 = 0$$

with $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \frac{\sqrt{2}}{2}$. By using this linearly independent set $\{e_1, e_2\}$, any $W = w_1 + jw_2 \in \mathbb{BC}$ can be written as a linear

combination of e_1 and e_2 uniquely. That is, $W = w_1 + jw_2$ can be written as

$$W = w_1 + jw_2 = e_1 z_1 + e_2 z_2 \quad (1.1)$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$ (Alpay et al., 2014). Here z_1 and z_2 are elements of $\mathbb{C}(i)$ and (1.1), the preceding formula, is named with the *idempotent representation* of W .

Besides the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$, another norm named with ($\mathbb{D}$ -valued) hyperbolic-valued norm $|W|_k$ of any bicomplex number $W = e_1 z_1 + e_2 z_2$ is defined as

$$|W|_k = e_1 |z_1| + e_2 |z_2|.$$

For any hyperbolic number $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, an idempotent representation can also be written as

$$\alpha = e_1 \alpha_1 + e_2 \alpha_2$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. If $\alpha_1 > 0$ and $\alpha_2 > 0$ for any $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, then we say that α is called a positive hyperbolic number. Thus, $\mathbb{D}^+ \cup \{0\}$, the set of non-negative hyperbolic numbers

$$\begin{aligned} \mathbb{D}^+ \cup \{0\} &= \{\alpha = \beta_1 + k\beta_2: \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\} \\ &= \{\alpha = e_1 \alpha_1 + e_2 \alpha_2: \alpha_1, \alpha_2 \geq 0\}. \end{aligned}$$

can be defined. Now, let α and γ be any two elements of $\mathbb{D}$. In (Alpay et al., 2014) and (Luna-Elizarrarás et al., 2015), a relation $\preceq$ is defined on $\mathbb{D}$ by

$$\alpha \preceq \gamma \Leftrightarrow \gamma - \alpha \in \mathbb{D}^+ \cup \{0\}.$$

It is showed in (Alpay et al., 2014) that this relation " $\preceq$ " has reflexive, anti-symmetric and transitive properties. Therefore " $\preceq$ "

can define a partial order relation on $\mathbb{D}$. If idempotent representations of the hyperbolic numbers α, γ are written as $\alpha = e_1\alpha_1 + e_2\alpha_2$ and $\gamma = e_1\gamma_1 + e_2\gamma_2$, then $\alpha \leq \gamma$ implies that $\alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$. By $\alpha < \gamma$, we mean $\alpha_1 < \gamma_1$ and $\alpha_2 < \gamma_2$. For more details on hyperbolic numbers $\mathbb{D}$ and partial order " $\leq$ ", one can refer to Section 1.5 of (Alpay et al., 2014), (Luna-Elizarrarás et al., 2015) and (Price, 2018).

Definition 1. Let U be a subset of $\mathbb{D}$. U is called a $\mathbb{D}$ -bounded above set if there is a hyperbolic number δ such that $\delta \geq \alpha$ for all $\alpha \in U$. If $U \subset \mathbb{D}$ is a $\mathbb{D}$ -bounded set from above, then the hyperbolic supremum of U is defined as the smallest member of the set of all upper bounds of U (Luna-Elizarrarás et al., 2015).

In other words, the hyperbolic number $\lambda = e_1\lambda_1 + e_2\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$, is the $\mathbb{D}$ -supremum of U if

- i. $e_1\alpha_1 + e_2\alpha_2 \leq e_1\lambda_1 + e_2\lambda_2$ for each $\alpha = e_1\alpha_1 + e_2\alpha_2 \in U$
- ii. For any $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 > 0$, there exists $\theta = e_1\theta_1 + e_2\theta_2 \in U$ such that $e_1\theta_1 + e_2\theta_2 > e_1(\lambda_1 - \varepsilon_1) + e_2(\lambda_2 - \varepsilon_2)$

are satisfied.

Remark 1. Let $e_1\theta_1 + e_2\theta_2$ be a $\mathbb{D}$ -bounded above subset of $\mathbb{D}$ and $U_1 := \{\gamma_1: e_1\gamma_1 + e_2\gamma_2 \in U\}$, $U_2 := \{\gamma_2: e_1\gamma_1 + e_2\gamma_2 \in U\}$. Then the $\sup_{\mathbb{D}} U$ is given by

$$\sup_{\mathbb{D}} U := e_1 \sup U_1 + e_2 \sup U_2.$$

Similarly, for any set U which is $\mathbb{D}$ -bounded from below, $\mathbb{D}$ -infimum of U can be defined as

$$\inf_{\mathbb{D}} U := e_1 \inf U_1 + e_2 \inf U_2$$

where U_1 and U_2 are as above, [Remark 1.5.2] (Alpay et al., 2014).

Definition 2. Let $(X, +)$ be an abelian group and $(X, +, \cdot)$ be a $\mathbb{BC}$ -module. If there is a topology τ_X in X , such that the operations $+: X \times X \rightarrow X$ and $\cdot: \mathbb{BC} \times X \rightarrow X$ are continuous, then $(X, +, \cdot)$ is called a topological $\mathbb{BC}$ -module.

Remark 2. A $\mathbb{BC}$ -module space or $\mathbb{D}$ -module space Y can be decomposed as

$$Y = e_1 Y_1 + e_2 Y_2 \quad (1.2)$$

where $Y_1 = e_1 Y$ and $Y_2 = e_2 Y$ are $\mathbb{R}$ -vector or $\mathbb{C}(i)$ -vector spaces. The spelling in (1.2) is called as the idempotent decomposition of the space Y . Therefore, any element y in Y can be uniquely inscribed as $y = e_1 y_1 + e_2 y_2$ with $y_1 \in Y_1$ and $y_2 \in Y_2$, (Alpay et al., 2014).

The following is known from (Saini, Sharma & Kumar, 2020).

Definition 3. Let $(X, \|\cdot\|_X)$ be a $\mathbb{BC}$ -module. If every Cauchy sequence in X converges to any element of it with respect to the norm, then $(X, \|\cdot\|_X)$ is called a bicomplex Banach module.

Proposition 1. $(X, \|\cdot\|_X)$ is a bicomplex Banach module, if and only if, the decomposition pairs of the space, $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ are complex Banach spaces (Kumar & Saini, 2016).

Definition 4. Let $T: X \rightarrow X$ be a map. Then T is called a $\mathbb{BC}$ -linear operator on X , if the following exist:

- i. $T(x + y) = T(x) + T(y)$,
- ii. $T(\alpha x) = \alpha T(x)$

for every $x, y \in X$ and $\alpha \in \mathbb{BC}$.

The following result is well known from (Colombo, Sabatini & Struppa, 2014).

Proposition 2. Let X be a bicomplex Banach module and $T: X \rightarrow X$ be a linear operator. Suppose that X has an idempotent

decomposition as $X = e_1X_1 + e_2X_2$. Then the operator T admits the idempotent representation $T = e_1T_1 + e_2T_2$, where

$$T_j: e_jX \rightarrow e_jX$$

$$x \rightarrow T_j(x) := e_jT(e_jx)$$

are linear operators for $j = 1, 2$ respectively.

Definition 5. Let $\mathcal{M}$ be a σ -algebra on a set Ω and $\mu = \mu_1e_1 + \mu_2e_2$ be a bicomplex-valued function defined on Ω . Then μ is called a bicomplex measure on $\mathcal{M}$ if μ_1 and μ_2 are both complex measures on $\mathcal{M}$. Nevertheless if μ_1 and μ_2 are positive measures on $\mathcal{M}$ namely, range of both μ_1, μ_2 are $[0, \infty]$, then μ is called a $\mathbb{D}$ -measure on $\mathcal{M}$. Also, μ is called a $\mathbb{D}^+$ -measure on $\mathcal{M}$, if μ_1, μ_2 are real measures on $\mathcal{M}$ i.e. $\mu_1(\cdot), \mu_2(\cdot) \in [0, \infty)$, (Ghosh & Mondal, 2022).

Assume that $\Omega = (\Omega, \mathcal{M}, \mu)$ is a σ -finite complete measure space and f_1, f_2 are complex-valued (real-valued) measurable functions on Ω . The function having idempotent decomposition $f = f_1e_1 + f_2e_2$ is called as a $\mathbb{BC}$ -measurable function and $|f|_k = |f_1|e_1 + |f_2|e_2$ is called a $\mathbb{D}$ -valued measurable function on Ω (Dubey, Kumar & Sharma, 2014). Thus, for any given complex valued function space $(F(\Omega), \|\cdot\|_\Omega)$, one can create a $\mathbb{BC}$ -valued function space $(F(\Omega, \mathbb{BC}), \|\cdot\|_{\mathbb{BC}})$ by combining all f_1, f_2 and bringing out functions of the type

$$f = f_1e_1 + f_2e_2$$

where f_1 and f_2 are in $(F(\Omega), \|\cdot\|_\Omega)$ with $\|f\|_{\mathbb{BC}}^2 = \frac{1}{2}(\|f_1\|_\Omega^2 + \|f_2\|_\Omega^2)$. Similar definition can be given for any hyperbolic measurable function.

For any $\mathbb{BC}$ -valued measurable function $f = f_1e_1 + f_2e_2$, it is easy to see that $|f|_k = |f_1|e_1 + |f_2|e_2$ is $\mathbb{D}$ -valued measurable. Because if $f = f_1e_1 + f_2e_2$ is a $\mathbb{BC}$ -valued measurable function,

then f_1 and f_2 are $\mathbb{C}$ -measurable functions. Therefore real and imaginary parts of f_1 and f_2 are $\mathbb{R}$ -valued measurable and so does $|f_1|$ and $|f_2|$. As a result, $|f|_k$ is $\mathbb{D}$ -measurable. Also, for any two $\mathbb{BC}$ -valued measurable functions f and g , it can be easily seen that their sum and multiplication functions are also $\mathbb{BC}$ -measurable functions, (Dubey, Kumar & Sharma, 2014) and (Ghosh & Mondal, 2022). More results on $\mathbb{D}$ -topology such as $\mathbb{D}$ -limit, $\mathbb{D}$ -continuity, $\mathbb{D}$ -Cauchy and $\mathbb{D}$ -convergence etc. can be found in (Değirmen & Sağır, 2023), (Ghosh & Mondal, 2022), (Toksoy & Sağır, 2023) and the references therein.

Definition 6. Let $\mathcal{M}$ be a σ -algebra and $\mu = e_1\mu_1 + e_2\mu_2$ be a $\mathbb{BC}$ -measure on $(\Omega, \mathcal{M})$. Then any two bicomplex valued $\mathbb{BC}$ -measurable functions $f = e_1f_1 + e_2f_2$ and $g = e_1g_1 + e_2g_2$ on Ω are called to be equal (μ -a.e.) if $f_1 = g_1$ (μ_1 -a.e.) and $f_2 = g_2$ (μ_2 -a.e.).

Definition 7. Let $\mu = e_1\mu_1 + e_2\mu_2$ be a $\mathbb{D}$ -measure on a measure space $(\Omega, \mathcal{M})$ and $1 \leq p < \infty$. Suppose $L^p(\Omega, \mu_1)$ and $L^p(\Omega, \mu_2)$ stand for the linear space of all (equivalence classes of) complex valued, measurable functions f_1 and f_2 defined on Ω with

$$\int_{\Omega} |f_1(x)|^p d\mu_1 < \infty \quad \text{and} \quad \int_{\Omega} |f_2(x)|^p d\mu_2 < \infty.$$

Then the space $L_{\mathbb{BC}}^p(\Omega, \mathcal{M}, \mu) = L_{\mathbb{BC}}^p(\mu)$ consist of equivalence classes all bicomplex valued, bicomplex measurable functions $f = e_1f_1 + e_2f_2$ on Ω such that $f_1 \in L^p(\Omega, \mu_1)$ and $f_2 \in L^p(\Omega, \mu_2)$ (Toksoy & Sağır, 2023).

Proposition 3. For $1 \leq p < \infty$, $L_{\mathbb{BC}}^p(\mu)$ is a $\mathbb{BC}$ -module under usual addition operation in functions and bicomplex scalar multiplication (Toksoy & Sağır, 2023).

Let $1 \leq p < \infty$. By using Definition 2 and Remark 2, we may write an idempotent decomposition

$$L_{\mathbb{BC}}^p(\mu) = e_1L^p(\mu_1) + e_2L^p(\mu_2)$$

for $L_{\mathbb{BC}}^p(\mu)$ where $L^p(\mu_1)$ and $L^p(\mu_2)$ are usual Lebesgue spaces, (Toksoy & Sağır, 2023). Therefore, a hyperbolic ($\mathbb{D}$ -valued) norm can be defined on the $\mathbb{BC}$ -module $L_{\mathbb{BC}}^p(\mu)$ with

$$\|f\|_{p,\mathbb{D}} = e_1 \|f_1\|_{p,\mu_1} + e_2 \|f_2\|_{p,\mu_2}$$

for any $e_1 f_1 + e_2 f_2 = f \in L_{\mathbb{BC}}^p(\mu)$.

Proposition 4. The space $(L_{\mathbb{BC}}^p(\mu), \|\cdot\|_{p,\mathbb{D}})$ is a bicomplex Banach module for $1 \leq p < \infty$ (Toksoy & Sağır, 2023).

In (Toksoy & Sağır, 2023), by using Definition 2.2 of (Değirmen & Sağır, 2023), a new functional

$$\begin{aligned} \|f\|_{p,k} &= \left(\int_{\Omega} |f(x)|_k^p d\mu \right)^{\frac{1}{p}} \\ &= \left(\int_{\Omega} |e_1 f_1(x) + e_2 f_2(x)|_k^p (e_1 d\mu_1 + e_2 d\mu_2) \right)^{\frac{1}{p}} \end{aligned} \quad (1.3)$$

is defined and showed that $\|f\|_{p,\mathbb{D}} = \|f\|_{p,k}$ for any $f \in L_{\mathbb{BC}}^p(\mu)$.

New Results on $\mathbb{BC}$ -Lebesgue spaces

Proposition 5. Let $1 \leq p < \infty$. The set

$$\mathbb{S} = \{s = s_1 e_1 + s_2 e_2 \mid s_1, s_2 \in S\}$$

is $\mathbb{D}$ – dense in $L_{\mathbb{BC}}^p(\Omega, \mathcal{M}, \mu)$ where S is the set of simple functions.

Proof. Let $\varepsilon = e_1 \varepsilon_1 + e_2 \varepsilon_2 > 0$ and $f = e_1 f_1 + e_2 f_2$ be any element of $L_{\mathbb{BC}}^p(\mu)$. By the definition of $L_{\mathbb{BC}}^p(\mu)$, the functions f_1 and f_2 belong to $L^p(\mu_1)$ and $L^p(\mu_2)$. Since the set of simple (step) functions S is dense subset of $L^p(\mu_1)$ and $L^p(\mu_2)$, then there exist simple functions h_1 and h_2 such that

$$\|f_1 - h_1\|_{p,\mu_1} < \varepsilon_1 \quad \text{and} \quad \|f_2 - h_2\|_{p,\mu_2} < \varepsilon_2.$$

If one call $e_1 h_1 + e_2 h_2$ as h , then $h \in \mathbb{S}$ and

$$\begin{aligned}\|f - h\|_{p,\mathbb{D}} &= e_1 \|f_1 - h_1\|_{p,\mu_1} + e_2 \|f_2 - h_2\|_{p,\mu_2} \\ &< e_1 \varepsilon_1 + e_2 \varepsilon_2 = \varepsilon.\end{aligned}$$

This means $\mathbb{S}$ is $\mathbb{D}$ -dense in $L^p_{\mathbb{BC}}(\mu)$.

Remark 3. If we define $C_c(\Omega, \mathbb{BC})$ as the set of all functions $f_1 e_1 + f_2 e_2$ where $f_1, f_2 \in C_c(\Omega)$, then $C_c(\Omega, \mathbb{BC})$ is $\mathbb{D}$ -dense in $L^p_{\mathbb{BC}}(\mu)$ by Lusin's theorem where $C_c(\Omega)$ is the set of all continuous complex functions on Ω whose support is compact.

The following theorem is Theorem 2.6 of (Toksoy & Sağır, 2023).

Theorem 1. (Hölder's inequality) Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p_{\mathbb{BC}}(\mu)$, $g \in L^q_{\mathbb{BC}}(\mu)$ with $f = f_1 e_1 + f_2 e_2$, $g = g_1 e_1 + g_2 e_2$. Then $fg \in L^1_{\mathbb{BC}}(\mu)$ and

$$\|fg\|_{1,\mathbb{D}} \leq \|f\|_{p,\mathbb{D}} \|g\|_{q,\mathbb{D}}.$$

Theorem 2. Let f be an element of $L^p_{\mathbb{BC}}(\Omega, \mathfrak{M}, \mu)$ for $1 < p < \infty$. Then

$$\|f\|_{p,\mathbb{D}} = \sup_{g \in L^q_{\mathbb{BC}}(\mu)} \left\{ \frac{\|fg\|_{1,\mathbb{D}}}{\|g\|_{q,\mathbb{D}}} : g \neq 0, \frac{1}{p} + \frac{1}{q} = 1 \right\}.$$

Proof. Using Hölder's inequality and (1.3), we can write that

$$\|fg\|_{1,\mathbb{D}} = \|fg\|_{1,k} = \int_{\Omega} |fg|_k d\mu \leq \|f\|_{p,\mathbb{D}} \|g\|_{q,\mathbb{D}}.$$

Then

$$\|fg\|_{1,\mathbb{D}} \|g\|^{-1}_{q,\mathbb{D}} \leq \|f\|_{p,\mathbb{D}}$$

for all $g \neq 0$ and this implies that

$$\sup_{g \in L^q_{\mathbb{BC}}(\mu)} \left\{ \frac{\|fg\|_{1,\mathbb{D}}}{\|g\|_{q,\mathbb{D}}} : g \neq 0, \frac{1}{p} + \frac{1}{q} = 1 \right\} \leq \|f\|_{p,\mathbb{D}}. \quad (1.4)$$

Now suppose that f is non-zero and $g = \beta |f|_k^{p-1}$ where β is a constant. In that case $|fg|_k = |\beta|_k |f|_k^p$ and $\|fg\|_{1,\mathbb{D}} = |\beta|_k \|f\|_{p,k}^p$. If one chooses $|\beta|_k$ as $\|f\|_{p,\mathbb{D}}^{1-p}$, then the equality

$$\|fg\|_{1,\mathbb{D}} = |\beta|_k \|f\|_{p,k}^p = \|f\|_{p,\mathbb{D}}^{1-p} \|f\|_{p,\mathbb{D}}^p = \|f\|_{p,\mathbb{D}} \quad (1.5)$$

is written. Since $|g|_k^q = |\beta|_k^q |f|_k^{(p-1)q}$ and $(p-1)q = q$, if we integrate the both sides of this equality, then

$$\begin{aligned} \|g\|_{q,\mathbb{D}} &= \|g\|_{q,k} = \left(\int_{\Omega} |g|_k^q d\mu \right)^{\frac{1}{q}} = \left(\int_{\Omega} |\beta|_k^q |f|_k^{(p-1)q} d\mu \right)^{\frac{1}{q}} \\ &= |\beta|_k \|f\|_{p,k}^{\frac{p}{q}} = \|f\|_{p,\mathbb{D}}^{1-p} \|f\|_{p,k}^{\frac{p}{q}} = 1 \end{aligned}$$

and $\|g\|_{q,\mathbb{D}}^{-1} = 1$ are obtained. Thus by using (1.5), we can write

$$\|f\|_{p,\mathbb{D}} = \|fg\|_{1,\mathbb{D}} \|g\|_{q,\mathbb{D}}^{-1} \leq \sup_{g \in L_{\mathbb{B}\mathbb{C}}^q(\mu)} \left\{ \|fg\|_{1,\mathbb{D}} \|g\|_{q,\mathbb{D}}^{-1} : g \neq 0, \frac{1}{p} + \frac{1}{q} = 1 \right\}. \quad (1.6)$$

Combining this (1.6) with (1.4), we get the result.

The following theorem, Minkowski inequality for $\mathbb{B}\mathbb{C}$ –Lebesgue space, is Theorem 2.8 of (Toksoy & Sağır, 2023).

Theorem 3. Let $f = f_1 e_1 + f_2 e_2$ and $g = g_1 e_1 + g_2 e_2$ be any two elements of $L_{\mathbb{B}\mathbb{C}}^p(\mu)$. Then

$$\|f + g\|_{p,\mathbb{D}} \leq \|f\|_{p,\mathbb{D}} + \|g\|_{p,\mathbb{D}}$$

for all $1 \leq p \leq \infty$.

To show the duality of $\mathbb{B}\mathbb{C}$ -Lebesgue spaces for $1 < p < \infty$, following similar arguments are adapted from the book (Castillo & Rafeiro, 2016).

Theorem 4. Each function $g \in L_{\mathbb{B}\mathbb{C}}^q(\mu)$ defines a linear functional F which is $\mathbb{D}$ -bounded in $L_{\mathbb{B}\mathbb{C}}^p(\mu)$ given by

$$F(f) = \int_{\Omega} f g d\mu$$

and $\|F\| = \|g\|_{q, \mathbb{D}}$.

Proof. Let $g = e_1 g_1 + e_2 g_2$ be a fixed function in $L^q_{\mathbb{BC}}(\mu)$. We will show that F given by

$$F(f) = \int_{\Omega} f g d\mu$$

is a linear functional in $L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$. Indeed, let α and β be bicomplex numbers and f, h be elements of $L^p_{\mathbb{BC}}(\Omega, \mathcal{M}, \mu)$ where $f = e_1 f_1 + e_2 f_2$ and $h = e_1 h_1 + e_2 h_2$. Then

$$\begin{aligned} F(\alpha f + \beta h) &= \int_{\Omega} (\alpha f + \beta h) g d\mu \\ &= \int_{\Omega} [(\alpha_1 f_1 + \beta_1 h_1)e_1 + (\alpha_2 f_2 + \beta_2 h_2)e_2](e_1 g_1 + e_2 g_2) d\mu \\ &= e_1 \int_{\Omega} (\alpha_1 f_1 + \beta_1 h_1) g_1 d\mu_1 + e_2 \int_{\Omega} (\alpha_2 f_2 + \beta_2 h_2) g_2 d\mu_2 \\ &= e_1 \int_{\Omega} \alpha_1 f_1 g_1 d\mu_1 + e_1 \int_{\Omega} \beta_1 h_1 g_1 d\mu_1 + e_2 \int_{\Omega} \alpha_2 f_2 g_2 d\mu_2 + e_2 \int_{\Omega} \beta_2 h_2 g_2 d\mu_2 \\ &= e_1 \int_{\Omega} \alpha_1 f_1 g_1 d\mu_1 + e_2 \int_{\Omega} \alpha_2 f_2 g_2 d\mu_2 + e_1 \int_{\Omega} \beta_1 h_1 g_1 d\mu_1 + e_2 \int_{\Omega} \beta_2 h_2 g_2 d\mu_2 \\ &= (e_1 \alpha_1 + e_2 \alpha_2) \int_{\Omega} (e_1 f_1 + e_2 f_2)(e_1 g_1 + e_2 g_2)(e_1 d\mu_1 + e_2 d\mu_2) \\ &\quad + (e_1 \beta_1 + e_2 \beta_2) \int_{\Omega} (e_1 f_1 + e_2 f_2)(e_1 g_1 + e_2 g_2)(e_1 d\mu_1 + e_2 d\mu_2) \\ &= \alpha \int_{\Omega} f g d\mu + \beta \int_{\Omega} h g d\mu \\ &= \alpha F(f) + \beta F(h). \end{aligned}$$

On the other hand

$$|F(f)|_k = \left| \int_{\Omega} f g d\mu \right|_k \leq \int_{\Omega} |f g|_k d\mu \leq \|f\|_{p, \mathbb{D}} \|g\|_{q, \mathbb{D}}$$

by Theorem 1. Then it can be written that

$$\frac{|F(f)|_k}{\|f\|_{p,\mathbb{D}}} \leq \|g\|_{q,\mathbb{D}}$$

which is meaning

$$\|F\| \leq \|g\|_{q,\mathbb{D}}. \quad (1.7)$$

This inequality and the previous ones show that F is a $\mathbb{D}$ –bounded linear operator. Furthermore, if we define a function f as

$$f = |g|_k^{q-2} \bar{g}_3 \quad (\dagger)$$

where $\bar{g}_3$ is the third conjugate of g , then

$$fg = |g|_k^{q-2} \bar{g}_3 g = |g|_k^{q-2} |g|_k^2 = |g|_k^q \quad (\dagger\dagger)$$

can be written by the known equality $W\bar{W}_3 = |W|_k^2$. Besides, we get

$$|f|_k = |g|_k^{q-2} |\bar{g}_3|_k = |g|_k^{q-1}$$

by $(\dagger)$ and

$$|f|_k^p = |g|_k^{p(q-1)} = |g|_k^q \quad (*)$$

can be obtained since $p(q-1) = q$. Then we have

$$F(f) = \int_{\Omega} fg d\mu = \int_{\Omega} |g|_k^q d\mu = \|g\|_{q,\mathbb{D}}^q, \quad (1.8)$$

by $(\dagger\dagger)$ and then

$$\int_{\Omega} |g|_k^q d\mu = \|g\|_{q,k}^q = \|g\|_{q,\mathbb{D}}^q = \|g\|_{q,\mathbb{D}}^{p(q-1)} = \frac{\|g\|_{q,\mathbb{D}}^{pq}}{\|g\|_{q,\mathbb{D}}^p},$$

where

$$\|g\|_{q,\mathbb{D}}^p \int_{\Omega} |g|_k^q d\mu = \|g\|_{q,\mathbb{D}}^p \|g\|_{q,k}^q = \|g\|_{q,\mathbb{D}}^{pq}.$$

(**)

Therefore, by using (*) and (**)

$$\|g\|_{q,\mathbb{D}}^p \int_{\Omega} |g|_k^q d\mu = \|g\|_{q,\mathbb{D}}^p \int_{\Omega} |f|_k^p d\mu = \|g\|_{q,\mathbb{D}}^{pq}$$

can be written. From here, we get

$$\|g\|_{q,\mathbb{D}}^q = \|f\|_{p,\mathbb{D}} \|g\|_{q,\mathbb{D}}.$$

As a result, by (1.8)

$$|F(f)|_k \geq F(f) = \|g\|_{q,\mathbb{D}} \|f\|_{p,\mathbb{D}},$$

and

$$\frac{|F(f)|_k}{\|f\|_{p,\mathbb{D}}} \geq \|g\|_{q,\mathbb{D}}$$

can be obtained. Therefore, there is a function $f = |g|_k^{q-2} \bar{g}_3$ satisfying

$$\|F\| \geq \|g\|_{q,\mathbb{D}}.$$

Consequently, the norm attains the supremum and $\|F\| = \|g\|_{q,\mathbb{D}}$ by (1.7).

Lemma 1. Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. Let $g \in L_{\mathbb{B}\mathbb{C}}^1(\Omega, \mathcal{M}, \mu)$ be such that for any $M > 0$ and for every simple function $s \in \mathbb{S} = \{s = s_1 e_1 + s_2 e_2 \mid s_1, s_2 \in \mathbb{S}\}$ the following inequality

$$\left| \int_{\Omega} s g d\mu \right|_k \leq M \|s\|_{p,\mathbb{D}}$$

holds for all $1 \leq p < \infty$. Then $g \in L_{\mathbb{B}\mathbb{C}}^q(\Omega, \mathcal{M}, \mu)$ with $\|g\|_{q,\mathbb{D}} \leq M$, where p and q are the conjugates.

Proof. The proof will be separated in two cases.

Case I. Let $p = 1$ and $B = \{x \in \Omega: M < |g(x)|_k\}$. It is easy to see that B is in $\mathcal{M}$. If we choose the function $s = e_1\chi_B + e_2\chi_B$, then by hypothesis we have

$$\begin{aligned} \left| \int_{\Omega} sg d\mu \right|_k &= \left| \int_{\Omega} (e_1\chi_B + e_2\chi_B)g d\mu \right|_k = \left| \int_B g d\mu \right|_k \\ &\leq \int_B |g|_k d\mu \leq M \|\chi_B\|_{1,\mathbb{D}} \end{aligned}$$

namely

$$\begin{aligned} \left| \int_B g d\mu \right|_k &\leq \int_B |g|_k d\mu \leq M \|\chi_B\|_{1,\mathbb{D}} = M(e_1\mu_1(B) + e_2\mu_2(B)) = \\ &\int_B M d\mu. \end{aligned}$$

Then one can see that

$$\int_B (|g|_k - M) d\mu \leq 0.$$

Since $|g|_k > M$, we can conclude that $\mu(B) = 0$ which means that $|g(x)|_k \leq M$ (μ -a.e.) and so $\|g\|_{\infty,\mathbb{D}} \leq M$. As a result, the lemma is proved for Case I.

Case II. Let $1 < p < \infty$. Since $|g|_k^q > 0$, there exists a sequence of nonnegative simple functions $\{s_n\}_{n \in \mathbb{N}}$ such that $s_n \xrightarrow{\mathbb{D}} |g|_k^q$ pointwise by Proposition 5. Let $t_n = s_n^{\frac{1}{p}} \cdot \text{sgn}_{\mathbb{D}}(g)$ be a sequence derived from $\{s_n\}_{n \in \mathbb{N}}$ where $\text{sgn}_{\mathbb{D}}(g) = \bar{g}_3/|g_3|_k$ and $\bar{g}_3$ is the third conjugate of g . Note that each t_n is a simple function and

$$\begin{aligned} \|t_n\|_{p,\mathbb{D}} &= \left(\int_{\Omega} |t_n|_k^p d\mu \right)^{\frac{1}{p}} = \left(\int_{\Omega} \left| s_n^{\frac{1}{p}} \cdot \text{sgn}_{\mathbb{D}}(g) \right|_k^p d\mu \right)^{\frac{1}{p}} = \left(\int_{\Omega} |s_n|_k d\mu \right)^{\frac{1}{p}} \\ &= \|s_n\|_{1,\mathbb{D}}^{\frac{1}{p}}. \end{aligned}$$

Since

$$gt_n = gs_n^{\frac{1}{p}} \cdot \text{sgn}_{\mathbb{D}}(g) = s_n^{\frac{1}{p}} \cdot |g|_k \geq s_n^{\frac{1}{p}} s_n^{\frac{1}{q}} = s_n,$$

it can be written by the hypothesis that

$$0 \leq \int_{\Omega} s_n d\mu \leq \int_{\Omega} g \cdot t_n d\mu \leq M \|t_n\|_{p, \mathbb{D}}.$$

Therefore, we get

$$\int_{\Omega} s_n d\mu \leq M^q.$$

By using the bicomplex monotone convergence theorem [Theorem 3.7] in (Ghosh & Mondal, 2022), we can conclude that

$$\int_{\Omega} |g|_k^q d\mu \leq M^q,$$

where $g \in L_{\mathbb{BC}}^q(\Omega, \mathcal{M}, \mu)$ and $\|g\|_{q, \mathbb{D}} \leq M$.

Theorem 5. (*Riesz Representation Theorem for $\mathbb{BC}$ -Lebesgue spaces*). Let $(\Omega, \mathcal{M}, \mu)$ be a σ -finite measure space and $1 \leq p < \infty$. If T is a linear functional in $L_{\mathbb{BC}}^p(\mu)$, then there exists a unique function g in $L_{\mathbb{BC}}^q(\mu)$ such that

$$T(f) = \int_{\Omega} f g d\mu \quad (1.9)$$

for all elements of $L_{\mathbb{BC}}^p(\mu)$ and

$$\|T\| = \|g\|_{q, \mathbb{D}} \quad (1.10)$$

where p, q are the conjugates.

Proof. At first, the uniqueness of g will be shown. For this, suppose that there exists functions $g_1, g_2 \in L_{\mathbb{BC}}^q(\mu)$ such that satisfy (1.9), namely

$$\int_E g_1 d\mu = \int_E g_2 d\mu$$

for all $E \in \mathcal{M}$ with $\mu(E) < \infty_{\mathbb{D}}$ where

$$g_1(x) = e_1 g_1^{(1)}(x) + e_2 g_1^{(2)}(x) \quad \text{and} \quad g_2(x) = e_1 g_2^{(1)}(x) + e_2 g_2^{(2)}(x).$$

Since $(\Omega, \mathcal{M}, \mu)$ is a σ -finite measure space, we can find a sequence of disjoint sets $\{\Omega_n\}_{n \in \mathbb{N}}$ in $\mathcal{M}$ such that $\mu(\Omega_n) = e_1\mu_1(\Omega_n) + e_2\mu_2(\Omega_n) < \infty_{\mathbb{D}}$ for all n and

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

Now, let $A := \{x \in \Omega : g_1(x) > g_2(x)\}$ and $B := \{x \in \Omega : g_1(x) < g_2(x)\}$. Then

$$\int_{\Omega_n \cap A} g_1 d\mu = \int_{\Omega_n \cap A} g_2 d\mu$$

and so

$$\begin{aligned} 0 &= \int_{\Omega_n \cap A} (g_1 - g_2) d\mu \\ &= e_1 \int_{\Omega_n \cap A} (g_1^{(1)} - g_2^{(1)}) d\mu_1 \\ &\quad + e_2 \int_{\Omega_n \cap A} (g_1^{(2)} - g_2^{(2)}) d\mu_2. \end{aligned}$$

Since $g_1(x) > g_2(x)$ i.e. $g_1^{(1)}(x) > g_2^{(1)}(x)$ and $g_1^{(2)}(x) > g_2^{(2)}(x)$ for all $x \in \Omega_n \cap A$, we can find that $\mu_1(\Omega_n \cap A) = 0$ and $\mu_2(\Omega_n \cap A) = 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \mu(A) &= e_1\mu_1(A) + e_2\mu_2(A) \\ &= e_1 \sum_{n=1}^{\infty} \mu_1(A \cap \Omega_n) + e_2 \sum_{n=1}^{\infty} \mu_2(A \cap \Omega_n) = 0. \end{aligned}$$

Similarly $\mu(B) = 0$ and $g_1^{(1)} = g_2^{(1)}$ (μ_1 -a.e.), $g_1^{(2)} = g_2^{(2)}$ (μ_2 -a.e.). Therefore $g_1 = g_2$ (μ -a.e.) and this proves the uniqueness.

Now we'll prove the existence of g by cases.

Case 1. Let $\mu(\Omega) < \infty_{\mathbb{D}}$ and define

$$e_1 v_1(E) + e_2 v_2(E) = v(E) = T(\chi_E) = e_1 T_1(\chi_E) + e_2 T_2(\chi_E)$$

for each $E \in \mathcal{M}$. Since $\mu(\Omega) < \infty_{\mathbb{D}}$, we get $\mu_1(E), \mu_2(E) < \infty$. Thus

$$\begin{aligned}\|\chi_E\|_{p,\mathbb{D}} &= e_1\|\chi_E\|_{p,\mu_1} + e_2\|\chi_E\|_{p,\mu_2} \\ &= e_1\mu_1(E)^{\frac{1}{p}} + e_2\mu_2(E)^{\frac{1}{p}} = \mu(E)^{\frac{1}{p}}\end{aligned}$$

by Definition 2.2 in (Değirmen & Sağır, 2023). This means $\chi_E \in L^p_{\mathbb{B}\mathbb{C}}(\mu)$. Now, it will be shown that $\nu = e_1\nu_1 + e_2\nu_2$ is a $\mathbb{D}$ -signed measure on $\mathcal{M}$. It is easy to see that $\chi_{\emptyset}$ is the zero function in $L^p_{\mathbb{B}\mathbb{C}}(\mu)$ and so $\nu(\emptyset) = T(\chi_{\emptyset})$. Since T is a bicomplex function, $\nu(\cdot)$ is also a bicomplex function. Likewise, let $\{E_n\}_{n \in \mathbb{N}}$ be a collection of disjoint sets in $\mathcal{M}$ and define

$$E = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad A_n = \bigcup_{i=1}^n E_i.$$

Then the sequence $\{A_n\}_{n \in \mathbb{N}}$ is increasing and

$$\bigcup_{n=1}^{\infty} A_n = E.$$

Therefore, one can write

$$\chi_{A_n} = \sum_{k=1}^n \chi_{E_k}$$

by induction and

$$\begin{aligned}T(\chi_{A_n}) &= e_1 T_1(\chi_{A_n}) + e_2 T_2(\chi_{A_n}) = e_1 \sum_{k=1}^n T_1(\chi_{E_k}) + e_2 \sum_{k=1}^n T_2(\chi_{E_k}) \\ &= e_1 \sum_{k=1}^n \nu_1(E_k) + e_2 \sum_{k=1}^n \nu_2(E_k) = \sum_{k=1}^n \nu(E_k)\end{aligned}$$

by the linearity of T . Since

$$\begin{aligned}
\|\chi_{A_n} - \chi_E\|_{p, \mathbb{D}} &= e_1 \|\chi_{A_n} - \chi_E\|_{p, \mu_1} + e_2 \|\chi_{A_n} - \chi_E\|_{p, \mu_2} \\
&= e_1 \left(\int_{\Omega} |\chi_{A_n} - \chi_E|^p d\mu_1 \right)^{\frac{1}{p}} + e_2 \left(\int_{\Omega} |\chi_{A_n} - \chi_E|^p d\mu_2 \right)^{\frac{1}{p}} \\
&= e_1 \mu_1(E \setminus A_n)^{\frac{1}{p}} + e_2 \mu_2(E \setminus A_n)^{\frac{1}{p}} = (\mu(E) - \mu(A_n))^{\frac{1}{p}}
\end{aligned}$$

and $\{A_n\}_{n \in \mathbb{N}}$ is an increasing sequence, we have $\mu(E) = \lim_{n \rightarrow \infty} \mu(A_n)$ namely

$$\lim_{n \rightarrow \infty} \|\chi_{A_n} - \chi_E\|_{p, \mathbb{D}} = 0.$$

Using the $\mathbb{D}$ -continuity of T in $L_{\mathbb{B}\mathbb{C}}^p(\mu)$, it follows that

$$\lim_{n \rightarrow \infty} T(\chi_{A_n}) = T(\chi_E)$$

and

$$\begin{aligned}
v(E) &= e_1 v_1(E) + e_2 v_2(E) = e_1 T_1(\chi_E) + e_2 T_2(\chi_E) \\
&= e_1 \lim_{n \rightarrow \infty} T_1(\chi_{A_n}) + e_2 \lim_{n \rightarrow \infty} T_2(\chi_{A_n}) = \lim_{n \rightarrow \infty} T(\chi_{A_n}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n v(E_k).
\end{aligned}$$

This last equality says that v is a $\mathbb{D}$ -signed measure. Now, $\mathbb{D}$ -absolute continuity of v with respect to μ will be proved ($v \ll_{\mathbb{D}} \mu$) by using Theorem 3.12 in (Ghosh & Mondal, 2022). Suppose that $E \in \mathcal{M}$ with $\mu(E) = 0$. Then

$$\begin{aligned}
\|\chi_E\|_{p, \mathbb{D}} &= e_1 \|\chi_E\|_{p, \mu_1} + e_2 \|\chi_E\|_{p, \mu_2} \\
&= e_1 \mu_1(E)^{\frac{1}{p}} + e_2 \mu_2(E)^{\frac{1}{p}} = \mu(E)^{\frac{1}{p}}.
\end{aligned}$$

This says that χ_E is equal to the zero function in $L_{\mathbb{B}\mathbb{C}}^p(\Omega, \mathcal{M}, \mu)$ and $T(\chi_E) = 0$, namely $v(E) = 0$ and so $v \ll_{\mathbb{D}} \mu$. By Theorem 3.16 of (Ghosh & Mondal, 2022), the bicomplex version of Lebesgue-Radon-Nikodym theorem for measures (signed) finite, there is a bicomplex measurable function g such that

$$v(E) = \int_E g d\mu$$

for all $E \in \mathcal{M}$. Therefore

$$\begin{aligned} \int_{\Omega} g d\mu &= \int_{\Omega} (e_1 g_1 + e_2 g_2)(e_1 d\mu_1 + e_2 d\mu_2) \\ &= e_1 \int_{\Omega} g_1 d\mu_1 + e_2 \int_{\Omega} g_2 d\mu_2 = e_1 v_1(\Omega) + e_2 v_2(\Omega) \\ &= v(\Omega) = T(\chi_{\Omega}) = T(1) < \infty_{\mathbb{D}}, \end{aligned}$$

and $g \in L^1_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$. Let us check whether g meets the hypotheses of Lemma 1. Let $s \in L^p_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$ be a simple $\mathcal{M}$ -measurable function with the following canonical representation

$$s = e_1 s_1 + e_2 s_2 = e_1 \sum_{k=1}^n \alpha_k^{(1)} \chi_{E_k} + e_2 \sum_{k=1}^n \alpha_k^{(2)} \chi_{E_k}.$$

Then, by using the definition of T , we get

$$\begin{aligned} T(s) &= e_1 T_1(s_1) + e_2 T_2(s_2) = e_1 T_1\left(\sum_{k=1}^n \alpha_k^{(1)} \chi_{E_k}\right) + e_2 T_2\left(\sum_{k=1}^n \alpha_k^{(2)} \chi_{E_k}\right) \\ &= e_1 \sum_{k=1}^n \alpha_k^{(1)} v_1(E_k) + e_2 \sum_{k=1}^n \alpha_k^{(2)} v_2(E_k) = e_1 \sum_{k=1}^n \alpha_k^{(1)} \int_{E_k} g_1 d\mu_1 + e_2 \sum_{k=1}^n \alpha_k^{(2)} \int_{E_k} g_2 d\mu_2 \\ &= e_1 \int_{\Omega} g_1 \left(\sum_{k=1}^n \alpha_k^{(1)} \chi_{E_k}\right) d\mu_1 + e_2 \int_{\Omega} g_2 \left(\sum_{k=1}^n \alpha_k^{(2)} \chi_{E_k}\right) d\mu_2 \\ &= e_1 \int_{\Omega} g_1 s_1 d\mu_1 + e_2 \int_{\Omega} g_2 s_2 d\mu_2 = \int_{\Omega} g s d\mu. \end{aligned}$$

Therefore, we have

$$T(s) = \int_{\Omega} g s d\mu$$

for all (step) simple function $s \in L^p_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$. As a result of this, $|T(s)|_k = \left| \int_{\Omega} g s d\mu \right|_k$.

If $M = \|T\|$, then M is $\mathbb{D}$ -finite and it demonstrates that g meets the criteria of Lemma 1. Therefore we can conclude that $g \in L_{\mathbb{BC}}^q(\Omega, \mathcal{M}, \mu)$ and

$$\|g\|_{q, \mathbb{D}} \leq \|T\| = M. \quad (1.11)$$

Now, we shall demonstrate that $T(f) = \int_{\Omega} g f d\mu$ for any $f \in L_{\mathbb{BC}}^p(\Omega, \mathcal{M}, \mu)$. Let $f \in L_{\mathbb{BC}}^p(\Omega, \mathcal{M}, \mu)$ and $\varepsilon = e_1 \varepsilon_1 + e_2 \varepsilon_2 > 0$. Since the set, obtained from simple functions, $\mathbb{S} = \{s = s_1 e_1 + s_2 e_2 \mid s_1, s_2 \in S\}$ is $\mathbb{D}$ -dense in $L_{\mathbb{BC}}^p(\Omega, \mathcal{M}, \mu)$, one can find a simple function $s = e_1 s_1 + e_2 s_2 \in \mathbb{S} \subset L_{\mathbb{BC}}^p(\Omega, \mathcal{M}, \mu)$ such that

$$\|f - s\|_{p, \mathbb{D}} < \frac{\varepsilon}{\|g\|_{q, \mathbb{D}} + \|T\| + 1}.$$

Then

$$\begin{aligned} |T(f) - \int_{\Omega} g f d\mu|_k &= |T(f) - T(s) + T(s) - \int_{\Omega} g f d\mu|_k \leq |T(f) - T(s)|_k + |\int_{\Omega} s g d\mu - \int_{\Omega} g f d\mu|_k \\ &= |T(f - s)|_k + |\int_{\Omega} s g d\mu - \int_{\Omega} g f d\mu|_k \leq |T(f - s)|_k + \int_{\Omega} |g|_k |s - f|_k d\mu \\ &\leq \|T\| \|f - s\|_{p, \mathbb{D}} + \|g\|_{q, \mathbb{D}} \|f - s\|_{p, \mathbb{D}} = \|f - s\|_{p, \mathbb{D}} (\|T\| + \|g\|_{q, \mathbb{D}}) \\ &\leq \|f - s\|_{p, \mathbb{D}} (\|T\| + \|g\|_{q, \mathbb{D}} + 1) < \varepsilon. \end{aligned}$$

Since ε is arbitrary, we can conclude that

$$T(f) = \int_{\Omega} g f d\mu$$

for all $f \in L_{\mathbb{BC}}^p(\Omega, \mathcal{M}, \mu)$. Finally by using the Hölder's inequality (Theorem 1), we can write $|T(f)|_k \leq \|f\|_{p, \mathbb{D}} \|g\|_{q, \mathbb{D}}$ where

$$\|T\| \leq \|g\|_{q, \mathbb{D}}. \quad (1.12)$$

Now from (1.11) and (1.12) we have $\|T\| = \|g\|_{q, \mathbb{D}}$, which is done for the Case 1.

Case 2. $\mu(\Omega) = \infty$. Under the σ -finiteness of μ , there exists a collection of measurable sets $\{\Omega_n\}_{n \in \mathbb{N}}$ such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, $\Omega_n \subset \Omega_{n+1}$ and $\mu(\Omega_n) < \infty_{\mathbb{D}}$ for all $n \in \mathbb{N}$. Therefore, Case 1 may be applied to the measure space $(\Omega_n, \mathcal{M} \cap \Omega_n, \mu_n)$ where μ_n is the restriction of μ to $\mathcal{M} \cap$

Ω_n . Now let $T_n = T|_{L^p_{\mathbb{B}\mathbb{C}}(\mu_n)}$. Then, there exists $g_n \in L^q_{\mathbb{B}\mathbb{C}}(\mu_n)$ for all $n \in \mathbb{N}$ such that

$$T_n(h) = \int_{\Omega_n} h g_n d\mu_n \quad (1.13)$$

for all $h \in L^p_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$ which vanishes outside Ω_n by Case 1. Also

$$\|T\| \geq \|T_n\| = \|g_n\|_{q, \mathbb{D}}. \quad (1.14)$$

If we define

$$\tilde{g}_n(x) = \begin{cases} g_n(x), & x \in \Omega_n \\ 0, & x \notin \Omega_n \end{cases}$$

then, the integral in (1.13) can be written as

$$T_n(h) = \int_{\Omega} h \tilde{g}_n d\mu_n \quad (1.15)$$

for all $h \in L^p_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$ which vanishes outside Ω_n . Since $\tilde{g}_{n+1}$ restricted to Ω_n have the same properties as $\tilde{g}_n$ under the uniqueness, we have $\tilde{g}_{n+1} = \tilde{g}_n$ in Ω_n . Now define $g(x) = g_n(x)$ if $x \in \Omega_n$. Since $|\tilde{g}_n(x)|_k \leq |\tilde{g}_{n+1}(x)|_k$ for all $x \in \Omega$ and

$$\lim_{n \rightarrow \infty} \tilde{g}_n(x) = g(x),$$

we can say that

$$\int_{\Omega} |g|_k^q d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} |\tilde{g}_n|_k^q d\mu \leq \|T\|^q$$

by the bicomplex monotone convergence theorem in (Ghosh & Mondal, 2022). This implies that $g \in L^q_{\mathbb{B}\mathbb{C}}(\Omega, \mathfrak{M}, \mu)$ and

$$\|T\| \geq \|g\|_{q, \mathbb{D}}. \quad (1.16)$$

Let f be any element of $L^p_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$ and $f_n = f \cdot \chi_{\Omega_n}$. Then f_n vanishes outside Ω_n and the pointwise convergence $f_n \xrightarrow{\mathbb{D}} f$ exists in Ω . It is easy to see that $|f_n - f|_k \leq |f|_k$ and so $|f_n - f|_k^p \leq |f|_k^p$. By the bicomplex Dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f|_k^p d\mu = 0.$$

The $\mathbb{D}$ – continuity of T implies that $T(f_n) \xrightarrow{\mathbb{D}} T(f)$ when $n \rightarrow \infty$. Moreover, we have $|f_n \cdot g|_k = |f_n|_k |g|_k \leq |f|_k |g|_k = |f \cdot g|_k$, $f \cdot g \in L^1_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$ and $\lim_{n \rightarrow \infty} f_n g = f g$. If we apply the bicomplex Dominated convergence theorem once more, we get

$$\begin{aligned} \int_{\Omega} f g d\mu &= \lim_{n \rightarrow \infty} \int_{\Omega} f_n g d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f \chi_{\Omega_n} g d\mu \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} (f \chi_{\Omega_n})(g \chi_{\Omega_n}) d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} (f_n) \tilde{g}_n d\mu \\ &= \lim_{n \rightarrow \infty} T(f_n) = T(f). \end{aligned}$$

Thus, we get (1.9) and applying the Hölder's inequality once more,

$$|T(f)|_k \leq \|f\|_{p, \mathbb{D}} \|g\|_{q, \mathbb{D}}$$

can be written. This means $\|T\| \leq \|g\|_{q, \mathbb{D}}$, and so $\|T\| = \|g\|_{q, \mathbb{D}}$ by (1.16), which ends the proof.

Corollary 1. The space $L^p_{\mathbb{B}\mathbb{C}}(\Omega, \mathcal{M}, \mu)$ with $1 < p < \infty$ is reflexive.

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CHAPTER IV

On Geometric Structure of Bicomplex Weighted $\mathbb{BC}$ –Modules $l_p^k(\mathbb{BC})$

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Introduction

In recent years, a novel number system known as bicomplex numbers has emerged as an expansion of the existing system of complex numbers. The book written by Price (Price, 1991) is the foremost recommended beginning work for bicomplex numbers. Next, we recommend books (Alpay & et al., 2014) and (Luna-Elizarraras & et al., 2015) that provide detailed explanations of the

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fundamental frameworks of bicomplex numbers, which concentrate on functional analysis.

The following fundamental details concerning bicomplex numbers are derived from books (Alpay & et al., 2014; Luna-Elizarraras & al., 2015; Price, 1991). Let us begin by examining two separate imaginary numbers, i and j , where $i^2 = j^2 = -1$. This results in the creation of two sets, $\mathbb{C}(i)$ and $\mathbb{C}(j)$, which are exactly the same as set $\mathbb{C}$. Additionally, let us take an imaginary number, denoted as k , which is the result of multiplying these two imaginary numbers and possesses the following characteristics:

$$ij = ji = k$$

$$ik = ki = -j$$

$$jk = kj = -i$$

$$k^2 = 1.$$

Consequently, the set of bicomplex numbers denoted by $\mathbb{BC}$ is given as

$$\mathbb{BC} = \{z + jw | z, w \in \mathbb{C}(i)\}.$$

The set $\mathbb{BC}$ can also be written as

$$\mathbb{BC} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}.$$

Addition and multiplication operations are defined in the following manners:

$$(z_1 + jw_1) + (z_2 + jw_2) = (z_1 + z_2) + j(w_1 + w_2)$$

and

$$(z_1 + jw_1)(z_2 + jw_2) = (z_1w_1 - w_1w_2) + j(z_1w_2 + w_1z_2).$$

By undertaking these operations, the set $\mathbb{BC}$ improves into a commutative ring and, as a result, becomes a module on itself. A valuable subset of $\mathbb{BC}$ is denoted by $\mathbb{D}$, which is called the hyperbolic numbers set, and is properly described as

$$\mathbb{D} = \{x + ky | x, y \in \mathbb{R}\}$$

by replacing z and w with x and iy in the description of $\mathbb{BC}$, whenever $x, y \in \mathbb{R}$. Now, let us examine two complex numbers,

$$e_1 = \frac{1+ij}{2} \text{ and } e_2 = \frac{1-ij}{2}.$$

These particular numbers possess the following properties, which are readily apparent:

$$e_1 e_2 = 0, e_1 + e_2 = 1, e_1 e_1 = e_1, e_2 e_2 = e_2.$$

Both the numbers e_1 and e_2 belong to the set of hyperbolic numbers $\mathbb{D}$. They constitute the idempotent basis of bicomplex numbers. A bicomplex number $z + jw$ can be expressed in a unique form:

$$z + jw = e_1 z_1 + e_2 z_2,$$

where $z_1 = z - iw$, $z_2 = z + iw \in \mathbb{C}(i)$. The given mathematical expression indicates the idempotent representation of bicomplex numbers. Therefore, it ensures that every hyperbolic number possesses an idempotent representation, that can be expressed as:

$$x + ky = e_1 a_1 + e_2 a_2,$$

where $a_1 = x + y$, $a_2 = x - y \in \mathbb{R}$. The non-negative hyperbolic numbers are represented as $\mathbb{D}^+$ and given by

$$\mathbb{D}^+ = \{e_1 a_1 + e_2 a_2 | a_1, a_2 \geq 0\}.$$

Consider $\kappa, \mu \in \mathbb{D}$. The hyperbolic numbers are equipped with a partial order relation denoted by $\lesssim$, which is given as:

$$\kappa \lesssim \mu \text{ if and only if } \mu - \kappa \in \mathbb{D}^+.$$

Moreover, we write

$$\kappa < \mu \text{ if and only if } \kappa \lesssim \mu \text{ but } \kappa \neq \mu.$$

To find out more concerning the features of the partial order relation $\lesssim$, please consult Section 2.6.2 of (Luna-Elizarraras & et al., 2015). Let us now provide a brief overview of $\mathbb{D}$ -boundedness as presented in Section 2.6.3 of (Luna-Elizarraras & et al., 2015). Let $X \subset \mathbb{D}$. Assuming X possesses a $\mathbb{D}$ -upper bound (or a $\mathbb{D}$ -lower bound) l , it

implies that for any $x \in X$, x can be compared to l and $x \lesssim l$ (or $l \lesssim x$). The set X can be described as $\mathbb{D}$ -bounded if it is $\mathbb{D}$ -bounded both from above and from below. If X is a set that has a $\mathbb{D}$ -upper bound, we determine the concept of its $\mathbb{D}$ -supremum, represented as

$$\sup_{\mathbb{D}} X,$$

as the least $\mathbb{D}$ -upper bound for X . Similarly, we establish its $\mathbb{D}$ -infimum

$$\inf_{\mathbb{D}} X,$$

as the greatest $\mathbb{D}$ -lower bound for X . In this context, the least $\mathbb{D}$ -upper bound denotes

$$\sup_{\mathbb{D}} X \lesssim l$$

for any $\mathbb{D}$ -upper bound l , even though not all $\mathbb{D}$ -upper bounds are comparable to one another. Likewise, the greatest $\mathbb{D}$ -lower bound signifies that

$$l \lesssim \inf_{\mathbb{D}} X$$

for any $\mathbb{D}$ -lower bound l , even though not all $\mathbb{D}$ -lower bounds are comparable to one another. Consider the sets

$$X_1 = \{x_1 : x_1 e_1 + x_2 e_2 \in X\}$$

and

$$X_2 = \{x_2 : x_1 e_1 + x_2 e_2 \in X\}.$$

If X is $\mathbb{D}$ -bounded from above, then

$$\sup_{\mathbb{D}} X = \sup_{\mathbb{D}} X_1 e_1 + \sup_{\mathbb{D}} X_2 e_2.$$

If X is $\mathbb{D}$ -bounded from below, then

$$\inf_{\mathbb{D}} X = \inf_{\mathbb{D}} X_1 e_1 + \inf_{\mathbb{D}} X_2 e_2.$$

Additionally, the following items satisfy:

(i) If both X and Y have a $\mathbb{D}$ -lower bound, then $X + Y$ also has a $\mathbb{D}$ -lower bound and

$$\inf_{\mathbb{D}}(X + Y) = \inf_{\mathbb{D}}X + \inf_{\mathbb{D}}Y.$$

(ii) If both X and Y have a $\mathbb{D}$ -upper bound, then $X + Y$ also has a $\mathbb{D}$ -upper bound and

$$\sup_{\mathbb{D}}(X + Y) = \sup_{\mathbb{D}}X + \sup_{\mathbb{D}}Y.$$

For a bicomplex number $z + jw$, $|z + jw|_k$ represents the hyperbolic modulus and is indicated by

$$|z + jw|_k^2 = (z + jw)(\bar{z} - j\bar{w}).$$

By representing the complex number $z + jw$ in idempotent form, we obtain

$$|z + jw|_k = |e_1z_1 + e_2z_2|_k = e_1|z_1| + e_2|z_2|. \quad (1)$$

Modules are defined as algebraic structures that are constructed over rings, similar to how vector spaces are constructed over fields. A module X is referred to as a $\mathbb{BC}$ -module if it is defined over the ring $\mathbb{BC}$ (see (Gervais Lavoie & et al., 2011; Rochon & Tremblay, 2006)). Consider E be a $\mathbb{BC}$ -module. Let us take $u, v \in E$ and $\lambda \in \mathbb{BC}$. A function $\|\cdot\|_{\mathbb{D}}: E \rightarrow \mathbb{D}^+$ is considered to be a hyperbolic ($\mathbb{D}$ -valued) norm on E if it satisfies the conditions listed below:

(i) $\|u\|_{\mathbb{D}} = 0$ if and only if $u = 0$.

(ii) $\|\lambda u\|_{\mathbb{D}} = |\lambda|_k \|u\|_{\mathbb{D}}$.

(iii) $\|u + v\|_{\mathbb{D}} \lesssim \|u\|_{\mathbb{D}} + \|v\|_{\mathbb{D}}$.

The hyperbolic modulus $|\cdot|_k$ with representation (1) refers to the hyperbolic ($\mathbb{D}$ -valued) norm of $z + jw \in \mathbb{BC}$. Let $v, \eta \in \mathbb{BC}$ and $\gamma \in \mathbb{D}^+$. The following characteristics are thereby associated with the hyperbolic ($\mathbb{D}$ -valued) norm $|\cdot|_k$:

(i) $|v|_k = 0$ if and only if $v = 0$.

(ii) $|v\eta|_k = |v|_k |\eta|_k$.

$$(iii) |v + \eta|_k \lesssim |v|_k + |\eta|_k.$$

$$(iv) |\gamma|_k = \gamma$$

$$(v) |\gamma v|_k = \gamma |v|_k$$

Consider X as a $\mathbb{BC}$ -module. Therefore, we possess

$$X = e_1 X_1 + e_2 X_2$$

such that $X_1 = e_1 X$ and $X_2 = e_2 X$ are complex vector spaces and also $\mathbb{BC}$ -modules. The equation presented here is commonly referred to as the idempotent decomposition of X . Consequently, every x belonging to the set X can be expressed uniquely as $x = e_1 x_1 + e_2 x_2$, where $x_1 \in X_1$ and $x_2 \in X_2$ (see (Gervais Lavoie et al., 2010)). Let X_1 and X_2 be normed spaces equipped with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. The $\mathbb{BC}$ -module can be equipped with the hyperbolic ($\mathbb{D}$ -valued) norm determined by

$$\|x\|_{\mathbb{D}} = e_1 \|x_1\|_1 + e_2 \|x_2\|_2.$$

Consider T is a $\mathbb{BC}$ -module with the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$. A sequence (t_n) belonging to T converges to $t_0 \in T$ concerning the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$ if for every $0 < \epsilon$ there exists $n_0 \in \mathbb{N}$ such that $\|t_n - t_0\|_{\mathbb{D}} < \epsilon$ for all $n \geq n_0$. In this study, the term used for describing this convergence is $\mathbb{D}$ -convergence (or $\mathbb{D}$ -converges). Also, the term $\mathbb{D}$ -divergent will be used for $\mathbb{D}$ -non-convergent sequences.

Consider T is a $\mathbb{BC}$ -module with hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$. A sequence (t_n) belonging to T is Cauchy sequence concerning the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$ if for every $0 < \epsilon$ there exists $n_0 \in \mathbb{N}$ such that $\|t_n - t_m\|_{\mathbb{D}} < \epsilon$ for all $n, m \geq n_0$. If every Cauchy sequence in X with respect to the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$ $\mathbb{D}$ -converges to $t_0 \in X$, then we say that the $\mathbb{BC}$ -module X is complete with respect to the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$.

A bicomplex Banach module is defined as a complete real-valued normed or hyperbolic ($\mathbb{D}$ -valued) normed $\mathbb{BC}$ -module.

(Kumar & et al., 2011; Gervais Lavoie & et al. 2010; Gervais Lavoie & et al., 2011; Kumar & et al., 2016).

The Lebesgue sequence spaces are a family of sequence spaces that are important in functional analysis and measure theory. They are named after the French mathematician Henri Lebesgue, who made significant contributions to both fields. These spaces are defined as the collection of all sequences of complex numbers that satisfy certain convergence properties. The weighted Lebesgue sequence spaces are a class of function spaces that generalize the classical Lebesgue sequence spaces. These spaces have been extensively studied in functional analysis and have applications in various areas of mathematics and engineering. In a recent study, (Güngör, 2020) examined the geometric characteristics of non-Newtonian Lebesgue sequence spaces. Also, (Oğur, 2019), (Sağır & Alşalvar, 2019) pertain to the examination of the geometric characteristics of the weighted Lebesgue function and sequence space, respectively. Furthermore, there have been new investigations into bicomplex function and sequence Lebesgue spaces, which can be seen as extensions of Lebesgue function and sequence spaces. There is a significant amount of research in the literature that examines the geometric characteristics of bicomplex Lebesgue function spaces and Lebesgue sequence spaces. (Değirmen & Sağır, 2023) and (Toksoy & Sağır, 2023) are among the notable examples. Besides, (Değirmen & Sağır, 2021) examined the D –topological duals of bicomplex Lebesgue sequence spaces with hyperbolic ($\mathbb{D}$ -valued) norm.

Recent work (Sağır & Güngör, 2023) introduces the weighted bicomplex sequence spaces $l_{p,\alpha}^k(\mathbb{BC})$ as a generalization of the bicomplex Lebesgue sequence spaces $l_p^k(\mathbb{BC})$ with hyperbolic ($\mathbb{D}$ -valued) norm, where α is a bicomplex weighted sequence. Initially, this study will address the findings of earlier studies, mostly focusing on the results of studies (Sağır & Güngör, 2023) and (Değirmen & Sağır, 2023), which will serve as the foundation for the main portion of this study.

To begin, let us summarize some valuable observations gained from the findings of the study (Değirmen & Sağır, 2023).

Definition 1.1. Consider $w, \lambda \in \mathbb{BC}$ and $w \neq 0$. The map given by $w^\lambda = e^{\lambda \text{Ln} w}$ is referred to as the principal value of the bicomplex power w^λ . Observe that for $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ and $w = w_1 e_1 + w_2 e_2$

$$w^\lambda = e^{\lambda \text{Ln} w} = w_1^{\lambda_1} e_1 + w_2^{\lambda_2} e_2$$

Particularly if $\lambda \in \mathbb{R}$, then we have

$$w^\lambda = w_1^\lambda e_1 + w_2^\lambda e_2.$$

Lemma 1.2. Consider $p \in \mathbb{R}$ with $2 \leq p < \infty$. Assume that u and v are any two bicomplex numbers. Then we have

$$|u + v|_k^p + |u - v|_k^p \lesssim 2^{p-1} (|u|_k^p + |v|_k^p).$$

Definition 1.3. Consider A is a subset of a $\mathbb{BC}$ -module E . Then A is described as a $\mathbb{BC}$ -convex set if $u, v \in E$ and $\gamma \in \mathbb{D}^+$ satisfying $0 \lesssim \gamma \lesssim 1$ implies that $\gamma u + (1 - \gamma)v \in A$.

Let us briefly outline the significant findings derived from (Toksoy and Sağır, 2023).

Definition 1.4. Assume that B is a bicomplex Banach module with the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{B, \mathbb{D}}$. Then B is described as $\mathbb{BC}$ -strictly convex, if $\|\gamma u + (1 - \gamma)v\|_{B, \mathbb{D}} < 1$ for all $0 \lesssim \gamma \lesssim 1$ with $\gamma \in \mathbb{D}^+$, $u, v \in B$ with $\|u\|_{B, \mathbb{D}} = \|v\|_{B, \mathbb{D}} = 1$ and $u \neq v$.

Definition 1.5. Assume that B is a bicomplex Banach module with the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{B, \mathbb{D}}$. Then B is described as $\mathbb{BC}$ -uniformly convex, if for any $\epsilon \in \mathbb{D}^+$ with $0 < \epsilon \lesssim 2$, the conditions $\|u\|_{B, \mathbb{D}} \lesssim 1$, $\|v\|_{B, \mathbb{D}} \lesssim 1$, $\|u - v\|_{B, \mathbb{D}} \gtrsim \epsilon$ imply there exists a $\delta = \delta(\epsilon) \in \mathbb{D}^+ \setminus \{0\}$ such that $\left\| \frac{u+v}{2} \right\|_{B, \mathbb{D}} < 1 - \delta$ for all $u, v \in B$.

Lemma 1.6. Consider $p \in \mathbb{R}$ with $p \in (1, 2]$ and $q = \frac{p}{p-1}$. Assume that u and v are any two bicomplex numbers. Then we have

$$|u + v|_k^q + |u - v|_k^q \lesssim 2(|u|_k^p + |v|_k^p)^{q-1}.$$

Let us present a significant theorem provided in (Değirmen & Sağır, 2023). Also, for the definition and properties of the bicomplex $\mathbb{BC}$ -modules $l_p^k(\mathbb{BC})$, see (Değirmen & Sağır, 2023).

Theorem 1.7. Let $1 < p < \infty$. Then $l_p^k(\mathbb{BC})$ is $\mathbb{BC}$ -strictly convex.

In the following, we briefly describe the properties and definitions of the weighted bicomplex sequence spaces $l_{p,\alpha}^k(\mathbb{BC})$ that have been extensively discussed in (Sağır & Güngör, 2023).

Definition 1.8. A bicomplex weighted sequence $\alpha = \{\alpha(n)\}_{n=0}^\infty$ is a sequence of positive hyperbolic numbers with $\alpha(0) = e_1 + e_2 = 1$, $\sum_{n=0}^\infty \alpha(n)$ is $\mathbb{D}$ -divergent and $\alpha(n) \gtrsim 1$. We define the sets $l_{p,\alpha}^k(\mathbb{BC})$ for $0 < p < \infty$ and $l_{\infty,\alpha}^k(\mathbb{BC})$ utilizing the hyperbolic ($\mathbb{D}$ -valued) norm $|\cdot|_k$ in the following manner:

$$l_{p,\alpha}^k(\mathbb{BC}) := \left\{ u = (u_n) \right. \\ \left. \in s(\mathbb{BC}) : \sum_{n=0}^\infty |u_n|_k^p \alpha(n) \text{ } \mathbb{D}\text{-converges} \right\}, 0 < p < \infty$$

and

$$l_{\infty,\alpha}^k(\mathbb{BC}) := \left\{ u = (u_n) \right. \\ \left. \in s(\mathbb{BC}) : \sup_{\mathbb{D}} \{|u_n|_k \alpha(n) : n \in \mathbb{N}\} \text{ is finite} \right\},$$

where $s(\mathbb{BC})$ is the $\mathbb{BC}$ -module of all bicomplex sequences. Let $u_n = u_{n1}e_1 + u_{n2}e_2$, $\alpha(n) = \alpha_1(n)e_1 + \alpha_2(n)e_2$. Then the complex series

$$\sum_{n=0}^{\infty} |u_{n1}|^p \alpha_1(n), \sum_{n=0}^{\infty} |u_{n2}|^p \alpha_2(n)$$

are convergent and so the sequences $\{u_{n1}\}_{n=0}^{\infty}$ and $\{u_{n2}\}_{n=0}^{\infty}$ belong to the weighted sequence spaces of complex numbers l_{p,α_1} and l_{p,α_2} respectively. Thus, $l_{p,\alpha}^k(\mathbb{BC})$ consists of all bicomplex sequences $u = (u_n) = (u_{n1}e_1 + u_{n2}e_2)$ such that $u_1 = (u_{n1}) \in l_{p,\alpha_1}$ and $u_2 = (u_{n2}) \in l_{p,\alpha_2}$ where $\alpha_1 = \alpha_1(n)$, $\alpha_2 = \alpha_2(n)$ are complex weighted sequences. Therefore $l_{p,\alpha}^k(\mathbb{BC})$ can be written as

$$l_{p,\alpha}^k(\mathbb{BC}) = l_{p,\alpha_1}e_1 + l_{p,\alpha_2}e_2$$

Consider $u = (u_n) = (u_{n1}e_1 + u_{n2}e_2) \in l_{\infty,\alpha}^k(\mathbb{BC})$. Then the sequences $\{u_{n1}\}_{n=0}^{\infty}$ and $\{u_{n2}\}_{n=0}^{\infty}$ belong to the weighted bounded sequence spaces of complex numbers l_{∞,α_1} and l_{∞,α_2} respectively. As a result, $l_{\infty,\alpha}^k(\mathbb{BC})$ consists of all $\mathbb{D}$ -bounded sequences $u = (u_n) = (u_{n1}e_1 + u_{n2}e_2)$ such that $u_1 = (u_{n1}) \in l_{\infty,\alpha_1}$ and $u_2 = (u_{n2}) \in l_{\infty,\alpha_2}$. Hence $l_{\infty,\alpha}^k(\mathbb{BC})$ can be written as

$$l_{\infty,\alpha}^k(\mathbb{BC}) = l_{\infty,\alpha_1}e_1 + l_{\infty,\alpha_2}e_2.$$

Proposition 1.9. The set $l_{\infty,\alpha}^k(\mathbb{BC})$ is a $\mathbb{BC}$ -module under usual addition operation in sequences and bicomplex scalar multiplication.

Remark 1.10. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Consider $u = (u_n) \in l_{\infty,\alpha}^k(\mathbb{BC})$, where $u = (u_n) = (u_{n1}e_1 + u_{n2}e_2)$ and $\{\alpha(n)\}_{n=0}^{\infty}$ are bicomplex weighted sequences. Clearly, the weighted spaces l_{∞,α_1} and l_{∞,α_2} are normed spaces with norms

$$\|u_1\|_{\infty,\alpha_1} = \sup\{|u_{n1}|\alpha_1(n): n \in \mathbb{N}\}$$

and

$$\|u_2\|_{\infty, \alpha_2} = \sup\{|u_{n2}| \alpha_2(n) : n \in \mathbb{N}\}$$

respectively. Hence, we can endow the $\mathbb{BC}$ -module $l_{\infty, \alpha}^k(\mathbb{BC})$ with hyperbolic ($\mathbb{D}$ -valued) norm that is given by

$$\|u\|_{\mathbb{D}, l_{\infty, \alpha}(\mathbb{BC})} = \|u_1\|_{\infty, \alpha_1} e_1 + \|u_2\|_{\infty, \alpha_2} e_2.$$

Let us take the function $\|\cdot\|_{\mathbb{D}, l_{\infty, \alpha}^k(\mathbb{BC})}$ that is defined as

$$\|u\|_{\mathbb{D}, l_{\infty, \alpha}^k(\mathbb{BC})} = \sup_{\mathbb{D}}\{|u_n|_k \alpha(n) : n \in \mathbb{N}\}.$$

Therefore, we have

$$\|u\|_{\mathbb{D}, l_{\infty, \alpha}^k(\mathbb{BC})} = \|u\|_{l_{\infty, \alpha}^k(\mathbb{BC})} \quad (2)$$

for all $u \in l_{\infty, \alpha}^k(\mathbb{BC})$. Also, we can write

$$\|u\|_{l_{\infty, \alpha}^k(\mathbb{BC})} = \|\alpha_1 u_1\|_{\infty} e_1 + \|\alpha_2 u_2\|_{\infty} e_2$$

Remark 1.11. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. For $0 < p < \infty$, we have

$$l_{p, \alpha}^k(\mathbb{BC}) = l_{p, \alpha_1} e_1 + l_{p, \alpha_2} e_2$$

where $\alpha(n) = \alpha_1(n)e_1 + \alpha_2(n)e_2$, $u_n = u_{n1}e_1 + u_{n2}e_2$ such that l_{p, α_i} are complex weighted Lebesgue sequence spaces in sense

$$l_{p, \alpha_i} = \left\{ u_i = (u_{ni}) \in s(\mathbb{C}) : \sum_{n=0}^{\infty} |u_{ni}|^p \alpha_i(n) < \infty, i = 1, 2. \right\}$$

Obviously, if $u = (u_n) \in l_{p, \alpha}^k(\mathbb{BC})$, then $\left(u_n \alpha(n)^{\frac{1}{p}}\right) \in l_p^k(\mathbb{BC})$.

Proposition 1.12. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then the set $l_{p, \alpha}^k(\mathbb{BC})$ for $0 < p < \infty$ is a $\mathbb{BC}$ -submodule of $s(\mathbb{BC})$.

Remark 1.13. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Consider $u \in l_{p, \alpha}^k(\mathbb{BC})$ with $u = (u_n) = (u_{n1}e_1 +$

$u_{n2}e_2$), $\alpha = (\alpha(n)) = (\alpha_1(n)e_1 + \alpha_2(n)e_2)$, for $0 < p < \infty$. We may write

$$l_{p,\alpha}^k(\mathbb{BC}) = l_{p,\alpha_1}e_1 + l_{p,\alpha_2}e_2.$$

Obviously, the weighted Lebesgue sequence spaces l_{p,α_1} and l_{p,α_2} are normed spaces with

$$\|u_i\|_{l_{p,\alpha_i}} = \left(\sum_{n=0}^{\infty} |u_{ni}|^p \alpha_i(n) \right)^{1/p}, \quad i = 1, 2, \\ \text{for } 1 \leq p < \infty.$$

Therefore, we can endow the $\mathbb{BC}$ -module $l_{p,\alpha}^k(\mathbb{BC})$ with hyperbolic ($\mathbb{D}$ -valued) norm that is given by

$$\|u\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} = \|u_1\|_{l_{p,\alpha_1}}e_1 + \|u_2\|_{l_{p,\alpha_2}}e_2$$

where $u_1 = (u_{n1}), u_2 = (u_{n2}), \alpha_1 = \alpha_1(n), \alpha_2 = \alpha_2(n)$. Let us take the function $\|u\|_{l_{p,\alpha}^k(\mathbb{BC})}$ defined as

$$\|u\|_{l_{p,\alpha}^k(\mathbb{BC})} = \left(\sum_{n=0}^{\infty} |u_n|^p \alpha(n) \right)^{1/p}.$$

Hence, we obtain

$$\|u\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} = \|u\|_{l_{p,\alpha}^k(\mathbb{BC})} \quad (3)$$

for all $u \in l_{p,\alpha}^k(\mathbb{BC})$

Let $1 \leq p < \infty$. Let us take $\alpha = \alpha_1e_1 + \alpha_2e_2$. Given that spaces $l_{p,\alpha}^k(\mathbb{BC})$ and $l_{\infty,\alpha}^k(\mathbb{BC})$ can be written as

$$l_{p,\alpha}^k(\mathbb{BC}) = l_{p,\alpha_1}e_1 + l_{p,\alpha_2}e_2$$

and

$$l_{\infty,\alpha}^k(\mathbb{BC}) = l_{\infty,\alpha_1}e_1 + l_{\infty,\alpha_2}e_2,$$

respectively, where l_{p,α_i} and l_{∞,α_i} are Banach spaces for $i = 1, 2$, and equations (2) and (3) are considered, it is clear that normed

spaces $l_{p,\alpha}^k(\mathbb{BC})$ and $l_{\infty,\alpha}^k(\mathbb{BC})$ are the bicomplex Banach modules with hyperbolic ($\mathbb{D}$ -valued) norms from Theorem 3.5 in (Kumar & Singh, 2015) and Theorem 1.1 in (Kumar & et al., 2016).

Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. This work demonstrates the $\mathbb{BC}$ -convexity of $l_{p,\alpha}^k(\mathbb{BC})$ and $l_{\infty,\alpha}^k(\mathbb{BC})$ for $1 \leq p < \infty$. It is asserted that $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -strictly convex for $1 < p < \infty$, and $\mathbb{BC}$ -uniformly convex for $2 \leq p < \infty$. The $\mathbb{D}$ -Hölder's and $\mathbb{D}$ -Minkowski inequalities were stated for the case of $p \in (0,1)$. Furthermore, it is established that $l_p^k(\mathbb{BC})$ and $l_{p,\alpha}^k(\mathbb{BC})$ are $\mathbb{BC}$ -uniformly convex for, $1 < p < 2$.

Main Results

Theorem 2.1. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence and $1 \leq p < \infty$. Then $l_{p,\alpha}^k(\mathbb{BC})$ and $l_{\infty,\alpha}^k(\mathbb{BC})$ are $\mathbb{BC}$ -convex.

Proof. In the beginning, let us demonstrate the $\mathbb{BC}$ -convexity of $l_{p,\alpha}^k(\mathbb{BC})$ for $1 \leq p < \infty$. Let $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{p,\alpha}^k(\mathbb{BC})$ and $\gamma \in \mathbb{D}^+$ where $0 \lesssim \gamma \lesssim 1$. Then $\sum_{n=0}^{\infty} |\zeta_n|_k^p \alpha(n)$ and $\sum_{n=0}^{\infty} |\eta_n|_k^p \alpha(n)$ converges. Also, we have

$$\begin{aligned} |\gamma \zeta_n + (1 - \gamma) \eta_n|_k^p \alpha(n) &= |(\gamma \zeta_n + (1 - \gamma) \eta_n) \alpha^{1/p}(n)|_k^p \\ &\lesssim \left(|\gamma \zeta_n \alpha^{1/p}(n)|_k + |(1 - \gamma) \eta_n \alpha^{1/p}(n)|_k \right)^p \\ &\lesssim \left(2 \sup_{\mathbb{D}} \left\{ |\gamma \zeta_n \alpha^{1/p}(n)|_k, |(1 - \gamma) \eta_n \alpha^{1/p}(n)|_k \right\} \right)^p \\ &= 2^p \sup_{\mathbb{D}} \left\{ |\gamma \zeta_n \alpha^{1/p}(n)|_k^p, |(1 - \gamma) \eta_n \alpha^{1/p}(n)|_k^p \right\} \\ &= 2^p \sup_{\mathbb{D}} \left\{ \gamma^p |\zeta_n \alpha^{1/p}(n)|_k^p, (1 - \gamma)^p |\eta_n \alpha^{1/p}(n)|_k^p \right\} \\ &\lesssim 2^p \left(\gamma^p |\zeta_n \alpha^{1/p}(n)|_k^p + (1 - \gamma)^p |\eta_n \alpha^{1/p}(n)|_k^p \right) \end{aligned}$$

for all $n = \mathbb{N} \cup \{0\}$. Hence we write

$$\begin{aligned}
& \sum_{n=0}^{\infty} |\gamma \zeta_n + (1 - \gamma) \eta_n|_k^p \alpha(n) \\
& \lesssim 2^p \left(\gamma^p \sum_{n=0}^{\infty} |\zeta_n \alpha^{1/p}(n)|_k^p \right. \\
& \quad \left. + (1 - \gamma)^p \sum_{n=0}^{\infty} |\eta_n \alpha^{1/p}(n)|_k^p \right) \\
& = 2^p \left(\gamma^p \sum_{n=0}^{\infty} |\zeta_n|_k^p \alpha(n) \right. \\
& \quad \left. + (1 - \gamma)^p \sum_{n=0}^{\infty} |\eta_n|_k^p \alpha(n) \right)
\end{aligned}$$

Therefore, based on the comparison test, it can be concluded that the series

$$\sum_{n=0}^{\infty} |\gamma \zeta_n + (1 - \gamma) \eta_n|_k^p \alpha(n)$$

is convergent. Thus, $\gamma \zeta + (1 - \gamma) \eta \in l_{p,\alpha}^k(\mathbb{BC})$ is achieved as intended. Hence, it can be concluded that $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -convex.

Now, let us demonstrate the $\mathbb{BC}$ -convexity of $l_{\infty,\alpha}^k(\mathbb{BC})$. Let $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{\infty,\alpha}^k(\mathbb{BC})$ and $\gamma \in \mathbb{D}^+$ where $0 \lesssim \gamma \lesssim 1$. Then

$$\sup_{\mathbb{D}} \{|\zeta_n|_k \alpha(n) : n \in \mathbb{N}\}$$

and

$$\sup_{\mathbb{D}} \{|\eta_n|_k \alpha(n) : n \in \mathbb{N}\}$$

are finite. Therefore, we obtain

$$\begin{aligned}
& \sup_{\mathbb{D}} \{ |\gamma \zeta_n + (1 - \gamma) \eta_n|_k \alpha(n) : n \in \mathbb{N} \} \\
& \lesssim \sup_{\mathbb{D}} \{ |\gamma \zeta_n|_k \alpha(n) + (1 - \gamma) |\eta_n|_k \alpha(n) : n \in \mathbb{N} \} \\
& = \gamma \sup_{\mathbb{D}} \{ |\zeta_n|_k \alpha(n) : n \in \mathbb{N} \} \\
& \quad + (1 - \gamma) \sup_{\mathbb{D}} \{ |\eta_n|_k \alpha(n) : n \in \mathbb{N} \}.
\end{aligned}$$

Consequently, $\gamma \zeta + (1 - \gamma) \eta \in l_{\infty, \alpha}^k(\mathbb{BC})$ is established. Hence, it can be concluded that $l_{\infty, \alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -convex.

Theorem 2.2. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then $l_{p, \alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -strictly convex for $1 < p < \infty$.

Proof. Let $\zeta, \eta \in l_{p, \alpha}^k(\mathbb{BC})$ with $\zeta \neq \eta$, $\|\zeta\|_{\mathbb{D}, l_{p, \alpha}^k(\mathbb{BC})} = \|\eta\|_{\mathbb{D}, l_{p, \alpha}^k(\mathbb{BC})} = 1$ and $\gamma \in \mathbb{D}^+$ with $0 \lesssim \gamma \lesssim 1$. By Remark 1.11, we get $\zeta \alpha^{\frac{1}{p}} \in l_p^k(\mathbb{BC})$ and $\eta \alpha^{\frac{1}{p}} \in l_p^k(\mathbb{BC})$. Then we have

$$\left\| \zeta \alpha^{\frac{1}{p}} \right\|_{\mathbb{D}, l_p^k(\mathbb{BC})} = 1 \text{ and } \left\| \eta \alpha^{\frac{1}{p}} \right\|_{\mathbb{D}, l_p^k(\mathbb{BC})} = 1.$$

As stated by Theorem 1.7, it has been established that $l_p^k(\mathbb{BC})$ has $\mathbb{BC}$ -strictly convexity for $1 < p < \infty$. Therefore, we obtain

$$\begin{aligned}
\|\gamma \zeta + (1 - \gamma) \eta\|_{\mathbb{D}, l_{p, \alpha}^k(\mathbb{BC})} &= \left\| (\gamma \zeta + (1 - \gamma) \eta) \alpha^{\frac{1}{p}} \right\|_{\mathbb{D}, l_p^k(\mathbb{BC})} \\
&= \left\| \gamma \zeta \alpha^{\frac{1}{p}} + (1 - \gamma) \eta \alpha^{\frac{1}{p}} \right\|_{\mathbb{D}, l_p^k(\mathbb{BC})} \\
&< 1
\end{aligned}$$

for $1 < p < \infty$. Consequently, the property of $\mathbb{BC}$ -strictly convexity is assured in $l_{p,\alpha}^k(\mathbb{BC})$ for $1 < p < \infty$.

Theorem 2.3. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -uniformly convex for $2 \leq p < \infty$.

Proof. Let $2 \leq p < \infty$. Let us take $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{p,\alpha}^k(\mathbb{BC})$, $\epsilon \in \mathbb{D}^+$ with $0 < \epsilon \lesssim 2$, $\|\zeta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} \lesssim 1$, $\|\eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} \lesssim 1$, $\|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} \gtrsim \epsilon$. Thus, we can write

$$\begin{aligned} & \|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p + \|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \\ &= \sum_{n=0}^{\infty} |\zeta_n + \eta_n|_k^p \alpha(n) + \sum_{n=0}^{\infty} |\zeta_n - \eta_n|_k^p \alpha(n) \\ &= \sum_{n=0}^{\infty} (|\zeta_n + \eta_n|_k^p \alpha(n) + |\zeta_n - \eta_n|_k^p \alpha(n)) \\ &= \sum_{n=0}^{\infty} \left(\left| (\zeta_n + \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^p + \left| (\zeta_n - \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^p \right). \end{aligned}$$

By applying the inequality stated in Lemma 1.2, we have

$$\begin{aligned} & \|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p + \|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \\ & \lesssim \sum_{n=0}^{\infty} 2^{p-1} \left| \zeta_n \alpha^{\frac{1}{p}}(n) \right|_k^p + \left| \eta_n \alpha^{\frac{1}{p}}(n) \right|_k^p \\ & \|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p + \|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \\ & \lesssim 2^{p-1} \left(\sum_{n=0}^{\infty} |\zeta_n|_k^p \alpha(n) + \sum_{n=0}^{\infty} |\eta_n|_k^p \alpha(n) \right) \end{aligned}$$

$$\begin{aligned}
&= 2^{p-1} \left(\|\zeta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} + \|\eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} \right) \\
&\lesssim 2^{p-1} \cdot 2 = 2^p.
\end{aligned}$$

Therefore, we get

$$\|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \lesssim 2^p - \|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \lesssim 2^p - \epsilon^p,$$

it follows that

$$\left\| \frac{\zeta + \eta}{2} \right\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} = \left(\frac{1}{2^p} \|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \right)^{\frac{1}{p}} \lesssim \left(1 - \left(\frac{\epsilon}{2} \right)^p \right)^{\frac{1}{p}}.$$

If an assumption is made that $\delta(\epsilon) = 1 - \left(1 - \left(\frac{\epsilon}{2} \right)^p \right)^{1/p}$, then it is possible to conclude that $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -uniformly convex for $2 \leq p < \infty$.

Lemma 2.4. ($\mathbb{D}$ -Holder's inequality for $p \in (0, 1)$) Let p and q be real numbers with $0 < p < 1$ and $-\infty < q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\zeta_m, \eta_m \in \mathbb{BC}$ for $m \in \{1, 2, \dots, n\}$. Then we have

$$\sum_{m=1}^n |\zeta_m \eta_m|_k \gtrsim \left(\sum_{m=1}^n |\zeta_m|_k^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^n |\eta_m|_k^q \right)^{\frac{1}{q}}.$$

Proof. Suppose that the conditions of the lemma hold. Let $\zeta_m = \zeta_{m1}e_1 + \zeta_{m2}e_2$ and $\eta_m = \eta_{m1}e_1 + \eta_{m2}e_2$ for $m \in \{1, 2, \dots, n\}$. By using the properties of $|\cdot|_k$, we can write

$$\sum_{m=1}^n |\zeta_m \eta_m|_k = \sum_{m=1}^n |\zeta_m|_k |\eta_m|_k$$

$$\begin{aligned}
&= \sum_{m=1}^n |\zeta_{m1}e_1 + \zeta_{m2}e_2|_k |\eta_{m1}e_1 + \eta_{m2}e_2|_k \\
&= \sum_{m=1}^n (|\zeta_{m1}|e_1 + |\zeta_{m2}|e_2)(|\eta_{m1}|e_1 + |\eta_{m2}|e_2) \\
&= \sum_{m=1}^n (|\zeta_{m1}||\eta_{m1}|e_1 + |\zeta_{m2}||\eta_{m2}|e_2) \\
&= \left(\sum_{m=1}^n |\zeta_{m1}||\eta_{m1}| \right) e_1 + \left(\sum_{m=1}^n |\zeta_{m2}||\eta_{m2}| \right) e_2.
\end{aligned}$$

By applying the usual Hölder's inequality for $p \in (0,1)$ in Section 2.8 of (Hardy & et al., 1952), we get

$$\begin{aligned}
\sum_{m=1}^n |\zeta_m \eta_m|_k &\lesssim \left(\left(\sum_{m=1}^n |\zeta_{m1}|^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^n |\eta_{m1}|^q \right)^{\frac{1}{q}} \right) e_1 \\
&\quad + \left(\left(\sum_{m=1}^n |\zeta_{m2}|^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^n |\eta_{m2}|^q \right)^{\frac{1}{q}} \right) e_2 \\
&= \left(\left(\sum_{m=1}^n |\zeta_{m1}|^p \right)^{\frac{1}{p}} e_1 + \left(\sum_{m=1}^n |\zeta_{m2}|^p \right)^{\frac{1}{p}} e_2 \right)
\end{aligned}$$

$$\times \left(\left(\sum_{m=1}^n |\eta_{m1}|^q \right)^{\frac{1}{q}} e_1 + \left(\sum_{m=1}^n |\eta_{m2}|^q \right)^{\frac{1}{q}} e_2 \right).$$

By considering the property stated in Definition 1.1, we obtain

$$\begin{aligned} \sum_{m=1}^n |\zeta_m \eta_m|_k &\gtrsim \left(\left(\sum_{m=1}^n |\zeta_{m1}|^p \right) e_1 + \left(\sum_{m=1}^n |\zeta_{m2}|^p \right) e_2 \right)^{\frac{1}{p}} \\ &\quad \times \left(\left(\sum_{m=1}^n |\eta_{m1}|^q \right) e_1 + \left(\sum_{m=1}^n |\eta_{m2}|^q \right) e_2 \right)^{\frac{1}{q}} \\ &= \left(\sum_{m=1}^n |\zeta_{m1}|^p e_1 + |\zeta_{m2}|^p e_2 \right)^{\frac{1}{p}} \left(\sum_{m=1}^n |\eta_{m1}|^q e_1 \right. \\ &\quad \left. + |\eta_{m2}|^q e_2 \right)^{\frac{1}{q}} \\ &= \left(\sum_{m=1}^n (|\zeta_{m1}| e_1 + |\zeta_{m2}| e_2)^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^n (|\eta_{m1}| e_1 \right. \\ &\quad \left. + |\eta_{m2}| e_2)^q \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{m=1}^n |\zeta_{m1}e_1 + \zeta_{m2}e_2|_k^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^n |\eta_{m1}e_1 \right. \\
&\quad \left. + \eta_{m2}e_2|_k^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{m=1}^n |\zeta_m|_k^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^n |\eta_m|_k^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Lemma 2.5. ($\mathbb{D}$ -Minkowski inequality for $p \in (0, 1)$) Let p be a real number with $0 < p < 1$. Assume that $\zeta_m, \eta_m \in \mathbb{B}\mathbb{C}$ for $m \in \{1, 2, \dots, n\}$. Then we have

$$\left(\sum_{m=1}^n |\zeta_m + \eta_m|_k^p \right)^{\frac{1}{p}} \gtrsim \left(\sum_{m=1}^n |\zeta_m|_k^p \right)^{\frac{1}{p}} + \left(\sum_{m=1}^n |\eta_m|_k^p \right)^{\frac{1}{p}}.$$

Proof. Suppose that the conditions of the lemma hold. Let $\zeta_m = \zeta_{m1}e_1 + \zeta_{m2}e_2$ and $\eta_m = \eta_{m1}e_1 + \eta_{m2}e_2$ for $m \in \{1, 2, \dots, n\}$. By using the properties of $|\cdot|_k$ and the property stated in Definition 1.1, we can write

$$\begin{aligned}
&\left(\sum_{m=1}^n |\zeta_m + \eta_m|_k^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^n |(\zeta_{m1}e_1 + \zeta_{m2}e_2) + (\eta_{m1}e_1 + \eta_{m2}e_2)|_k^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^n |(\zeta_{m1} + \eta_{m1})e_1 + (\zeta_{m2} + \eta_{m2})e_2|_k^p \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{m=1}^n |\zeta_{m1} + \eta_{m1}|^p e_1 + |\zeta_{m2} + \eta_{m2}|^p e_2 \right)^{\frac{1}{p}} \\
&= \left(\left(\sum_{m=1}^n |\zeta_{m1} + \eta_{m1}|^p \right) e_1 + \left(\sum_{m=1}^n |\zeta_{m2} + \eta_{m2}|^p \right) e_2 \right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^n |\zeta_{m1} + \eta_{m1}|^p \right)^{\frac{1}{p}} e_1 + \left(\sum_{m=1}^n |\zeta_{m2} + \eta_{m2}|^p \right)^{\frac{1}{p}} e_2.
\end{aligned}$$

By applying the usual Minkowski inequality for $p \in (0,1)$ in Section 2.11 of (Hardy & et al., 1952), we get

$$\begin{aligned}
\left(\sum_{m=1}^n |\zeta_m + \eta_m|^p \right)^{\frac{1}{p}} &\gtrsim \left(\left(\sum_{m=1}^n |\zeta_{m1}|^p \right)^{\frac{1}{p}} + \left(\sum_{m=1}^n |\eta_{m1}|^p \right)^{\frac{1}{p}} \right) e_1 \\
&\quad + \left(\left(\sum_{m=1}^n |\zeta_{m2}|^p \right)^{\frac{1}{p}} + \left(\sum_{m=1}^n |\eta_{m2}|^p \right)^{\frac{1}{p}} \right) e_2 \\
&= \left(\left(\sum_{m=1}^n |\zeta_{m1}|^p \right)^{\frac{1}{p}} e_1 + \left(\sum_{m=1}^n |\zeta_{m2}|^p \right)^{\frac{1}{p}} e_2 \right) \\
&\quad + \left(\left(\sum_{m=1}^n |\eta_{m1}|^p \right)^{\frac{1}{p}} e_1 \right. \\
&\quad \left. + \left(\sum_{m=1}^n |\eta_{m2}|^p \right)^{\frac{1}{p}} e_2 \right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\left(\sum_{m=1}^n |\zeta_{m1}|^p \right) e_1 + \left(\sum_{m=1}^n |\zeta_{m2}|^p \right) e_2 \right)^{\frac{1}{p}} \\
&\quad + \left(\left(\sum_{m=1}^n |\eta_{m1}|^p \right) e_1 + \left(\sum_{m=1}^n |\eta_{m2}|^p \right) e_2 \right)^{\frac{1}{p}} \\
&\left(\sum_{m=1}^n |\zeta_m + \eta_m|_k^p \right)^{\frac{1}{p}} \\
&\quad \gtrsim \left(\sum_{m=1}^n |\zeta_{m1}|^p e_1 + |\zeta_{m2}|^p e_2 \right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{m=1}^n |\eta_{m1}|^p e_1 + |\eta_{m2}|^p e_2 \right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^n (|\zeta_{m1}| e_1 + |\zeta_{m2}| e_2)^p \right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{m=1}^n (|\eta_{m1}| e_1 + |\eta_{m2}| e_2)^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^n |\zeta_{m1} e_1 + \zeta_{m2} e_2|_k^p \right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{m=1}^n |\eta_{m1} e_1 + \eta_{m2} e_2|_k^p \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \left(\sum_{m=1}^n |\zeta_m|_k^p \right)^{\frac{1}{p}} + \left(\sum_{m=1}^n |\eta_m|_k^p \right)^{\frac{1}{p}}.$$

Proposition 2.6. Let p and q be real numbers with $1 < p < 2$ and $q = \frac{p}{p-1}$. Assume that $\zeta = (\zeta_n), \eta = (\eta_n) \in l_p^k(\mathbb{BC})$. Then we have

$$\begin{aligned} \|\zeta + \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q \\ \lesssim 2 \left(\|\zeta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p \right)^{q-1}. \end{aligned}$$

Proof. Suppose that the conditions of the theorem hold. Firstly, we can write

$$\begin{aligned} \|\zeta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q &= \left(\sum_{n=1}^{\infty} |\zeta_n|_k^p \right)^{\frac{q}{p}} = \left(\sum_{n=1}^{\infty} |\zeta_n|_k^{q(p-1)} \right)^{\frac{1}{p-1}} \quad (4) \\ &= \|\zeta\|_{\mathbb{D}, l_{p-1}^k(\mathbb{BC})}^q. \end{aligned}$$

By applying the equality (4) to $\|\zeta + \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q$ and $\|\zeta - \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q$, respectively, and using $\mathbb{D}$ -Minkowski inequality for $p - 1 \in (0, 1)$, we get

$$\begin{aligned} \|\zeta + \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q \\ = \|\zeta + \eta\|_{\mathbb{D}, l_{p-1}^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D}, l_{p-1}^k(\mathbb{BC})}^q \\ \lesssim \|\zeta + \eta\|_{\mathbb{D}, l_{p-1}^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D}, l_{p-1}^k(\mathbb{BC})}^q. \end{aligned}$$

By utilizing the expression provided in Lemma 1.6, we obtain

$$\begin{aligned}
& \|\zeta + \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q \\
& \lesssim 2 \left\| (|\zeta|_k^p + |\eta|_k^p)^{q-1} \right\|_{\mathbb{D}, l_{p-1}^k(\mathbb{BC})} \\
& = 2 \left(\sum_{n=1}^{\infty} (|\zeta_n|_k^p + |\eta_n|_k^p)^{(q-1)(p-1)} \right)^{\frac{1}{p-1}} \\
& = 2 \left(\sum_{n=1}^{\infty} (|\zeta_n|_k^p + |\eta_n|_k^p) \right)^{\frac{1}{p-1}} \\
& = 2 \left(\|\zeta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p \right)^{q-1}.
\end{aligned}$$

Theorem 2.7. Let $1 < p < 2$. Then $l_p^k(\mathbb{BC})$ is $\mathbb{BC}$ -uniformly convex.

Proof. Let $1 < p < 2$. Let us take $\zeta = (\zeta_n), \eta = (\eta_n) \in l_p^k(\mathbb{BC})$, $\epsilon \in \mathbb{D}^+$ with $0 < \epsilon \lesssim 2$, $\|\zeta\|_{\mathbb{D}, l_p^k(\mathbb{BC})} \lesssim 1$, $\|\eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})} \lesssim 1$, $\|\zeta - \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})} \gtrsim \epsilon$. By Proposition 2.6, we can write

$$\begin{aligned}
& \|\zeta + \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q \\
& \lesssim 2 \left(\|\zeta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p \right)^{q-1} \\
& \quad - \|\zeta - \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^q \\
& \lesssim 2^q - \epsilon^q = 2^q \left(1 - \left(\frac{\epsilon}{2} \right)^q \right).
\end{aligned}$$

Then, we have

$$\|\zeta + \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p \lesssim 2^p \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{p}{q}}$$

Hence, it follows that

$$\left\|\frac{\zeta + \eta}{2}\right\|_{\mathbb{D}, l_p^k(\mathbb{BC})} = \left(\frac{1}{2^p} \|\zeta + \eta\|_{\mathbb{D}, l_p^k(\mathbb{BC})}^p\right)^{\frac{1}{p}} \lesssim \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}.$$

If an assumption is made that $\delta(\epsilon) = 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{1/q}$, then it is possible to conclude that $l_p^k(\mathbb{BC})$ is $\mathbb{BC}$ -uniformly convex for $1 < p < 2$.

Proposition 2.8. Let $\{\alpha(n)\}_{n=0}^\infty$ be a bicomplex weighted sequence. Let p and q be real numbers with $1 < p \leq 2$ and $q = \frac{p}{p-1}$.

Assume that $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{p,\alpha}^k(\mathbb{BC})$. Then we have

$$\begin{aligned} \|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^q \\ \lesssim 2 \left(\|\zeta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \right)^{q-1}. \end{aligned}$$

Proof. Let $v = (v_n), s = (s_n) \in l_p^k(\mathbb{BC})$. By $\mathbb{D}$ -Minkowski inequality for $r \in (0,1)$, we can write

$$\left(\sum_{n=0}^{\infty} |v_n|_k^r \right)^{\frac{1}{r}} + \left(\sum_{n=0}^{\infty} |s_n|_k^r \right)^{\frac{1}{r}} \lesssim \left(\sum_{n=0}^{\infty} |v_n + s_n|_k^r \right)^{\frac{1}{r}}. \quad (5)$$

Let $\{\alpha(n)\}_{n=0}^\infty$ be a bicomplex weighted sequence. Let p and q be real numbers with $1 < p \leq 2$ and $q = \frac{p}{p-1}$. Assume that $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{p,\alpha}^k(\mathbb{BC})$. Thus $\zeta \alpha^{\frac{1}{p}}, \eta \alpha^{\frac{1}{p}} \in l_p^k(\mathbb{BC})$ by Remark 1.11. Let us replace r by $\frac{p}{q}$ and in (5) for

$$v_n = \left| (\zeta_n + \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^q$$

and

$$s_n = \left| (\zeta_n - \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^q.$$

Then we write

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} \left| (\zeta_n + \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^p \right)^{\frac{q}{p}} + \left(\sum_{n=0}^{\infty} \left| (\zeta_n - \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^p \right)^{\frac{q}{p}} \\ & \lesssim \left(\sum_{n=0}^{\infty} \left(\left| (\zeta_n + \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^q + \left| (\zeta_n - \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^q \right)^{\frac{p}{q}} \right)^{\frac{q}{p}} \\ & = \left(\sum_{n=0}^{\infty} \left(\left| \left(\zeta_n \alpha^{\frac{1}{p}}(n) + \eta_n \alpha^{\frac{1}{p}}(n) \right) \right|_k^q \right. \right. \\ & \quad \left. \left. + \left| \left(\zeta_n \alpha^{\frac{1}{p}}(n) - \eta_n \alpha^{\frac{1}{p}}(n) \right) \right|_k^q \right)^{\frac{p}{q}} \right)^{\frac{q}{p}}. \end{aligned}$$

By using Lemma 1.6, we get

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} \left| (\zeta_n + \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^p \right)^{\frac{q}{p}} + \left(\sum_{n=0}^{\infty} \left| (\zeta_n - \eta_n) \alpha^{\frac{1}{p}}(n) \right|_k^p \right)^{\frac{q}{p}} \\ & \lesssim \left(\sum_{n=0}^{\infty} \left(2 \left(\left| \zeta_n \alpha^{\frac{1}{p}}(n) \right|_k^p + \left| \eta_n \alpha^{\frac{1}{p}}(n) \right|_k^p \right)^{q-1} \right)^{\frac{p}{q}} \right)^{\frac{q}{p}} \end{aligned}$$

$$\begin{aligned}
&= 2 \left(\sum_{n=0}^{\infty} \left(\left| \zeta_n \alpha^{\frac{1}{p}}(n) \right|_k^p + \left| \eta_n \alpha^{\frac{1}{p}}(n) \right|_k^p \right) \right)^{\frac{q}{p}} \\
&= 2 \left(\sum_{n=0}^{\infty} |\zeta_n|_k^p \alpha(n) + \sum_{n=0}^{\infty} |\eta_n|_k^p \alpha(n) \right)^{\frac{q}{p}} \\
&= 2 \left(\|\zeta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \right)^{q-1}
\end{aligned}$$

where $q - 1 = \frac{p}{q}$. Therefore, we obtain

$$\begin{aligned}
&\|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^q + \|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^q \\
&\lesssim 2 \left(\|\zeta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \right)^{q-1}.
\end{aligned}$$

Theorem 2.9. Let $\{\alpha(n)\}_{n=0}^{\infty}$ be a bicomplex weighted sequence. Then $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -uniformly convex for $1 < p < 2$.

Proof. Let $1 < p < 2$. Let us take $\zeta = (\zeta_n), \eta = (\eta_n) \in l_{p,\alpha}^k(\mathbb{BC})$, $\epsilon \in \mathbb{D}^+$ with $0 < \epsilon \lesssim 2$, $\|\zeta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} \lesssim 1$, $\|\eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} \lesssim 1$, $\|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} \gtrsim \epsilon$. By Proposition 2.8, we can write

$$\begin{aligned}
&\|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^q \\
&\lesssim 2 \left(\|\zeta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p + \|\eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \right)^{q-1} \\
&\quad - \|\zeta - \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^q \\
&\lesssim 2^q - \epsilon^q = 2^q \left(1 - \left(\frac{\epsilon}{2} \right)^q \right).
\end{aligned}$$

Then, we have

$$\|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p \lesssim 2^p \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{p}{q}}$$

Hence, it follows that

$$\left\|\frac{\zeta + \eta}{2}\right\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})} = \left(\frac{1}{2^p} \|\zeta + \eta\|_{\mathbb{D}, l_{p,\alpha}^k(\mathbb{BC})}^p\right)^{\frac{1}{p}} \lesssim \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}.$$

If an assumption is made that $\delta(\epsilon) = 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{1/q}$, then it is possible to conclude that $l_{p,\alpha}^k(\mathbb{BC})$ is $\mathbb{BC}$ -uniformly convex for $1 < p < 2$.

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CHAPTER V

Degree Based Topological Descriptors and Polynomials of Certain Dendrimers

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Introduction

Nowadays, computer-aided design for productions in every field is a trend investigation area. Owing to this, it is possible to avoid from losses such as time and budget, arising from long and costly experimental processes. Using graph theory applications is one of the methods contributing to this aim.

Graph theory is related to the modeling of objects and relations of each object with each other by representing each object as a vertex and each relation of objects as an edge. The concept of topological index, which takes place in chemical graph theory, succeeds to

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obtain numeric values from molecular graphs of compounds, and these numerical datas have a big importance for the creation and efficiency of existing and also new designs. Every time it is not impossible to do computing stages manually via formulas. There are hundreds of topological indices and also various of them have complex formulas. To avoid from difficulties of calculations, there exist some methods applied via algebraic polynomials such as "*M*-polynomial", "*CoM*-polynomial" etc... (See from [1-3]) Each of them is a degree-based polynomial derivated by the edge and vertex partition technique from the graph and complement graph of a compound.

In [17] authors handled the optical transpose interconnection system swapped network via topological indices. In [18], polynomials of degree-based topological indices for the OTIS and swapped networks have been studied.

Dendrimers are multibranched composite structures that have been studied extensively [4-9] and are of many applications in drug delivery [10, 11], catalysis [12], and light harvesting [13]. Nanostar dendrimer is a part of the new group of macro-particles that appear to be photon funnels just like artificial antennas and are used in the formation of nanotubes, micro and macro-capsules, chemical sensors, colored glasses, and modified electrodes [14-16].

The Bibliometric analysis of the keywords in publications of dendrimers in different fields using the WOS data base (<https://www.webofscience.com/wos/woscc/basic-search>) is presented on in Figure 2.

Following the invention of dendrimers the possibility was recognized of using them for improving optical sensor performance [19]. Dendritic macromolecules are a new category of hyper-structured material and have recently been introduced into optical and optoelectronic applications. Here, functional chromophores can be replaced at branches, cores, and the end of the dendrimers to control their optical properties [20].

In this study, we calculate various Banhatti topological (co)indices, which are defined in the near past [21-24], for two dendrimers: Tetrathiafulvalene and Organosilicon dendrimers via two algebraic polynomials mentioned above. We give numerical and graphical comparisons to indicate the performance of indices and coindices. We hope these representations and numeric data may be helpful for testing the efficiency of optical applications of dendrimers in the future.

Preliminaries

The complement of a graph G , signed by $\bar{G}$, is a simple graph with the same vertex set $V(G)$ provided that any two vertices $v_1v_2 \in E(\bar{G})$ if and only if $v_1v_2 \notin E(G)$ [25]. Over time, researchers have begun to incorporate the nonadjacent pairing of vertices into consideration while calculating some topological indices of molecular graphs, resulting in degree-based topological indices known as coindices. In [3] Kirmani constructed *CoM*-polynomials as an alternative to *M*-polynomials using Berhe's Lemma [26] as follows:

Lemma 1. The following statement holds for a connected graph G of order n .

$$\bar{m}_{ij} = |\bar{E}_{ij}| = \begin{cases} \frac{n_i(n_i - 1)}{2} - m_{ii} & , i = j \\ n_i n_j - m_{ij} & , i < j \end{cases}$$

The following representations will be followed in the rest of the study for a graph G .

$$n_i = |V_i| \text{ for } V_i = \{v \in V(G), d(v) = i\}$$

$$m_{ij} = |E_{ij}| \text{ for } E_{ij} = \{uv \in E(G), d(u) = i \text{ and } d(v) = j\}$$

$$\bar{m}_{ij} = |\bar{E}_{ij}| \text{ for } \bar{E}_{ij} = \{uv \in E(\bar{G}), d(u) = i \text{ and } d(v) = j\}$$

where $d(u), d(v)$ indicate the degrees of vertices that are adjacent to u and v , respectively.

Definition 1. For a simple connected graph G , M and CoM -polynomials are defined as,

$$M(G; x, y) = \sum_{i \leq j} m_{ij} x^i y^j$$

$$CoM(G; x, y) = \sum_{i \leq j} \bar{m}_{ij} x^i y^j$$

where $\bar{m}_{ij}$ represents the number of edges $uv \notin E(G)$ such that $\{(d(u), d(v)) = \{i, j\}\}$.

Models and Methods

In this part of the study we will present some tables (Table 1 and Table 2) to construct our M -polynomials and CoM -polynomials

by edge and vertex partition technique for our dendrimer molecules given in Figure 3.

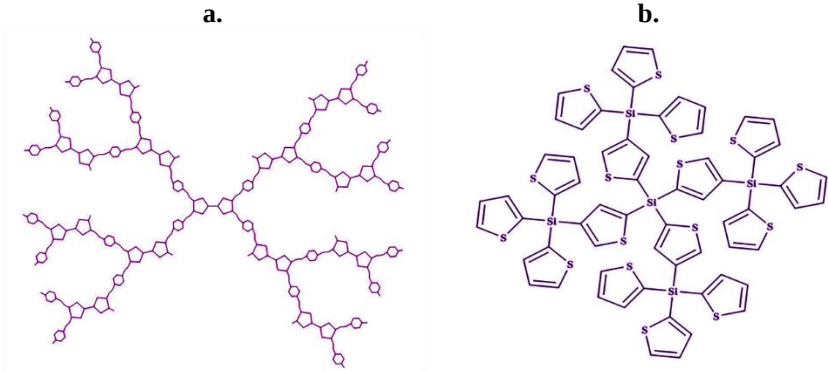


Figure 3. a.
Tetrathiafulvalene
dendrimer of generations
 G_n has grown 2 stages

*b. The second member of the
Organosilicon dendrimer
structure*

Tetrathiafulvalene Dendrimer

It has three type of vertices with the degrees 1, 2 and 3. Hence the vertex partition of the structure such that $n_1 = 6.2^n + 4$, $n_2 = 8.2^n + 8$, $n_3 = 30.2^n + 8$. Using these, we obtain the following edge partition table (Table 1) by Lemma 1.

Table 1. Edge partition for Tetrathiafulvalene dendrimer

$(d(u),d(v)); uv \in E(G)$	(1,3)	(2,1)	(2,2)	(2,3)	(3,3)
$ E_{ij} = m_{ij}$	$4.2^n - 4$	$8.2^n - 4$	$28.2^n - 16$	$92.2^n - 56$	$12.2^n - 9$
$ \overline{E}_{ij} = \overline{m}_{ij}$	$180.2^{2n} + 164.2^n + 36$	$48.2^{2n} + 72.2^n + 36$	$32.2^{2n} + 32.2^n + 44$	$240.2^{2n} + 212.2^n + 120$	$450.2^{2n} + 213.2^n + 37$

Hence related polynomials for Tetrathiafulvalene dendrimer obtained as

$$M(x, y) = (4 \cdot 2^n - 4)xy^3 + (8 \cdot 2^n - 4)x^2y + (28 \cdot 2^n - 16)x^2y^2 \\ + (92 \cdot 2^n - 56)x^2y^3 + (12 \cdot 2^n - 9)x^3y^3$$

$$CoM(x, y) = (180 \cdot 2^{2n} + 164 \cdot 2^n + 36)xy^3 \\ + (48 \cdot 2^{2n} + 72 \cdot 2^n + 36)x^2y \\ + (32 \cdot 2^{2n} + 32 \cdot 2^n + 44)x^2y^2 \\ + (240 \cdot 2^{2n} + 212 \cdot 2^n + 120)x^2y^3 \\ + (450 \cdot 2^{2n} + 213 \cdot 2^n + 37)x^3y^3$$

Organosilicon Dendrimer

It has three type of vertices with the degrees 2, 3 and 4. Hence the vertex partition of the structure such that $n_2 = 2 \cdot 3^n - 5$, $n_3 = \frac{10}{3} \cdot 3^n - 2$, $n_4 = \frac{16}{3} \cdot 3^n - 4$. Using these, we obtain the following edge partition table (Table 2) by Lemma 1.

Table 2. Edge partition for Organosilicon Dendrimer

$(d(u), d(v)); uv \in E(G)$	(2, 2)	(2, 3)	(3, 4)
$ E_{ij} = m_{ij}$	$6 \cdot 3^n - 6$	$4 \cdot 3^n - 4$	$\frac{8}{3} \cdot 3^n - 4$
$ \overline{E}_{ij} = \overline{m}_{ij}$	$2 \cdot 3^{2n} - 17 \cdot 3^n + 21$	$\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14$	$\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12$

Hence related polynomials for Tetrathiafulvalene dendrimer obtained as

$$M(x, y) = (6 \cdot 3^n - 6)x^2y^2 + (4 \cdot 3^n - 4)x^2y^3 \\ + \left(\frac{8}{3} \cdot 3^n - 4\right)x^3y^4$$

$$CoM(x, y) = (2 \cdot 3^{2n} - 17 \cdot 3^n + 21)x^2y^2 \\ + \left(\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14\right)x^2y^3 \\ + \left(\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12\right)x^3y^4$$

Table 3. Topological indices

Topological Index	Formula	Derivation From $M(G; x, y)$
1. First K Banhatti Index ($B_1(G)$)	$\sum_{uv \in E(G)} (d_u + d_{uv})$	$[D_x + D_y + 2D_x Q_{-2}J](f(x, y))_{x=y=1}$
2. Second K Banhatti Index ($B_2(G)$)	$\sum_{uv \in E(G)} (d_u \cdot d_{uv})$	$[D_x Q_{-2}J(D_x + D_y)](f(x, y))_{x=1}$
3. First K Hyper Banhatti Index ($HB_1(G)$)	$\sum_{uv \in E(G)} (d_u + d_{uv})^2$	$[D_x^2 + D_y^2 + 2D_x^2 Q_{-2}J + 2D_x Q_{-2}J(D_x + D_y)](f(x, y))_{x=y=1}$
4. Second K Hyper Banhatti Index ($HB_2(G)$)	$\sum_{uv \in E(G)} (d_u \cdot d_{uv})^2$	$[D_x^2 Q_{-2}J(D_x^2 + D_y^2)](f(x, y))_{x=1}$
5. Modified First K Banhatti Index ($mB_1(G)$)	$\sum_{uv \in E(G)} \frac{1}{(d_u + d_{uv})}$	$[S_x Q_{-2}J(L_x + L_y)](f(x, y))_{x=1}$
6. Modified Second K Banhatti Index ($mB_2(G)$)	$\sum_{uv \in E(G)} \frac{1}{(d_u \cdot d_{uv})}$	$[S_x Q_{-2}J(S_x + S_y)](f(x, y))_{x=1}$

Here motivated by Banhatti indices given in Table 3 we define various Banhatti coindices in Table 4. After this section we will calculate the related topological indices and coindices via derivations from our constructed M and CoM -polynomials.

Table 4. Topological Coindices

Topological Coindex	Formula	Derivation From $CoM(G; x, y)$
1. First K Banhatti Coindex ($\overline{B_1}(G)$)	$\sum_{uv \in E(G)} (d_u + d_{uv})$	$[D_x + D_y + 2D_x Q_{-2}J](f(x, y))_{x=y=1}$
2. Second K Banhatti Coindex ($\overline{B_2}(G)$)	$\sum_{uv \in E(G)} (d_u \cdot d_{uv})$	$[D_x Q_{-2}J(D_x + D_y)](f(x, y))_{x=1}$
3. First K Hyper Banhatti Coindex ($\overline{HB_1}(G)$)	$\sum_{uv \in E(G)} (d_u + d_{uv})^2$	$[D_x^2 + D_y^2 + 2D_x^2 Q_{-2}J + 2D_x Q_{-2}J(D_x + D_y)](f(x, y))_{x=y=1}$
4. Second K Hyper Banhatti Coindex ($\overline{HB_2}(G)$)	$\sum_{uv \in E(G)} (d_u \cdot d_{uv})^2$	$[D_x^2 Q_{-2}J(D_x^2 + D_y^2)](f(x, y))_{x=1}$
5. Modified First K Banhatti Coindex ($\overline{mB_1}(G)$)	$\sum_{uv \in E(G)} \frac{1}{(d_u + d_{uv})}$	$[S_x Q_{-2}J(L_x + L_y)](f(x, y))_{x=1}$

6. Modified Second K Banhatti Coindex ($\overline{mB_2}(G)$)	$\sum_{uv \in E(G)} \frac{1}{(d_u \cdot d_{uv})}$	$[S_x Q_{-2} J(S_x + S_y)](f(x, y))_{x=1}$
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$$\begin{aligned}
D_x f(x, y) &= x \frac{\partial(f(x, y))}{\partial x} & D_y f(x, y) &= y \frac{\partial(f(x, y))}{\partial y} \\
L_x(f(x, y)) &= f(x^2, y) & L_y(f(x, y)) &= f(x, y^2) \\
S_x f(x, y) &= \int_0^x \frac{f(t, y)}{t} dt & S_y f(x, y) &= \int_0^y \frac{f(x, t)}{t} dt \\
Jf(x, y) &= f(x, x) & Q_\alpha f(x, y) &= x^\alpha f(x, y)
\end{aligned}$$

Computation Results

Computation of Topological Indices with the help of M -polynomials on Tetrathiafulvalene and Organosilicon Dendrimers

Theorem 1. The topological indices for Tetrathiafulvalene are given by,

$$\begin{aligned}
B_1(G) &= 1476 \cdot 2^n - 922, & B_2(G) &= 1948 \cdot 2^n - 1236 \\
HB_1(G) &= 7900 \cdot 2^n - 4990, & HB_2(G) &= 15220 \cdot 2^n - 9740 \\
mB_1(G) &= \frac{12592}{210} \cdot 2^n - \frac{1536}{42}, & mB_2(G) &= \frac{1012}{18} \cdot 2^n - \frac{855}{18}.
\end{aligned}$$

Proof.

The M -polynomial of Tetrathiafulvalene is of the form

$$\begin{aligned}
M(G; x, y) &= (4 \cdot 2^n - 4)xy^3 + (8 \cdot 2^n - 4)x^2y \\
&\quad + (28 \cdot 2^n - 16)x^2y^2 + (92 \cdot 2^n - 56)x^2y^3 \\
&\quad + (12 \cdot 2^n - 9)x^3y^3.
\end{aligned}$$

Then,

$$D_x = (4 \cdot 2^n - 4)xy^3 + (16 \cdot 2^n - 8)x^2y + (56 \cdot 2^n - 32)x^2y^2 \\ + (184 \cdot 2^n - 112)x^2y^3 + (36 \cdot 2^n - 27)x^3y^3$$

$$D_y = (12 \cdot 2^n - 12)xy^3 + (8 \cdot 2^n - 4)x^2y + (56 \cdot 2^n - 32)x^2y^2 \\ + (276 \cdot 2^n - 168)x^2y^3 + (36 \cdot 2^n - 27)x^3y^3$$

$$D_x + D_y = (16 \cdot 2^n - 16)xy^3 + (24 \cdot 2^n - 12)x^2y \\ + (112 \cdot 2^n - 64)x^2y^2 + (460 \cdot 2^n - 280)x^2y^3 \\ + (72 \cdot 2^n - 54)x^3y^3$$

$$J = (8 \cdot 2^n - 4)x^3 + (32 \cdot 2^n - 20)x^4 + (92 \cdot 2^n - 56)x^5 \\ + (12 \cdot 2^n - 9)x^6$$

$$2D_x Q_{-2} J = (16 \cdot 2^n - 8)x^3 + (128 \cdot 2^n - 80)x^4 \\ + (552 \cdot 2^n - 336)x^5 + (96 \cdot 2^n - 72)x^6$$

$$D_x Q_{-2} J (D_x + D_y) \\ = (24 \cdot 2^n - 12)x^2 + (256 \cdot 2^n - 160)x^2 \\ + (1380 \cdot 2^n - 840)x^3 + (288 \cdot 2^n - 224)x^4$$

$$2D_x Q_{-2} J (D_x + D_y) \\ = (48 \cdot 2^n - 24)x^2 + (512 \cdot 2^n - 320)x^2 \\ + (2760 \cdot 2^n - 1680)x^3 + (576 \cdot 2^n - 448)x^4$$

$$2D_x^2 Q_{-2} J = (16 \cdot 2^n - 8)x^2 + (256 \cdot 2^n - 160)x^2 \\ + (1656 \cdot 2^n - 1008)x^3 + (384 \cdot 2^n - 288)x^4$$

$$D_x^2 = (4 \cdot 2^n - 4)xy^3 + (32 \cdot 2^n - 16)x^2y + (112 \cdot 2^n - 64)x^2y^2 \\ + (368 \cdot 2^n - 224)x^2y^3 + (108 \cdot 2^n - 81)x^3y^3$$

$$D_y^2 = (12 \cdot 2^n - 12)xy^3 + (8 \cdot 2^n - 4)x^2y + (112 \cdot 2^n - 64)x^2y^2 \\ + (828 \cdot 2^n - 504)x^2y^3 + (108 \cdot 2^n - 81)x^3y^3$$

$$D_x^2 + D_y^2 = (16 \cdot 2^n - 16)xy^3 + (40 \cdot 2^n - 20)x^2y \\ + (224 \cdot 2^n - 128)x^2y^2 + (1196 \cdot 2^n - 728)x^2y^3 \\ + (216 \cdot 2^n - 162)x^3y^3$$

$$\begin{aligned}
Q_{-2}J(D_x^2 + D_y^2) \\
&= (40 \cdot 2^n - 20)x + (240 \cdot 2^n - 144)x^2 \\
&\quad + (1196 \cdot 2^n - 728)x^3 + (216 \cdot 2^n - 162)x^4
\end{aligned}$$

$$\begin{aligned}
D_x^2 Q_{-2}J(D_x^2 + D_y^2) \\
&= (40 \cdot 2^n - 20)x + (960 \cdot 2^n - 576)x^2 \\
&\quad + (10764 \cdot 2^n - 6552)x^3 + (3456 \cdot 2^n - 2592)x^4
\end{aligned}$$

$$\begin{aligned}
L_x &= (4 \cdot 2^n - 4)x^2y^3 + (8 \cdot 2^n - 4)x^4y + (28 \cdot 2^n - 16)x^4y^2 \\
&\quad + (92 \cdot 2^n - 56)x^4y^3 + (12 \cdot 2^n - 9)x^6y^3
\end{aligned}$$

$$\begin{aligned}
L_y &= (4 \cdot 2^n - 4)xy^6 + (8 \cdot 2^n - 4)x^2y^2 + (28 \cdot 2^n - 16)x^2y^4 \\
&\quad + (92 \cdot 2^n - 56)x^2y^6 + (12 \cdot 2^n - 9)x^3y^6
\end{aligned}$$

$$\begin{aligned}
J(L_x + L_y) &= (8 \cdot 2^n - 4)x^4 + (12 \cdot 2^n - 8)x^5 + (56 \cdot 2^n - 32)x^6 \\
&\quad + (96 \cdot 2^n - 60)x^7 + (92 \cdot 2^n - 56)x^8 \\
&\quad + (24 \cdot 2^n - 18)x^9
\end{aligned}$$

$$\begin{aligned}
S_x Q_{-2}J(L_x + L_y) \\
&= (4 \cdot 2^n - 2)x^2 + \left(4 \cdot 2^n - \frac{8}{3}\right)x^3 + (14 \cdot 2^n - 8)x^4 \\
&\quad + \left(\frac{96}{5} \cdot 2^n - 12\right)x^5 + \left(\frac{92}{6} \cdot 2^n - \frac{56}{6}\right)x^6 \\
&\quad + \left(\frac{24}{7} \cdot 2^n - \frac{18}{7}\right)x^7
\end{aligned}$$

$$\begin{aligned}
S_x &= (4 \cdot 2^n - 4)xy^3 + (4 \cdot 2^n - 2)x^2y + (14 \cdot 2^n - 8)x^2y^2 \\
&\quad + (46 \cdot 2^n - 28)x^2y^3 + (4 \cdot 2^n - 3)x^3y^3
\end{aligned}$$

$$\begin{aligned}
S_y &= \left(\frac{4}{3} \cdot 2^n - \frac{4}{3}\right)xy^3 + (8 \cdot 2^n - 4)x^2y + (14 \cdot 2^n - 8)x^2y^2 \\
&\quad + \left(\frac{92}{3} \cdot 2^n - \frac{56}{3}\right)x^2y^3 + (4 \cdot 2^n - 3)x^3y^3
\end{aligned}$$

$$\begin{aligned}
S_x + S_y &= \left(\frac{16}{3} \cdot 2^n - \frac{16}{3}\right)xy^3 + (12 \cdot 2^n - 6)x^2y \\
&\quad + (28 \cdot 2^n - 16)x^2y^2 + \left(\frac{230}{3} \cdot 2^n - \frac{140}{3}\right)x^2y^3 \\
&\quad + (8 \cdot 2^n - 6)x^3y^3
\end{aligned}$$

$$\begin{aligned}
Q_{-2}J(S_x + S_y) &= (12 \cdot 2^n - 16)x + \left(\frac{100}{3} \cdot 2^n - \frac{100}{3}\right)x^2 \\
&\quad + \left(\frac{230}{3} \cdot 2^n - \frac{140}{3}\right)x^3 + (8 \cdot 2^n - 6)x^4
\end{aligned}$$

$$\begin{aligned}
S_x Q_{-2}J(S_x + S_y) &= (12 \cdot 2^n - 16)x + \left(\frac{100}{6} \cdot 2^n - \frac{100}{6}\right)x^2 \\
&\quad + \left(\frac{230}{9} \cdot 2^n - \frac{140}{9}\right)x^3 + \left(2 \cdot 2^n - \frac{3}{2}\right)x^4.
\end{aligned}$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as $B_1(G) = 1476 \cdot 2^n - 922$, $B_2(G) = 1948 \cdot 2^n - 1236$, $HB_1(G) = 7900 \cdot 2^n - 4990$, $HB_2(G) = 15220 \cdot 2^n - 9740$, $mB_1(G) = \frac{12592}{210} \cdot 2^n - \frac{1536}{42}$, $mB_2(G) = \frac{1012}{18} \cdot 2^n - \frac{855}{18}$.

Theorem 2. The topological indices for Organosilicon are given by,

$$\begin{aligned}
B_1(G) &= \frac{568}{3} \cdot 3^n - 232, & B_2(G) &= \frac{604}{3} \cdot 3^n - 248 \\
HB_1(G) &= \frac{1208}{3} \cdot 3^n - 496, & HB_2(G) &= \frac{6980}{3} \cdot 3^n - 3160 \\
mB_1(G) &= \frac{554}{135} \cdot 3^n - \frac{487}{90}, & mB_2(G) &= \frac{199}{45} \cdot 3^n - \frac{206}{45}.
\end{aligned}$$

Proof.

The M - polynomial of Organosilicon is of the form

$$M(G; x, y) = (6 \cdot 3^n - 6)x^2y^2 + (4 \cdot 3^n - 4)x^2y^3 \\ + \left(\frac{8}{3} \cdot 3^n - 4\right)x^3y^4.$$

Then,

$$D_x = (12 \cdot 3^n - 12)x^2y^2 + (8 \cdot 3^n - 8)x^2y^3 + (8 \cdot 3^n - 12)x^3y^4$$

$$D_y = (12 \cdot 3^n - 12)x^2y^2 + (12 \cdot 3^n - 12)x^2y^3 \\ + \left(\frac{32}{3} \cdot 3^n - 16\right)x^3y^4$$

$$D_x + D_y = (24 \cdot 3^n - 24)x^2y^2 + (20 \cdot 3^n - 20)x^2y^3 \\ + \left(\frac{56}{3} \cdot 3^n - 28\right)x^3y^4$$

$$J = (6 \cdot 3^n - 6)x^4 + (4 \cdot 3^n - 4)x^5 + \left(\frac{8}{3} \cdot 3^n - 4\right)x^7$$

$$2D_xQ_{-2}J = (24 \cdot 3^n - 24)x^2 + (36 \cdot 3^n - 36)x^3 \\ + \left(\frac{200}{3} \cdot 3^n - 100\right)x^5$$

$$D_xQ_{-2}J(D_x + D_y) \\ = (48 \cdot 3^n - 48)x^2 + (60 \cdot 3^n - 60)x^3 \\ + \left(\frac{280}{3} \cdot 3^n - 140\right)x^5$$

$$2D_xQ_{-2}J(D_x + D_y) \\ = (96 \cdot 3^n - 96)x^2 + (120 \cdot 3^n - 120)x^3 \\ + \left(\frac{560}{3} \cdot 3^n - 280\right)x^5$$

$$2D_x^2Q_{-2}J = (48 \cdot 3^n - 48)x^2 + (108 \cdot 3^n - 108)x^3 \\ + \left(\frac{1000}{3} \cdot 3^n - 500\right)x^5$$

$$D_x^2 = (24 \cdot 3^n - 24)x^2y^2 + (16 \cdot 3^n - 16)x^2y^3 \\ + (24 \cdot 3^n - 36)x^3y^4$$

$$D_y^2 = (24 \cdot 3^n - 24)x^2y^2 + (36 \cdot 3^n - 36)x^2y^3 \\ + \left(\frac{128}{3} \cdot 3^n - 64\right)x^3y^4$$

$$D_x^2 + D_y^2 = (48 \cdot 3^n - 48)x^2y^2 + (52 \cdot 3^n - 52)x^2y^3 \\ + \left(\frac{200}{3} \cdot 3^n - 100\right)x^3y^4$$

$$Q_{-2}J(D_x^2 + D_y^2) \\ = (48 \cdot 3^n - 48)x^2 + (52 \cdot 3^n - 52)x^3 \\ + \left(\frac{200}{3} \cdot 3^n - 100\right)x^5$$

$$D_x^2 Q_{-2}J(D_x^2 + D_y^2) \\ = (192 \cdot 3^n - 192)x^2 + (468 \cdot 3^n - 468)x^3 \\ + \left(\frac{5000}{3} \cdot 3^n - 2500\right)x^5$$

$$L_x = (6 \cdot 3^n - 6)x^4y^2 + (4 \cdot 3^n - 4)x^4y^3 + \left(\frac{8}{3} \cdot 3^n - 4\right)x^6y^4$$

$$L_y = (6 \cdot 3^n - 6)x^2y^4 + (4 \cdot 3^n - 4)x^2y^6 + \left(\frac{8}{3} \cdot 3^n - 4\right)x^3y^8$$

$$J(L_x + L_y) = (12 \cdot 3^n - 12)x^6 + (4 \cdot 3^n - 4)x^7 + (4 \cdot 3^n - 4)x^8 \\ + \left(\frac{8}{3} \cdot 3^n - 4\right)x^{10} + \left(\frac{8}{3} \cdot 3^n - 4\right)x^{11}$$

$$\begin{aligned}
S_x Q_{-2} J(L_x + L_y) &= (3 \cdot 3^n - 3)x^4 + \left(\frac{4}{5} \cdot 3^n - \frac{4}{5}\right)x^5 \\
&+ \left(\frac{2}{3} \cdot 3^n - \frac{2}{3}\right)x^6 + \left(\frac{1}{3} \cdot 3^n - \frac{1}{2}\right)x^8 \\
&+ \left(\frac{8}{27} \cdot 3^n - \frac{4}{9}\right)x^9
\end{aligned}$$

$$S_x = (3 \cdot 3^n - 3)x^2y^2 + (2 \cdot 3^n - 2)x^2y^3 + \left(\frac{8}{9} \cdot 3^n - \frac{4}{3}\right)x^3y^4$$

$$S_y = (3 \cdot 3^n - 3)x^2y^2 + \left(\frac{4}{3} \cdot 3^n - \frac{4}{3}\right)x^2y^3 + \left(\frac{2}{3} \cdot 3^n - 1\right)x^3y^4$$

$$\begin{aligned}
S_x + S_y &= (6 \cdot 3^n - 6)x^2y^2 + \left(\frac{10}{3} \cdot 3^n - \frac{10}{3}\right)x^2y^3 \\
&+ \left(\frac{14}{9} \cdot 3^n - \frac{7}{3}\right)x^3y^4
\end{aligned}$$

$$\begin{aligned}
Q_{-2} J(S_x + S_y) &= (6 \cdot 3^n - 6)x^2 + \left(\frac{10}{3} \cdot 3^n - \frac{10}{3}\right)x^3 \\
&+ \left(\frac{14}{9} \cdot 3^n - \frac{7}{3}\right)x^5
\end{aligned}$$

$$\begin{aligned}
S_x Q_{-2} J(S_x + S_y) &= (3 \cdot 3^n - 3)x^2 + \left(\frac{10}{9} \cdot 3^n - \frac{10}{9}\right)x^3 \\
&+ \left(\frac{14}{45} \cdot 3^n - \frac{7}{45}\right)x^5.
\end{aligned}$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as $B_1(G) = \frac{568}{3} \cdot 3^n - 232$, $B_2(G) = \frac{604}{3} \cdot 3^n - 248$,
 $HB_1(G) = \frac{1208}{3} \cdot 3^n - 496$, $HB_2(G) = \frac{6980}{3} \cdot 3^n - 3160$,
 $mB_1(G) = \frac{554}{135} \cdot 2^n - \frac{487}{90}$, $mB_2(G) = \frac{199}{45} \cdot 2^n - \frac{206}{45}$.

Computation of Topological Coindices with the help of *CoM*-Polynomials on Tetrathiafulvalene and Organosilicon Dendrimers

Theorem 3. The topological indices for Tetrathiafulvalene are given by,

$$\begin{aligned}\overline{B_1}(G) &= 10876 \cdot 2^{2n} + 7232 \cdot 2^n + 2658, & \overline{B_2}(G) &= 16240 \cdot 2^{2n} + 10880 \cdot 2^n + 2658, \\ \overline{HB_1}(G) &= 67560 \cdot 2^{2n} + 42009 \cdot 2^n + 14807, & \overline{HB_2}(G) &= 144544 \cdot 2^{2n} + 92160 \cdot 2^n + 14807, \\ \overline{mB_1}(G) &= \frac{5835}{14} \cdot 2^{2n} + \frac{102387}{420} \cdot 2^n + \frac{18163}{140}, & \overline{mB_2}(G) &= \frac{1049}{3} \cdot 2^{2n} + \frac{10880}{3} \cdot 2^n + \frac{2658}{3}.\end{aligned}$$

Proof.

The *CoM*- polynomial of Tetrathiafulvalene is of the form

$$\begin{aligned}CoM(G; x, y) &= (180 \cdot 2^{2n} + 164 \cdot 2^n + 36)xy^3 \\ &\quad + (48 \cdot 2^{2n} + 72 \cdot 2^n + 36)x^2y \\ &\quad + (32 \cdot 2^{2n} + 32 \cdot 2^n + 44)x^2y^2 \\ &\quad + (240 \cdot 2^{2n} + 212 \cdot 2^n + 120)x^2y^3 \\ &\quad + (450 \cdot 2^{2n} + 213 \cdot 2^n + 37)x^3y^3.\end{aligned}$$

Then,

$$\begin{aligned}D_x &= (180 \cdot 2^{2n} + 164 \cdot 2^n + 36)xy^3 \\ &\quad + (96 \cdot 2^{2n} + 144 \cdot 2^n + 72)x^2y \\ &\quad + (64 \cdot 2^{2n} + 64 \cdot 2^n + 88)x^2y^2 \\ &\quad + (480 \cdot 2^{2n} + 414 \cdot 2^n + 240)x^2y^3 \\ &\quad + (1350 \cdot 2^{2n} + 639 \cdot 2^n + 111)x^3y^3\end{aligned}$$

$$\begin{aligned}D_y &= (540 \cdot 2^{2n} + 492 \cdot 2^n + 108)xy^3 \\ &\quad + (48 \cdot 2^{2n} + 72 \cdot 2^n + 36)x^2y \\ &\quad + (64 \cdot 2^{2n} + 64 \cdot 2^n + 88)x^2y^2 \\ &\quad + (720 \cdot 2^{2n} + 636 \cdot 2^n + 360)x^2y^3 \\ &\quad + (1350 \cdot 2^{2n} + 639 \cdot 2^n + 111)x^3y^3\end{aligned}$$

$$\begin{aligned}
D_x + D_y &= (720 \cdot 2^{2n} + 656 \cdot 2^n + 144)xy^3 \\
&\quad + (144 \cdot 2^{2n} + 216 \cdot 2^n + 108)x^2y \\
&\quad + (128 \cdot 2^{2n} + 128 \cdot 2^n + 176)x^2y^2 \\
&\quad + (1200 \cdot 2^{2n} + 1050 \cdot 2^n + 600)x^2y^3 \\
&\quad + (2700 \cdot 2^{2n} + 1278 \cdot 2^n + 222)x^3y^3
\end{aligned}$$

$$\begin{aligned}
J &= (48 \cdot 2^{2n} + 72 \cdot 2^n + 36)x^3 + (212 \cdot 2^{2n} + 196 \cdot 2^n + 80)x^4 \\
&\quad + (240 \cdot 2^{2n} + 212 \cdot 2^n + 120)x^5 \\
&\quad + (450 \cdot 2^{2n} + 213 \cdot 2^n + 37)x^6
\end{aligned}$$

$$\begin{aligned}
2D_x Q_{-2} J &= (96 \cdot 2^{2n} + 144 \cdot 2^n + 72)x \\
&\quad + (848 \cdot 2^{2n} + 784 \cdot 2^n + 320)x^2 \\
&\quad + (1440 \cdot 2^{2n} + 1272 \cdot 2^n + 720)x^3 \\
&\quad + (3600 \cdot 2^{2n} + 1704 \cdot 2^n + 296)x^4
\end{aligned}$$

$$\begin{aligned}
D_x Q_{-2} J(D_x + D_y) &= (144 \cdot 2^{2n} + 216 \cdot 2^n + 108)x \\
&\quad + (1696 \cdot 2^{2n} + 1568 \cdot 2^n + 640)x^2 \\
&\quad + (3600 \cdot 2^{2n} + 3150 \cdot 2^n + 1800)x^3 \\
&\quad + (10800 \cdot 2^{2n} + 5112 \cdot 2^n + 888)x^4
\end{aligned}$$

$$\begin{aligned}
2D_x Q_{-2} J(D_x + D_y) &= (288 \cdot 2^{2n} + 432 \cdot 2^n + 216)x \\
&\quad + (3392 \cdot 2^{2n} + 3136 \cdot 2^n + 1280)x^2 \\
&\quad + (7200 \cdot 2^{2n} + 6300 \cdot 2^n + 3600)x^3 \\
&\quad + (21600 \cdot 2^{2n} + 10224 \cdot 2^n + 1776)x^4
\end{aligned}$$

$$\begin{aligned}
2D_x^2 Q_{-2} J &= (192 \cdot 2^{2n} + 288 \cdot 2^n + 144)x \\
&\quad + (1696 \cdot 2^{2n} + 1568 \cdot 2^n + 720)x^2 \\
&\quad + (4320 \cdot 2^{2n} + 3816 \cdot 2^n + 2160)x^3 \\
&\quad + (14400 \cdot 2^{2n} + 6816 \cdot 2^n + 1184)x^4
\end{aligned}$$

$$\begin{aligned}
D_x^2 &= (180.2^{2n} + 164.2^n + 36)xy^3 \\
&\quad + (192.2^{2n} + 288.2^n + 144)x^2y \\
&\quad + (128.2^{2n} + 128.2^n + 176)x^2y^2 \\
&\quad + (960.2^{2n} + 828.2^n + 480)x^2y^3 \\
&\quad + (2700.2^{2n} + 1278.2^n + 222)x^3y^3
\end{aligned}$$

$$\begin{aligned}
D_y^2 &= (1620.2^{2n} + 1476.2^n + 324)xy^3 \\
&\quad + (48.2^{2n} + 72.2^n + 36)x^2y \\
&\quad + (128.2^{2n} + 128.2^n + 176)x^2y^2 \\
&\quad + (2160.2^{2n} + 3150.2^n + 1800)x^2y^3 \\
&\quad + (4050.2^{2n} + 1917.2^n + 333)x^3y^3
\end{aligned}$$

$$\begin{aligned}
D_x^2 + D_y^2 &= (1800.2^{2n} + 1640.2^n + 360)xy^3 \\
&\quad + (240.2^{2n} + 360.2^n + 180)x^2y \\
&\quad + (256.2^{2n} + 256.2^n + 352)x^2y^2 \\
&\quad + (3120.2^{2n} + 3978.2^n + 2280)x^2y^3 \\
&\quad + (6750.2^{2n} + 3195.2^n + 555)x^3y^3
\end{aligned}$$

$$\begin{aligned}
Q_{-2}J(D_x^2 + D_y^2) &= (240.2^{2n} + 360.2^n + 180)x \\
&\quad + (2056.2^{2n} + 1896.2^n + 712)x^2 \\
&\quad + (3120.2^{2n} + 3978.2^n + 2280)x^3 \\
&\quad + (6750.2^{2n} + 3195.2^n + 555)x^4
\end{aligned}$$

$$\begin{aligned}
D_x^2 Q_{-2}J(D_x^2 + D_y^2) &= (240.2^{2n} + 360.2^n + 180)x \\
&\quad + (8224.2^{2n} + 7584.2^n + 2848)x^2 \\
&\quad + (28080.2^{2n} + 35802.2^n + 20520)x^3 \\
&\quad + (108000.2^{2n} + 51120.2^n + 8880)x^4
\end{aligned}$$

$$\begin{aligned}
L_x = & (180.2^{2n} + 164.2^n + 36)x^2y^3 \\
& + (48.2^{2n} + 72.2^n + 36)x^4y \\
& + (32.2^{2n} + 32.2^n + 44)x^4y^2 \\
& + (240.2^{2n} + 212.2^n + 120)x^4y^3 \\
& + (450.2^{2n} + 213.2^n + 37)x^6y^3
\end{aligned}$$

$$\begin{aligned}
L_y = & (180.2^{2n} + 164.2^n + 36)xy^6 \\
& + (48.2^{2n} + 72.2^n + 36)x^2y^2 \\
& + (32.2^{2n} + 32.2^n + 44)x^2y^4 \\
& + (240.2^{2n} + 212.2^n + 120)x^2y^6 \\
& + (450.2^{2n} + 213.2^n + 37)x^3y^6
\end{aligned}$$

$$\begin{aligned}
J(L_x + L_y) = & (48.2^{2n} + 72.2^n + 36)x^4 \\
& + (228.2^{2n} + 236.2^n + 72)x^5 \\
& + (514.2^{2n} + 277.2^n + 125)x^6 \\
& + (420.2^{2n} + 376.2^n + 156)x^7 \\
& + (240.2^{2n} + 212.2^n + 120)x^8 \\
& + (450.2^{2n} + 213.2^n + 37)x^9
\end{aligned}$$

$$\begin{aligned}
S_x Q_{-2} J(L_x + L_y) \\
= & (24.2^{2n} + 36.2^n + 18)x^2 \\
& + \left(76.2^{2n} + \frac{236}{3}.2^n + 24\right)x^3 \\
& + \left(\frac{257}{2}2^{2n} + \frac{277}{4}.2^n + \frac{125}{4}\right)x^4 \\
& + \left(84.2^{2n} + \frac{376}{5}.2^n + \frac{156}{5}\right)x^5 \\
& + \left(40.2^{2n} + \frac{212}{6}.2^n + 20\right)x^6 \\
& + \left(\frac{450}{7}.2^{2n} + \frac{213}{7}.2^n + \frac{37}{7}\right)x^7
\end{aligned}$$

$$\begin{aligned}
S_x = & (180.2^{2n} + 164.2^n + 36)xy^3 \\
& + (24.2^{2n} + 36.2^n + 18)x^2y \\
& + (16.2^{2n} + 16.2^n + 22)x^2y^2 \\
& + (120.2^{2n} + 106.2^n + 60)x^2y^3 \\
& + \left(150.2^{2n} + 71.2^n + \frac{37}{3}\right)x^3y^3
\end{aligned}$$

$$\begin{aligned}
S_y = & \left(60.2^{2n} + \frac{164}{3}.2^n + 12\right)xy^3 \\
& + (48.2^{2n} + 72.2^n + 36)x^2y \\
& + (16.2^{2n} + 16.2^n + 22)x^2y^2 \\
& + \left(80.2^{2n} + \frac{212}{3}.2^n + 40\right)x^2y^3 \\
& + \left(150.2^{2n} + 71.2^n + \frac{37}{3}\right)x^3y^3
\end{aligned}$$

$$\begin{aligned}
S_x + S_y = & \left(240.2^{2n} + \frac{656}{3}.2^n + 48\right)xy^3 \\
& + (72.2^{2n} + 108.2^n + 54)x^2y \\
& + (32.2^{2n} + 32.2^n + 44)x^2y^2 \\
& + \left(200.2^{2n} + \frac{530}{3}.2^n + 100\right)x^2y^3 \\
& + \left(300.2^{2n} + 142.2^n + \frac{74}{3}\right)x^3y^3
\end{aligned}$$

$$\begin{aligned}
Q_{-2}J(S_x + S_y) = & (72.2^{2n} + 108.2^n + 54)x^2 \\
& + \left(272.2^{2n} + \frac{752}{3}.2^n + 92\right)x^2 \\
& + \left(200.2^{2n} + \frac{530}{3}.2^n + 100\right)x^3 \\
& + \left(300.2^{2n} + 142.2^n + \frac{74}{3}\right)x^4
\end{aligned}$$

$$\begin{aligned}
S_x Q_{-2} J(S_x + S_y) &= (72 \cdot 2^{2n} + 108 \cdot 2^n + 54)x \\
&+ \left(136 \cdot 2^{2n} + \frac{376}{3} \cdot 2^n + 46\right)x^2 \\
&+ \left(\frac{200}{3} \cdot 2^{2n} + \frac{530}{9} \cdot 2^n + \frac{100}{3}\right)x^3 \\
&+ \left(75 \cdot 2^{2n} + \frac{71}{2} \cdot 2^n + \frac{37}{6}\right)x^4 x^4.
\end{aligned}$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as $\overline{B_1}(G) = 10876 \cdot 2^{2n} + 7232 \cdot 2^n + 2658$, $\overline{B_2}(G) = 16240 \cdot 2^{2n} + 10046 \cdot 2^n + 3436$, $\overline{HB_1}(G) = 67560 \cdot 2^{2n} + 42009 \cdot 2^n + 14807$, $\overline{HB_2}(G) = 144544 \cdot 2^{2n} + 94866 \cdot 2^n + 32428$, $\overline{mB_1}(G) = \frac{5835}{14} \cdot 2^{2n} + \frac{102387}{420} \cdot 2^n + \frac{18163}{140}$, $\overline{mB_2}(G) = \frac{1049}{3} \cdot 2^{2n} + \frac{5899}{18} \cdot 2^n + \frac{279}{2}$.

Theorem 4. The topological coindices for Organosilicon are given by,

$$\begin{aligned}
\overline{B_1}(G) &= 3524 \cdot 3^{2n-2} + 2582 \cdot 3^{n-1} + 526, & \overline{B_2}(G) &= 6644 \cdot 3^{2n-2} - \\
\overline{HB_1}(G) &= 27436 \cdot 3^{2n-2} - 17542 \cdot 3^{n-1} + 3182, & \overline{HB_2}(G) &= 107596 \cdot 3^{2n-2} - \\
\overline{mB_1}(G) &= 619 \cdot 3^{2n-4} + \frac{6437}{10} \cdot 3^{n-3} + \frac{277}{15}, & \overline{mB_2}(G) &= 187 \cdot 3^{2n-2} -
\end{aligned}$$

Proof.

The *CoM*-polynomial of Organosilicon is of the form

$$\begin{aligned}
CoM(G; x, y) &= (2 \cdot 3^{2n} - 17 \cdot 3^n + 21)x^2 y^2 \\
&+ \left(\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14\right)x^2 y^3 \\
&+ \left(\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12\right)x^3 y^4.
\end{aligned}$$

Then,

$$\begin{aligned}
D_x &= (4 \cdot 3^{2n} - 34 \cdot 3^n + 42)x^2y^2 \\
&\quad + \left(\frac{40}{3} \cdot 3^{2n} - \frac{148}{3} \cdot 3^n + 28\right)x^2y^3 \\
&\quad + \left(\frac{480}{9} \cdot 3^{2n} - 80 \cdot 3^n + 36\right)x^3y^4
\end{aligned}$$

$$\begin{aligned}
D_y &= (4 \cdot 3^{2n} - 34 \cdot 3^n + 42)x^2y^2 + (20 \cdot 3^{2n} - 74 \cdot 3^n + 42)x^2y^3 \\
&\quad + \left(\frac{640}{9} \cdot 3^{2n} - \frac{320}{3} \cdot 3^n + 48\right)x^3y^4
\end{aligned}$$

$$\begin{aligned}
D_x + D_y &= (8 \cdot 3^{2n} - 68 \cdot 3^n + 84)x^2y^2 \\
&\quad + \left(\frac{100}{3} \cdot 3^{2n} - \frac{370}{3} \cdot 3^n + 70\right)x^2y^3 \\
&\quad + \left(\frac{1120}{9} \cdot 3^{2n} - \frac{5600}{3} \cdot 3^n + 84\right)x^3y^4
\end{aligned}$$

$$\begin{aligned}
J &= (2 \cdot 3^{2n} - 17 \cdot 3^n + 21)x^4 + \left(\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14\right)x^5 \\
&\quad + \left(\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12\right)x^7
\end{aligned}$$

$$\begin{aligned}
2D_x Q_{-2} J &= (8 \cdot 3^{2n} - 68 \cdot 3^n + 84)x^2 \\
&\quad + \left(40 \cdot 3^{2n} - \frac{148}{3} \cdot 3^n + 84\right)x^3 \\
&\quad + \left(\frac{1600}{9} \cdot 3^{2n} - \frac{800}{3} \cdot 3^n + 120\right)x^5
\end{aligned}$$

$$\begin{aligned}
D_x Q_{-2} J (D_x + D_y) &= (16 \cdot 3^{2n} - 136 \cdot 3^n + 168)x^2 \\
&\quad + (100 \cdot 3^{2n} - 370 \cdot 3^n + 210)x^3 \\
&\quad + \left(\frac{5600}{9} \cdot 3^{2n} - \frac{2800}{3} \cdot 3^n + 420\right)x^5
\end{aligned}$$

$$\begin{aligned}
2D_x Q_{-2} J(D_x + D_y) &= (32 \cdot 3^{2n} - 272 \cdot 3^n + 336)x^2 \\
&+ (200 \cdot 3^{2n} - 740 \cdot 3^n + 420)x^3 \\
&+ \left(\frac{11200}{9} \cdot 3^{2n} - \frac{5600}{3} \cdot 3^n + 840 \right) x^5
\end{aligned}$$

$$\begin{aligned}
2D_x^2 Q_{-2} J &= (16 \cdot 3^{2n} - 68 \cdot 3^n + 84)x^2 \\
&+ (120 \cdot 3^{2n} - 444 \cdot 3^n + 252)x^3 \\
&+ \left(\frac{8000}{9} \cdot 3^{2n} - \frac{4000}{3} \cdot 3^n + 600 \right) x^5
\end{aligned}$$

$$\begin{aligned}
D_x^2 &= (8 \cdot 3^{2n} - 68 \cdot 3^n + 84)x^2 y^2 \\
&+ \left(\frac{80}{3} \cdot 3^{2n} - \frac{296}{3} \cdot 3^n + 56 \right) x^2 y^3 \\
&+ \left(\frac{1440}{9} \cdot 3^{2n} - 240 \cdot 3^n + 108 \right) x^3 y^4
\end{aligned}$$

$$\begin{aligned}
D_y^2 &= (8 \cdot 3^{2n} - 68 \cdot 3^n + 84)x^2 y^2 \\
&+ (60 \cdot 3^{2n} - 222 \cdot 3^n + 126)x^2 y^3 \\
&+ \left(\frac{2560}{9} \cdot 3^{2n} - \frac{1280}{3} \cdot 3^n + 192 \right) x^3 y^4
\end{aligned}$$

$$\begin{aligned}
D_x^2 + D_y^2 &= (16 \cdot 3^{2n} - 136 \cdot 3^n + 168)x^2 y^2 \\
&+ \left(\frac{260}{3} \cdot 3^{2n} - \frac{962}{3} \cdot 3^n + 182 \right) x^2 y^3 \\
&+ \left(\frac{4000}{9} \cdot 3^{2n} - \frac{2000}{3} \cdot 3^n + 300 \right) x^3 y^4
\end{aligned}$$

$$\begin{aligned}
Q_{-2} J(D_x^2 + D_y^2) &= (16 \cdot 3^{2n} - 136 \cdot 3^n + 168)x^2 \\
&+ \left(\frac{260}{3} \cdot 3^{2n} - \frac{962}{3} \cdot 3^n + 182 \right) x^3 \\
&+ \left(\frac{4000}{9} \cdot 3^{2n} - \frac{2000}{3} \cdot 3^n + 300 \right) x^5
\end{aligned}$$

$$\begin{aligned}
D_x^2 Q_{-2} J(D_x^2 + D_y^2) &= (64 \cdot 3^{2n} - 544 \cdot 3^n + 672)x^2 \\
&+ (780 \cdot 3^{2n} - 2886 \cdot 3^n + 546)x^3 \\
&+ \left(\frac{100000}{9} \cdot 3^{2n} - \frac{500000}{3} \cdot 3^n + 7500 \right) x^5
\end{aligned}$$

$$\begin{aligned}
L_x &= (2 \cdot 3^{2n} - 17 \cdot 3^n + 21)x^4 y^2 \\
&+ \left(\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14 \right) x^4 y^3 \\
&+ \left(\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12 \right) x^6 y^4
\end{aligned}$$

$$\begin{aligned}
L_y &= (2 \cdot 3^{2n} - 17 \cdot 3^n + 21)x^2 y^4 \\
&+ \left(\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14 \right) x^2 y^6 \\
&+ \left(\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12 \right) x^3 y^8
\end{aligned}$$

$$\begin{aligned}
J(L_x + L_y) &= (4 \cdot 3^{2n} - 34 \cdot 3^n + 42)x^6 \\
&+ \left(\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14 \right) x^7 \\
&+ \left(\frac{20}{3} \cdot 3^{2n} - \frac{74}{3} \cdot 3^n + 14 \right) x^8 \\
&+ \left(\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12 \right) x^{10} \\
&+ \left(\frac{160}{9} \cdot 3^{2n} - \frac{80}{3} \cdot 3^n + 12 \right) x^{11}
\end{aligned}$$

$$\begin{aligned}
S_x Q_{-2} J(L_x + L_y) &= \left(3 \cdot 3^{2n} - \frac{34}{3} \cdot 3^n + \frac{42}{4}\right) x^4 \\
&+ \left(\frac{4}{3} \cdot 3^{2n} - \frac{74}{15} \cdot 3^n + \frac{14}{5}\right) x^7 \\
&+ \left(\frac{10}{3} \cdot 3^{2n} - \frac{74}{18} \cdot 3^n + \frac{7}{3}\right) x^8 \\
&+ \left(\frac{20}{9} \cdot 3^{2n} - \frac{10}{3} \cdot 3^n + \frac{3}{2}\right) x^{10} \\
&+ \left(\frac{160}{81} \cdot 3^{2n} - \frac{80}{27} \cdot 3^n + \frac{4}{3}\right) x^{11}
\end{aligned}$$

$$\begin{aligned}
S_x &= \left(3 \cdot 3^{2n} - \frac{17}{2} \cdot 3^n + \frac{21}{2}\right) x^2 y^2 \\
&+ \left(\frac{10}{3} \cdot 3^{2n} - \frac{37}{3} \cdot 3^n + 7\right) x^2 y^3 \\
&+ \left(\frac{160}{27} \cdot 3^{2n} - \frac{80}{9} \cdot 3^n + 4\right) x^3 y^4
\end{aligned}$$

$$\begin{aligned}
S_y &= \left(3 \cdot 3^{2n} - \frac{17}{2} \cdot 3^n + \frac{21}{2}\right) x^2 y^2 \\
&+ \left(\frac{20}{9} \cdot 3^{2n} - \frac{74}{9} \cdot 3^n + \frac{14}{3}\right) x^2 y^3 \\
&+ \left(\frac{40}{9} \cdot 3^{2n} - \frac{20}{3} \cdot 3^n + 3\right) x^3 y^4
\end{aligned}$$

$$\begin{aligned}
S_x + S_y &= (6 \cdot 3^{2n} - 17 \cdot 3^n + 21) x^2 y^2 \\
&+ \left(\frac{50}{9} \cdot 3^{2n} - \frac{185}{9} \cdot 3^n + \frac{35}{3}\right) x^2 y^3 \\
&+ \left(\frac{280}{27} \cdot 3^{2n} - \frac{140}{9} \cdot 3^n + 7\right) x^3 y^4
\end{aligned}$$

$$\begin{aligned}
Q_{-2}J(S_x + S_y) &= (6 \cdot 3^{2n} - 17 \cdot 3^n + 21)x^2 \\
&+ \left(\frac{50}{9} \cdot 3^{2n} - \frac{185}{9} \cdot 3^n + \frac{35}{3}\right)x^3 \\
&+ \left(\frac{280}{27} \cdot 3^{2n} - \frac{140}{9} \cdot 3^n + 7\right)x^5
\end{aligned}$$

$$\begin{aligned}
S_x Q_{-2}J(S_x + S_y) &= \left(3 \cdot 3^{2n} - \frac{17}{2} \cdot 3^n + 21\right)x^2 \\
&+ \left(\frac{50}{27} \cdot 3^{2n} - \frac{185}{27} \cdot 3^n + \frac{35}{9}\right)x^3 \\
&+ \left(\frac{56}{27} \cdot 3^{2n} - \frac{28}{9} \cdot 3^n + \frac{7}{5}\right)x^5
\end{aligned}$$

Hence it is easy to calculate the given topological indices $x = 1 = y = 1$, as

$$\begin{aligned}
B_1(G) &= 3524 \cdot 3^{2n-2} + 2582 \cdot 3^{n-1} + 526, \overline{B_2}(G) = 6644 \cdot 3^{2n-2} - 4318 \cdot 3^{n-1} + 798, \\
\overline{HB_1}(G) &= 27436 \cdot 3^{2n-2} - 17542 \cdot 3^{n-1} + 3182, \overline{HB_2}(G) = 107596 \cdot 3^{2n-2} - 60290 \cdot 3^{n-1} + 8718, \overline{mB_1}(G) = 619 \cdot 3^{2n-4} - \frac{6437}{10} \cdot 3^{n-3} + \frac{277}{15}, \\
\overline{mB_2}(G) &= 187 \cdot 3^{2n-3} - \frac{997}{2} \cdot 3^{n-3} + \frac{1421}{90}.
\end{aligned}$$

Numerical and Graphical Comparison

In this section we aim to show the relation between related Banhatti indices and coindices perceptibly, via numerical and graphical comparison figures in Figure 4 and Figure 5. It is seen that in each graphic, new defined Banhatti topological coindices increase/decreases rapidly than existing Banhatti topological indices. For Tetrathiafulvalene dendrimer, each graphic has increasing curvilinear model for both related indices and coindices. But for Organosilicon dendrimer, while only (modified) first K Banhatti indices and coindices have been increasing at the same time, the other ones are of decreased coindex lines while indices are of increased lines. These results are of guiding significance to the engineering application.

Figure 4. Numerical and Graphical Comparison of Organosilicon

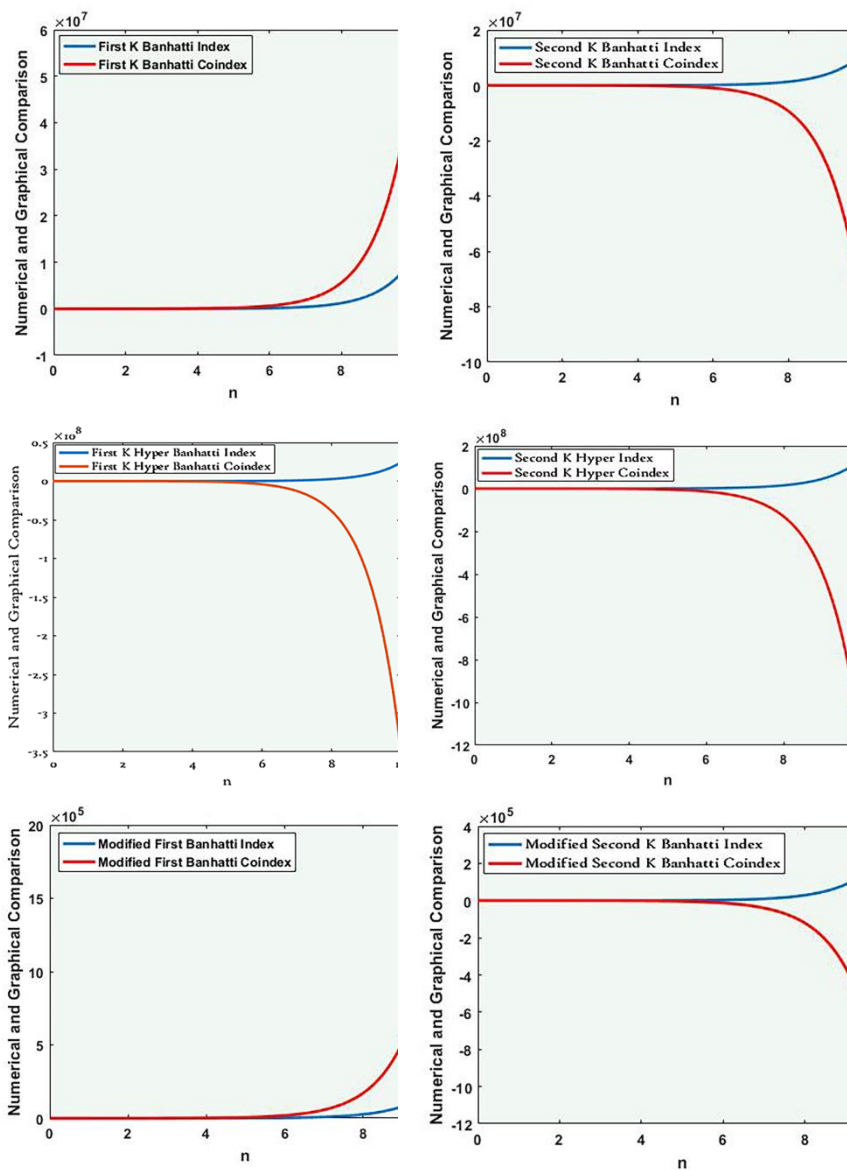
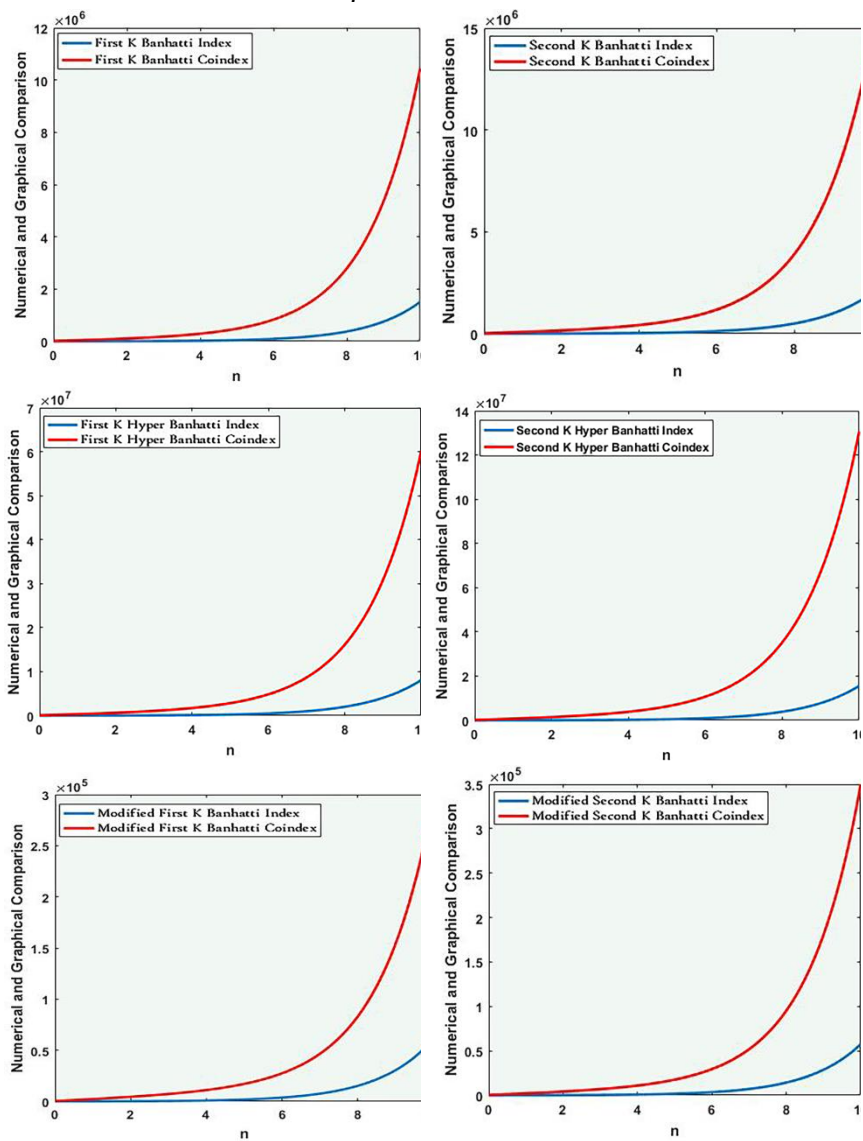


Figure 5. Numerical and Graphical Comparison of
Tetrathiafulvalene Dendrimer



6. Summarization and Conclusion

Topological indices are widely used calculation tools to predict physicochemical and bioactivity properties of chemical compounds. They have many applications in medical and material sciences via QSPR/QSAR analysis to design high performance, low cost products. Nowadays researchers have been interested in studies on optical applications via topological approach. Also, dendrimers are investigated as an effective conductor because of its more branched structure. Motivated by these, in this work we handle six types of Banhatti indices and we define topological coindex versions via complement graph theory. Afterward, we produce the M and CoM -polynomials of two specific dendritic compounds from molecular graphs of each compound via vertex and each partition technique. By algebraic polynomial approach, we compute all indices and coindices. At the end we give comparisonal figures showing that the performance between topological indices and coindices. We hope these representations and numerical data may be helpful for testing the efficiency of optical applications of dendrimers in the future.

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CHAPTER VI

On the Higher Order Leonardo Quaternions

Kübra GÜL¹

Introduction

Number sequences have an important place and their applications in various scientific fields in the literature (Abrate, Barbero, Cerruti, & Murru, 2014; Amannah & Nanwin, 2014; Falcon, & Plaza, 2007; Koshy, 2001; Shannon, Deveci & Erdağ, 2019). The most famous of these sequences are the Fibonacci sequences $\{F_n\}_{n=0}^{\infty}$ and Lucas sequences $\{L_n\}_{n=0}^{\infty}$ defined by, $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}$$

and

$$L_n = L_{n-1} + L_{n-2}$$

, where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

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In (Catarino & Borges, 2019), Leonardo sequence $\{\mathcal{L}_n\}_{n=0}^{\infty}$ is defined by

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1 \quad n \geq 2, \quad \mathcal{L}_0 = \mathcal{L}_1 = 1.$$

Also, Leonardo sequence is given by the equation

$$\mathcal{L}_{n+1} = 2\mathcal{L}_n - \mathcal{L}_{n-2}.$$

The characteristic equation of the equation (4) is

$$\lambda^3 - 2\lambda^2 + 1 = 0.$$

The Binet's formula of the Leonardo sequence is

$$\mathcal{L}_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta} \quad (1)$$

where α and β are roots of characteristic equation.

There are the relationships between Leonardo numbers, Fibonacci numbers and Lucas numbers as follows:

$$\begin{aligned} \mathcal{L}_n &= 2F_{n+1} - 1, \\ \mathcal{L}_{n-1} + \mathcal{L}_{n+1} &= 2L_{n+1} - 2, \\ \mathcal{L}_n + 2F_n &= \mathcal{L}_{n+1}, \\ \mathcal{L}_n + F_n + L_n &= 2\mathcal{L}_n + 1, \\ \mathcal{L}_{n+1}^2 + \mathcal{L}_n^2 &= 2(\mathcal{L}_{2n+2} - \mathcal{L}_{n+2} + 1). \end{aligned}$$

Several identities for Leonardo numbers were obtained by authors. Furthermore, they provided Leonardo numbers with a matrix representation. Some recent research on Leonardo numbers can be seen in (Alp & Koçer, 2021; Catarino & Borges, 2019).

In recent years, several authors investigated on higher order sequences associated with the well known number sequences. The study of higher order sequences began with the earlier work of Randić et al. (Randić, Morales & Araujo, 1996) where the authors investigated higher order Fibonacci numbers and its various algebraic properties. Cook et al. (Cook & Bacon, 2013) defined Jacobsthal higher order numbers. In (Prasad, Kumari, Mohanta, & Mahato, 2023), the authors introduced Mersenne higher order numbers and gave various algebraic properties. In the literature, there are several studies on higher order sequences associated with the different sequences and their generalizations. Gül (Gül, 2023) introduced higher order Leonardo numbers. Kizilates and Kone (Kizilates & Kone, 2021a) expanded the quaternion algebraic

research to higher order Fibonacci numbers. Then, the authors investigated hyper complex numbers and quaternions with the higher order sequences (Gül, 2022; Kizilates & Kone, 2021b; Ozimamoğlu, 2023; Özkan & Uysal, 2023).

Quaternions were investigated by Hamilton (Hamilton, 1866) as an extension to the complex numbers. A quaternion is defined by

$$q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad (2)$$

where a_0, a_1, a_2, a_3 are real numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are quaternionic units which satisfy the following rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1 \text{ and } \mathbf{i}\mathbf{j} = \mathbf{k} = -\mathbf{j}\mathbf{i}, \mathbf{j}\mathbf{k} = \mathbf{i} = -\mathbf{k}\mathbf{j}, \mathbf{k}\mathbf{i} = \mathbf{j} = -\mathbf{i}\mathbf{k}. \quad (3)$$

Several authors studied on different quaternions and their generalizations, some of which can be found in (Asci & Aydinyuz, 2021; Gül, 2019; Gül, 2020; Halici, 2012 Horadam, 1963). Pauli Leonardo quaternion was defined by Leonardo numbers coefficients (Isbilir & Tosun, 2023). Nurkan et al. (Nurkan & Güven, 2023) introduced ordered Leonardo quadruple numbers by dual quaternions and Leonardo numbers. Then, Yılmaz et al. (Yılmaz & Saçlı, 2023) studied the dual quaternions with the k -generalized Leonardo sequence.

In this study, we introduce a new type of quaternions, called the higher order Leonardo quaternions. We consider the coefficients of these quaternions as the higher order Leonardo numbers. We give Binet formula, generating function, several identities for this sequence.

Main Results

For positive integer k , the higher order Leonardo numbers $\{\mathcal{L}_n^{(k)}\}$ are defined by

$$\mathcal{L}_n^{(k)} = \frac{\mathcal{L}_{kn}}{\mathcal{L}_k}, \quad n = 0, 1, 2, \dots \quad (4)$$

In (Gül, 2023), the Binet's formula of the higher order Leonardo numbers $\mathcal{L}_n^{(k)}$ can be written as

$$\mathcal{L}_n^{(k)} = \frac{2\alpha^{kn+1} - 2\beta^{kn+1} - \alpha + \beta}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta}. \quad (5)$$

In (Mangueira, Alves & Catarino, 2022), Leonardo quaternion, denoted by $\mathcal{L}Q_n$, are defined by

$$\mathcal{L}Q_n = \mathcal{L}_n + \mathcal{L}_{n+1}\mathfrak{i} + \mathcal{L}_{n+2}\mathfrak{j} + \mathcal{L}_{n+3}\mathfrak{k}.$$

The Binet's formula of the Leonardo quaternions

$$\mathcal{L}Q_n = \frac{2\alpha^{n+1}\hat{\alpha} - 2\beta^{n+1}\hat{\beta} + w(-\alpha + \beta)}{\alpha - \beta} \quad (6)$$

where $\hat{\alpha} = 1 + \alpha^k\mathfrak{i} + \alpha^{2k}\mathfrak{j} + \alpha^{3k}\mathfrak{k}$, $\hat{\beta} = 1 + \beta^k\mathfrak{i} + \beta^{2k}\mathfrak{j} + \beta^{3k}\mathfrak{k}$ and $w = 1 + \mathfrak{i} + \mathfrak{j} + \mathfrak{k}$.

Definition 1. The higher order Leonardo quaternions $\{\mathcal{L}Q_n^{(k)}\}$, $n \in \mathbb{R}$, is defined as

$$\mathcal{L}Q_n^{(k)} = \mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k} \quad (7)$$

where $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ are quaternionic units and $\mathcal{L}_n^{(k)}$ is n th the higher order Leonardo numbers.

If it is taken as $k = 1$, the higher-order Leonardo quaternions $\mathcal{L}Q_n^{(1)}$ is called as the Leonardo quaternions.

The real and imaginary parts of the higher-order Leonardo quaternions are as follows:

$$\text{Re}(\mathcal{L}Q_n^{(k)}) = \mathcal{L}_n^{(k)},$$

$$\text{Im}(\mathcal{L}Q_n^{(k)}) = \mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k}$$

, respectively.

The conjugate of the higher-order Leonardo quaternion, denoted by

$$\overline{\mathcal{L}Q_n^{(k)}}, \text{ is given by } \overline{\mathcal{L}Q_n^{(k)}} = \mathcal{L}_n^{(k)} - \mathcal{L}_{n+1}^{(k)}\mathfrak{i} - \mathcal{L}_{n+2}^{(k)}\mathfrak{j} - \mathcal{L}_{n+3}^{(k)}\mathfrak{k}.$$

The norm of the higher-order Leonardo quaternion, denoted by $N(\mathcal{L}Q_n^{(k)})$, is given by

$$\begin{aligned} N(\mathcal{L}Q_n^{(k)}) &= \mathcal{L}Q_n^{(k)} \overline{\mathcal{L}Q_n^{(k)}} \\ &= (\mathcal{L}_n^{(k)})^2 + (\mathcal{L}_{n+1}^{(k)})^2 + (\mathcal{L}_{n+2}^{(k)})^2 + (\mathcal{L}_{n+3}^{(k)})^2. \end{aligned}$$

Proposition 2. The higher-order Leonardo quaternions satisfy the following identity:

$$(\mathcal{L}Q_n^{(k)})^2 = -\mathcal{L}Q_n^{(k)} \overline{\mathcal{L}Q_n^{(k)}} + 2\mathcal{L}_n^{(k)} \mathcal{L}Q_n^{(k)}.$$

Proof. From the definition of $\mathcal{L}Q_n^{(k)}$, we have

$$\begin{aligned} (\mathcal{L}Q_n^{(k)})^2 &= (\mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k})(\mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k}) \\ &= (\mathcal{L}_n^{(k)})^2 - (\mathcal{L}_{n+1}^{(k)})^2 - (\mathcal{L}_{n+2}^{(k)})^2 - (\mathcal{L}_{n+3}^{(k)})^2 + \\ &\quad 2\mathcal{L}_n^{(k)}(\mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k}) \\ &= (\mathcal{L}_n^{(k)})^2 - (\mathcal{L}_{n+1}^{(k)})^2 - (\mathcal{L}_{n+2}^{(k)})^2 - (\mathcal{L}_{n+3}^{(k)})^2 + \\ &\quad 2\mathcal{L}_n^{(k)}(\mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k}) \\ &= -(\mathcal{L}_n^{(k)})^2 - (\mathcal{L}_{n+1}^{(k)})^2 - (\mathcal{L}_{n+2}^{(k)})^2 - (\mathcal{L}_{n+3}^{(k)})^2 + \\ &\quad 2\mathcal{L}_n^{(k)}(\mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k}) \\ &= -\mathcal{L}Q_n^{(k)} \overline{\mathcal{L}Q_n^{(k)}} + 2\mathcal{L}_n^{(k)} \mathcal{L}Q_n^{(k)}. \end{aligned}$$

Theorem 3. The Binet's formula of the higher order Leonardo quaternions $\mathcal{L}_n^{(k)}$ is given by

$$\mathcal{L}Q_n^{(k)} = \frac{2\alpha^{kn+1}\hat{\alpha} - 2\beta^{kn+1}\hat{\beta} + w(-\alpha + \beta)}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \quad (8)$$

where $\hat{\alpha} = 1 + \alpha^k\mathfrak{i} + \alpha^{2k}\mathfrak{j} + \alpha^{3k}\mathfrak{k}$, $\hat{\beta} = 1 + \beta^k\mathfrak{i} + \beta^{2k}\mathfrak{j} + \beta^{3k}\mathfrak{k}$ and $w = 1 + \mathfrak{i} + \mathfrak{j} + \mathfrak{k}$.

Proof. Using the equations (6) and (7), we obtain

$$\begin{aligned}
\mathcal{L}Q_n^{(k)} &= \mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)}\mathfrak{i} + \mathcal{L}_{n+2}^{(k)}\mathfrak{j} + \mathcal{L}_{n+3}^{(k)}\mathfrak{k} \\
&= \frac{2\alpha^{kn+1}-2\beta^{kn+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} + \frac{2\alpha^{kn+k+1}-2\beta^{kn+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathfrak{i} + \\
&\frac{2\alpha^{kn+2k+1}-2\beta^{kn+2k+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathfrak{j} + \frac{2\alpha^{kn+3k+1}-2\beta^{kn+3k+1}-\alpha+\beta}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}\mathfrak{k} \\
&= \frac{1}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} (2\alpha^{kn+1}(1 + \alpha^k\mathfrak{i} + \alpha^{2k}\mathfrak{j} + \\
&\alpha^{3k}\mathfrak{k}) - 2\beta^{kn+1}(1 + \beta^k\mathfrak{i} + \beta^{2k}\mathfrak{j} + \beta^{3k}\mathfrak{k}) - \alpha + \beta(1 + \mathfrak{i} + \mathfrak{j} + \mathfrak{k})) \\
&= \frac{2\alpha^{kn+1}\hat{\alpha}-2\beta^{kn+1}\hat{\beta}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}.
\end{aligned}$$

Theorem 4. For the higher order Leonardo quaternions, the following relation holds:

$$\mathcal{L}Q_{n+1}^{(k)} = \frac{1}{2}(\mathcal{L}_k\mathcal{L}Q_n^{(k)} - (-1)^k\mathcal{L}Q_{n-1}^{(k)} + \frac{1}{5}\mathcal{L}_k^{-1}[-4K_{kn+k} + 2(-1)^{k+1}K_{kn-k+1}] + \mathcal{L}Q_n^{(k)} + w(1 - \mathcal{L}_k^{-1}(3 - (-1)^k))).$$

where K_n is the n th Lucas quaternion.

Proof. From the Binet formulas (1) and (8), we obtain as follows:

$$\begin{aligned}
&\frac{\mathcal{L}_k\mathcal{L}Q_n^{(k)} - (-1)^k\mathcal{L}Q_{n-1}^{(k)}}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \frac{2\alpha^{kn+1}\hat{\alpha}-2\beta^{kn+1}\hat{\beta}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} - \\
&(\alpha\beta)^k \frac{2\alpha^{kn-k+1}\hat{\alpha}-2\beta^{kn-k+1}\hat{\beta}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \\
&= \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)} \left(\hat{\alpha}(4\alpha^{kn+k+2} - 4\alpha^{kn+1}\beta^{k+1} - \right. \\
&2\alpha^{kn+2} + 2\beta\alpha^{kn+1} - 2\beta^k\alpha^{kn+2} + 2\beta^{k+1}\alpha^{kn+1}) + \\
&\hat{\beta}(4\beta^{kn+k+2} - 4\alpha^{k+1}\beta^{kn+1} - 2\beta^{kn+2} + 2\alpha\beta^{kn+1} + \\
&2\beta^{kn+1}\alpha^{k+1} - 2\beta^{kn+2}\alpha^k) + w(-\alpha + \beta)(2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \\
&\beta - \alpha^{k+1}\beta^k + \alpha^k\beta^{k+1}) \Big) \\
&= \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)} \left(2(\alpha - \beta)(2\hat{\alpha}\alpha^{kn+k+1} - \right. \\
&2\hat{\beta}\beta^{kn+k+1}) + 4\hat{\beta}\alpha\beta^{kn+k+1} + 4\hat{\alpha}\beta\alpha^{kn+k+1} -
\end{aligned}$$

$$\begin{aligned}
& 2\hat{\alpha}(\alpha\beta)^{k+1}\alpha^{kn-k} + 2\hat{\alpha}(\alpha\beta)^{k+1}\alpha^{kn-k+2} - 2\hat{\alpha}\alpha^{kn+1}(\alpha - \beta) - \\
& 2\hat{\beta}(\alpha\beta)^{k+1}\beta^{kn-k} + 2\hat{\beta}(\alpha\beta)^{k+1}\beta^{kn-k+2} - 2\hat{\beta}\beta^{kn+1}(-\alpha + \beta) + \\
& w(-\alpha + \beta)\left(2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta - (\alpha\beta)^k(\alpha - \beta)\right) \\
& = \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)}\left(2(\alpha - \beta)\left(2\hat{\alpha}\alpha^{kn+k+1} - \right.\right. \\
& \left.2\hat{\beta}\beta^{kn+k+1} + w(-\alpha + \beta)\right) - 4(\hat{\beta}\beta^{kn+k} + \hat{\alpha}\alpha^{kn+k}) - \\
& 2(\alpha\beta)^{k+1}(\hat{\alpha}\alpha^{kn-k} + \hat{\beta}\beta^{kn-k}) + 2(\alpha\beta)^{k+1}(\hat{\alpha}\alpha^{kn-k+2} + \\
& \hat{\beta}\beta^{kn-k+2}) - 2(\alpha - \beta)(\hat{\alpha}\alpha^{kn+1} - \hat{\beta}\beta^{kn+1}) + w(-\alpha + \\
& \beta)\left(2\alpha^{k+1} - 2\beta^{k+1} - 3\alpha + 3\beta - (\alpha\beta)^k(\alpha - \beta)\right) \\
& = 2\frac{2\hat{\alpha}\alpha^{kn+k+1}-2\hat{\beta}\beta^{kn+k+1}+w(-\alpha+\beta)}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)} - \\
& \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)}\left[\frac{-4(\hat{\beta}\beta^{kn+k}+\hat{\alpha}\alpha^{kn+k})}{(\alpha-\beta)} - \right. \\
& \left.2(\alpha\beta)^{k+1}\frac{(\hat{\alpha}\alpha^{kn-k}+\hat{\beta}\beta^{kn-k})}{(\alpha-\beta)} + 2(\alpha\beta)^{k+1}\frac{\hat{\alpha}\alpha^{kn-k+2}+\hat{\beta}\beta^{kn-k+2}}{\alpha-\beta}\right] - \\
& \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)(\alpha-\beta)}\left[(\alpha - \beta)\left(2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1} + \right.\right. \\
& \left.w(-\alpha + \beta)\right) + w(-\alpha + \beta)(2\alpha^{k+1} - 2\beta^{k+1} - 4\alpha + 4\beta - \\
& (\alpha\beta)^k(\alpha - \beta)) \\
& = 2\mathcal{L}Q_{n+1}^{(k)} - \frac{1}{5}\mathcal{L}_k^{-1}[-4K_{kn+k} - 2(-1)^{k+1}(K_{kn-k} - K_{kn-k+2})] - \\
& \mathcal{L}Q_n^{(k)} - w(1 - \mathcal{L}_k^{-1}(3 - (-1)^k)).
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathcal{L}Q_{n+1}^{(k)} &= \frac{1}{2}(\mathcal{L}_k\mathcal{L}Q_n^{(k)} - (-1)^k\mathcal{L}Q_{n-1}^{(k)} + \frac{1}{5}\mathcal{L}_k^{-1}[-4K_{kn+k} + \\
& 2(-1)^{k+1}K_{kn-k+1}] + \mathcal{L}Q_n^{(k)} + w(1 - \mathcal{L}_k^{-1}(3 - (-1)^k))).
\end{aligned}$$

Theorem 5. The generating function of the higher-order Leonardo quaternions is given by

$$\begin{aligned}
& G(x, k) \\
& = \frac{\mathcal{L}_k^{-1}(\mathcal{L}Q_0 + (-\mathcal{L}Q_0 - \mathcal{L}Q_{k-2} + w(L_k - 2))x) + (-\mathcal{L}Q_{k-2} - w((-1)^k + 1)x^2)}{1 - (1 + L_k)x - (L_k + (-1)^k)x^2 - (-1)^kx^3}.
\end{aligned}$$

Proof. Suppose that the generating function for the the higher-order Leonardo quaternions is

$$G(x, k) = \sum_{n=0}^{\infty} \mathcal{L}Q_n^{(k)} x^n = \mathcal{L}Q_0^{(k)} + \mathcal{L}Q_1^{(k)} x + \mathcal{L}Q_2^{(k)} x^2 + \dots + \mathcal{L}Q_n^{(k)} x^n + \dots.$$

By using Binet's formula of $\mathcal{L}_n^{(k)}$ (5), we can rewrite the generating function as follows:

$$\begin{aligned} G(x, k) &= \sum_{n=0}^{\infty} \mathcal{L}Q_n^{(k)} x^n = \sum_{n=0}^{\infty} (\mathcal{L}_n^{(k)} + \mathcal{L}_{n+1}^{(k)} \mathbb{I} + \mathcal{L}_{n+2}^{(k)} \mathbb{J} + \mathcal{L}_{n+3}^{(k)} \mathbb{K}) x^n \\ &= \sum_{n=0}^{\infty} \left(\frac{2\alpha^{kn+1} - 2\beta^{kn+1} - \alpha + \beta}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} + \frac{2\alpha^{kn+k+1} - 2\beta^{kn+k+1} - \alpha + \beta}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \mathbb{I} + \right. \\ &\quad \left. \frac{2\alpha^{kn+2k+1} - 2\beta^{kn+2k+1} - \alpha + \beta}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \mathbb{J} + \frac{2\alpha^{kn+3k+1} - 2\beta^{kn+3k+1} - \alpha + \beta}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \mathbb{K} \right) \\ &= \frac{1}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} (2 \sum_{n=0}^{\infty} \alpha^{kn+1} (1 + \alpha^k \mathbb{I} + \alpha^{2k} \mathbb{J} + \alpha^{3k} \mathbb{K}) x^n - 2 \sum_{n=0}^{\infty} \beta^{kn+1} (1 + \beta^k \mathbb{I} + \beta^{2k} \mathbb{J} + \beta^{3k} \mathbb{K}) x^n + (-\alpha + \beta) \sum_{n=0}^{\infty} (1 + \mathbb{I} + \mathbb{J} + \mathbb{K}) x^n) \\ &= \frac{1}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} (2\alpha \hat{\alpha} \sum_{n=0}^{\infty} (\alpha^k x)^n - 2\beta \hat{\beta} \sum_{n=0}^{\infty} (\beta^k x)^n + (-\alpha + \beta) w \sum_{n=0}^{\infty} x^n) \\ &= \frac{1}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \left(\frac{2\alpha \hat{\alpha}}{1 - \alpha^k x} - \frac{2\beta \hat{\beta}}{1 - \beta^k x} + (-\alpha + \beta) w \frac{1}{1 - x} \right) \\ &= \frac{1}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \frac{(2\alpha \hat{\alpha} (1 - \beta^k x) (1 - x) - 2\beta \hat{\beta} (1 - \alpha^k x) (1 - x) + (-\alpha + \beta) w (1 - \alpha^k x) (1 - \beta^k x))}{1 - (1 + \alpha^k + \beta^k) x - (\alpha^k + \beta^k + (\alpha \beta)^k) x^2 - (\alpha \beta)^k x^3} \\ &= \frac{1}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \frac{2\alpha \hat{\alpha} - 2\beta \hat{\beta} + (-\alpha + \beta) w + (-2\alpha \hat{\alpha} - 2\hat{\alpha} \alpha \beta^k + 2\hat{\beta} \beta + 2\hat{\beta} \beta \alpha^k + w(-\alpha + \beta)(-\alpha^k - \beta^k)) x}{1 - (1 + \alpha^k + \beta^k) x - (\alpha^k + \beta^k + (\alpha \beta)^k) x^2 - (\alpha \beta)^k x^3} \\ &\quad + \frac{1}{2\alpha^{k+1} - 2\beta^{k+1} - \alpha + \beta} \frac{(2\hat{\alpha} \alpha \beta^k - 2\hat{\beta} \beta \alpha^k + (-\alpha + \beta) w \alpha^k \beta^k) x^2}{1 - (1 + \alpha^k + \beta^k) x - (\alpha^k + \beta^k + (\alpha \beta)^k) x^2 - (\alpha \beta)^k x^3} \\ &= \frac{\mathcal{L}_k^{-1} (\mathcal{L}Q_0 + (-\mathcal{L}Q_0 + (-1)^{k+1} \mathcal{L}Q_{-k+1} + w(L_k + (-1)^{k+1} - 1)) x + ((-1)^k \mathcal{L}Q_{-k+1} + w((-1)^k - (-1)^k)) x^2)}{1 - (1 + L_k) x - (L_k + (-1)^k) x^2 - (-1)^k x^3} \end{aligned}$$

$$= \frac{\mathcal{L}_k^{-1}(\mathcal{L}Q_0 + (-\mathcal{L}Q_0 + (-1)^{k+1}\mathcal{L}Q_{-k+1} + w(L_k + (-1)^{k+1} - 1))x + (-1)^k\mathcal{L}Q_{-k+1}x^2)}{1 - (1 + L_k)x - (L_k + (-1)^k)x^2 - (-1)^kx^3}.$$

Theorem 6. The sum of the higher-order Leonardo quaternions is given by

$$\sum_{n=0}^{\infty} \mathcal{L}Q_n^{(k)} = \frac{w\mathcal{L}_k^{-1}(L_k - (-1)^k - 1)}{-2(L_k + (-1)^k)}.$$

Proof. If we take for $x = 1$ in Theorem 5, the proof is completed.

Lemma 7. Let $\hat{\alpha} = 1 + \alpha^k\mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k}$, $\hat{\beta} = 1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k}$ and $w = 1 + \mathbb{i} + \mathbb{j} + \mathbb{k}$. Then, there are the following equations

$$\hat{\alpha}\hat{\beta} = u - lv \quad (9)$$

$$\hat{\beta}\hat{\alpha} = u + lv \quad (10)$$

where $u = -2(-1)^k + L_k\mathbb{i} + L_{2k}\mathbb{j} + L_{3k}\mathbb{k}$, $v = \mathbb{i} - (-1)^kL_k\mathbb{j} + (-1)^k\mathbb{k}$ and $l = \alpha^k - \beta^k$.

Proof. By using the equations (2) and (3), we obtain as follows:

$$\begin{aligned} \hat{\alpha}\hat{\beta} &= (1 + \alpha^k\mathbb{i} + \alpha^{2k}\mathbb{j} + \alpha^{3k}\mathbb{k})(1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k}) \\ &= 1 + \beta^k\mathbb{i} + \beta^{2k}\mathbb{j} + \beta^{3k}\mathbb{k} + \alpha^k\mathbb{i} - (\alpha\beta)^k + \alpha^k\beta^{2k}\mathbb{i}\mathbb{j} + \\ &\quad \alpha^k\beta^{3k}\mathbb{i}\mathbb{k} + \alpha^{2k}\mathbb{j} + \alpha^{2k}\beta^k\mathbb{j}\mathbb{i} - \alpha^{2k}\beta^{2k} + \alpha^{2k}\beta^{3k}\mathbb{j}\mathbb{k} + \alpha^{3k}\mathbb{k} + \\ &\quad \alpha^{3k}\beta^k\mathbb{k}\mathbb{i} + \alpha^{3k}\beta^{2k}\mathbb{k}\mathbb{j} - \alpha^{3k}\beta^{3k} \\ &= 1 - (-1)^k - (-1)^{2k} - (-1)^{3k} + (\beta^k + \alpha^k + \alpha^{2k}\beta^{3k} - \\ &\quad \alpha^{3k}\beta^{2k})\mathbb{i} + (\alpha^{2k} + \beta^{2k} - \alpha^k\beta^{3k} + \alpha^{3k}\beta^k)\mathbb{j} + (\alpha^{3k} + \beta^{3k} + \\ &\quad \alpha^k\beta^{2k} - \alpha^{2k}\beta^k)\mathbb{k} \\ &= -(-1)^k - (-1)^{3k} + (\beta^k + \alpha^k - \alpha^{2k}\beta^{2k}(\alpha^k - \beta^k))\mathbb{i} + \\ &\quad (\alpha^{2k} + \beta^{2k} + \alpha^k\beta^k(\alpha^{2k} - \beta^{2k}))\mathbb{j} + (\alpha^{3k} + \beta^{3k} - \alpha^k\beta^k(\alpha^k - \beta^k))\mathbb{k} \end{aligned}$$

$$\begin{aligned}
&= -2(-1)^k + (\alpha^k + \beta^k)\mathfrak{i} + (\alpha^{2k} + \beta^{2k})\mathfrak{j} + (\alpha^{3k} + \beta^{3k})\mathfrak{k} \\
&\quad - (\alpha^{2k}\beta^{2k}(\alpha^k - \beta^k)\mathfrak{i} - \alpha^k\beta^k(\alpha^{2k} - \beta^{2k})\mathfrak{j} \\
&\quad + \alpha^k\beta^k(\alpha^k - \beta^k)\mathfrak{k}) \\
&= -2(-1)^k + L_k\mathfrak{i} + L_{2k}\mathfrak{j} + L_{3k}\mathfrak{k} - (\alpha^k - \beta^k)(\mathfrak{i} - \\
&(-1)^k L_k\mathfrak{j} + (-1)^k \mathfrak{k})) \\
&= u - lv \\
\hat{\beta}\hat{\alpha} &= (1 + \beta^k\mathfrak{i} + \beta^{2k}\mathfrak{j} + \beta^{3k}\mathfrak{k})(1 + \alpha^k\mathfrak{i} + \alpha^{2k}\mathfrak{j} + \alpha^{3k}\mathfrak{k}) \\
&= 1 + \alpha^k\mathfrak{i} + \alpha^{2k}\mathfrak{j} + \alpha^{3k}\mathfrak{k} + \beta^k\mathfrak{i} - (\alpha\beta)^k + \alpha^{2k}\beta^k\mathfrak{i}\mathfrak{j} + \\
&\alpha^{3k}\beta^k\mathfrak{i}\mathfrak{k} + \beta\mathfrak{j}^{2k} + \alpha^k\beta^{2k}\mathfrak{j}\mathfrak{i} - \alpha^{2k}\beta^{2k} + \alpha^{3k}\beta^{2k}\mathfrak{j}\mathfrak{k} + \beta^{3k}\mathfrak{k} + \\
&\alpha^k\beta^{3k}\mathfrak{k}\mathfrak{i} + \alpha^{2k}\beta^{3k}\mathfrak{k}\mathfrak{j} - \alpha^{3k}\beta^{3k} \\
&= 1 - (-1)^k - (-1)^{2k} - (-1)^{3k} + (\alpha^k + \beta^k \\
&\quad + (-1)^{2k}(\alpha^k - \beta^k))\mathfrak{i} + (\alpha^{2k} + \beta^{2k} \\
&\quad - (-1)^k(\alpha^{2k} - \beta^{2k})\mathfrak{j} + (\alpha^{3k} + \beta^{3k} + \alpha^k\beta^k(\alpha^k \\
&\quad - \beta^k))\mathfrak{k} \\
&= -2(-1)^k + (\alpha^k + \beta^k)\mathfrak{i} + (\alpha^{2k} + \beta^{2k})\mathfrak{j} + (\alpha^{3k} + \\
&\beta^{3k})\mathfrak{k} + ((-1)^{2k}(\alpha^k - \beta^k)\mathfrak{i} - (-1)^k(\alpha^{2k} - \beta^{2k})\mathfrak{j} + (-1)^k(\alpha^k - \\
&\beta^k)\mathfrak{k}) \\
&= -2(-1)^k + L_k\mathfrak{i} + L_{2k}\mathfrak{j} + L_{3k}\mathfrak{k} + (\alpha^k - \beta^k)((\mathfrak{i} - \\
&(-1)^k L_k\mathfrak{j} + (-1)^k \mathfrak{k})) \\
&= u + lv.
\end{aligned}$$

Theorem 8. For any $n, m, r \in \mathbb{Z}$, Vajda identity for the higher order Leonardo quaternions is given by

$$\begin{aligned}
&\mathcal{L}Q_{n+m}^{(k)}\mathcal{L}Q_{n+r}^{(k)} - \mathcal{L}Q_n^{(k)}\mathcal{L}Q_{n+m+r}^{(k)} = \\
&\mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{km}(uF_{kr} + vF_{kr}L_{kr}) - w(\mathcal{L}Q_{kn+kr} - \\
&\mathcal{L}Q_{kn+km+kr}) + (\mathcal{L}Q_{kn+km} - \mathcal{L}Q_{kn})w).
\end{aligned}$$

Proof. Left side of the equation is rewritten with the help of the Binet formula of $\mathcal{L}Q_n^{(k)}$ as follows:

$$\begin{aligned}
& \mathcal{L}Q_{n+m}^{(k)} \mathcal{L}Q_{n+r}^{(k)} = \\
& \frac{2\hat{\alpha}\alpha^{kn+km+1}-2\hat{\beta}\beta^{kn+km+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \cdot \frac{2\hat{\alpha}\alpha^{kn+kr+1}-2\hat{\beta}\beta^{kn+kr+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \\
& = \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)^2} \left(4\hat{\alpha}\hat{\alpha}\alpha^{2kn+km+kr+2} + \right. \\
& 4\hat{\beta}\hat{\beta}\beta^{2kn+km+kr+2} - 4\hat{\alpha}\hat{\beta}\alpha^{kn+km+1}\beta^{kn+kr+1} - \\
& 4\hat{\beta}\hat{\alpha}\beta^{kn+km+1}\alpha^{kn+kr+1} + 2\hat{\alpha}w\alpha^{kn+km+1}(-\alpha+\beta) - \\
& 2\hat{\beta}w\beta^{kn+km+1}(-\alpha+\beta) + w(-\alpha+\beta)(2\hat{\alpha}\alpha^{kn+kr+1} - \\
& \left. 2\hat{\beta}\beta^{kn+kr+1}) + w^2(-\alpha+\beta)^2 \right). \quad (11)
\end{aligned}$$

$$\begin{aligned}
& \mathcal{L}Q_n^{(k)} \mathcal{L}Q_{n+m+r}^{(k)} = \\
& \frac{2\hat{\alpha}\alpha^{kn+1}-2\hat{\beta}\beta^{kn+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \cdot \frac{2\hat{\alpha}\alpha^{kn+km+kr+1}-2\hat{\beta}\beta^{kn+km+kr+1}+w(-\alpha+\beta)}{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta} \\
& = \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)^2} \left(4\hat{\alpha}\hat{\alpha}\alpha^{2kn+km+kr+2} + \right. \\
& 4\hat{\beta}\hat{\beta}\beta^{2kn+km+kr+2} - 4\hat{\alpha}\hat{\beta}\alpha^{kn+1}\beta^{kn+km+kr+1} - \\
& 4\hat{\beta}\hat{\alpha}\beta^{kn+1}\alpha^{kn+km+kr+1} + 2\hat{\alpha}w\alpha^{kn+1}(-\alpha+\beta) - \\
& 2\hat{\beta}w\beta^{kn+1}(-\alpha+\beta) + w(-\alpha+\beta)(2\hat{\alpha}\alpha^{kn+km+kr+1} - \\
& \left. 2\hat{\beta}\beta^{kn+km+kr+1}) + w^2(-\alpha+\beta)^2 \right). \quad (12)
\end{aligned}$$

Subtracting the equation (11) from the equation (12), we obtain

$$\begin{aligned}
& \mathcal{L}Q_{n+m}^{(k)} \mathcal{L}Q_{n+r}^{(k)} - \mathcal{L}Q_n^{(k)} \mathcal{L}Q_{n+m+r}^{(k)} = \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)^2} - \\
& 4(\alpha\beta)^{kn+1} \left(\hat{\alpha}\hat{\beta}\beta^{kr}(\alpha^{km} - \beta^{km}) - \hat{\beta}\hat{\alpha}\alpha^{kr}(\alpha^{km} - \beta^{km}) \right) + \\
& 2w(-\alpha+\beta) \left((\hat{\alpha}\alpha^{kn+kr+1} - \hat{\beta}\beta^{kn+kr+1}) - (\hat{\alpha}\alpha^{kn+km+kr+1} - \right. \\
& \left. \hat{\beta}\beta^{kn+km+kr+1}) \right) + 2(-\alpha+\beta) [(\hat{\alpha}\alpha^{kn+km+1} - 2\hat{\beta}\beta^{kn+km+1})w - \\
& (2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1})w] \\
& = \frac{1}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)^2} \left(-4(-1)^{kn+1}(\alpha^{km} - \beta^{km})(\hat{\alpha}\hat{\beta}\beta^{kr} - \right. \\
& \hat{\beta}\hat{\alpha}\alpha^{kr}) + 2(-\alpha+\beta)[w \left((\hat{\alpha}\alpha^{kn+kr+1} - \hat{\beta}\beta^{kn+kr+1}) - \right. \\
& \left. (\hat{\alpha}\alpha^{kn+km+kr+1} - \hat{\beta}\beta^{kn+km+kr+1}) \right) + (\hat{\alpha}\alpha^{kn+km+1} - \\
& \left. 2\hat{\beta}\beta^{kn+km+1})w - (2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1})w] \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\alpha-\beta)^2}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)^2} \left(-4(-1)^{kn+1} \frac{(\alpha^{km}-\beta^{km})}{(\alpha-\beta)^2} (u\beta^{kr} - \right. \\
&lv\beta^{kr} - u\alpha^{kr} - lv\alpha^{kr}) + 2(-\alpha + \beta) \frac{1}{(\alpha-\beta)^2} [w \left((\hat{\alpha}\alpha^{kn+kr+1} - \right. \\
&\hat{\beta}\beta^{kn+kr+1} - \alpha + \beta) - (\hat{\alpha}\alpha^{kn+km+kr+1} - \hat{\beta}\beta^{kn+km+kr+1} - \alpha + \beta) \left. \right) + (\hat{\alpha}\alpha^{kn+km+1} - 2\hat{\beta}\beta^{kn+km+1} - \alpha + \beta)w - (2\hat{\alpha}\alpha^{kn+1} - \\
&2\hat{\beta}\beta^{kn+1} - \alpha + \beta)w] \left. \right) \\
&= \frac{(\alpha-\beta)^2}{(2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta)^2} \left(-4(-1)^{kn+1} \frac{(\alpha^{km}-\beta^{km})}{(\alpha-\beta)^2} (-u(\alpha^{kr} - \right. \\
&\beta^{kr}) - lv(\alpha^{kr} + \beta^{kr})) + 2(-\alpha + \beta) \frac{1}{(\alpha-\beta)^2} [w \left((\hat{\alpha}\alpha^{kn+kr+1} - \right. \\
&\hat{\beta}\beta^{kn+kr+1} + w(-\alpha + \beta)) - (\hat{\alpha}\alpha^{kn+km+kr+1} - \hat{\beta}\beta^{kn+km+kr+1} + \\
&w(-\alpha + \beta)) \left. \right) + (\hat{\alpha}\alpha^{kn+km+1} - 2\hat{\beta}\beta^{kn+km+1} + w(-\alpha + \beta))w - \\
&(2\hat{\alpha}\alpha^{kn+1} - 2\hat{\beta}\beta^{kn+1} + w(-\alpha + \beta))w] \left. \right).
\end{aligned}$$

By using the equations (1), (6), (9) and (10), we have

$$\begin{aligned}
&\mathcal{L}Q_{n+m}^{(k)}\mathcal{L}Q_{n+r}^{(k)} - \mathcal{L}Q_n^{(k)}\mathcal{L}Q_{n+m+r}^{(k)} = \\
&\mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{km}(uF_{kr} + vF_{kr}L_{kr}) - w(\mathcal{L}Q_{kn+kr} - \\
&\mathcal{L}Q_{kn+km+kr}) + (\mathcal{L}Q_{kn+km} - \mathcal{L}Q_{kn})w).
\end{aligned}$$

Theorem 9. For any $r \leq n \in \mathbb{Z}$, Catalan's identity for the higher order Leonardo quaternions is

$$\mathcal{L}Q_{n-r}^{(k)}\mathcal{L}Q_{n+r}^{(k)} - \left(\mathcal{L}Q_n^{(k)}\right)^2 = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{-kr}(uF_{kr} + vF_{kr}L_{kr}) - w(\mathcal{L}Q_{kn+kr} - \mathcal{L}Q_{kn}) + (\mathcal{L}Q_{kn-kr} - \mathcal{L}Q_{kn})w).$$

Proof. The proof of Catalan's identity is done for the special case $m = -r$ of Vajda identity.

Theorem 10. Cassini's identity for the higher order Leonardo quaternions is

$$\mathcal{L}Q_{n-1}^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \left(\mathcal{L}Q_n^{(k)}\right)^2 = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{-k}(uF_k + vF_kL_k) - w(\mathcal{L}Q_{kn+k} - \mathcal{L}Q_{kn}) + (\mathcal{L}Q_{kn-k} - \mathcal{L}Q_{kn})w).$$

Proof. When it is taken as $r = 1$ and $m = -1$ in Vajda identity, we have

$$\mathcal{L}Q_{n-1}^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \left(\mathcal{L}Q_n^{(k)}\right)^2 = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{-k}(uF_k + vF_kL_k) - w(\mathcal{L}Q_{kn+k} - \mathcal{L}Q_{kn}) + (\mathcal{L}Q_{kn-k} - \mathcal{L}Q_{kn})w).$$

Theorem 11. D’Ocagne’s identity for the higher-order Leonardo quaternions is

$$\mathcal{L}Q_k^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \mathcal{L}Q_n^{(k)}\mathcal{L}Q_{k+1}^{(k)} = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{k(k-n)}(uF_k + vF_kL_k) - w(\mathcal{L}Q_{k(n+1)} - \mathcal{L}Q_{k(k+1)}) + (\mathcal{L}Q_{k^2} - \mathcal{L}Q_{kn})w).$$

Proof. If we take as $m + n = k$ and $r = 1$ in Vajda identity, we obtain as follows:

$$\mathcal{L}Q_k^{(k)}\mathcal{L}Q_{n+1}^{(k)} - \mathcal{L}Q_n^{(k)}\mathcal{L}Q_{k+1}^{(k)} = \mathcal{L}_k^{-2}(4(-1)^{kn+1}F_{k(k-n)}(uF_k + vF_kL_k) - w(\mathcal{L}Q_{k(n+1)} - \mathcal{L}Q_{k(k+1)}) + (\mathcal{L}Q_{k^2} - \mathcal{L}Q_{kn})w).$$

Conclusion

In this paper, we have defined the higher order Leonardo quaternions with the higher order Leonardo number components. We have introduced basic definitions and properties of these quaternions. Then, we have obtained fundamental properties such as Binet formula and generating function. Moreover, we have proved several identities such as Vajda identity, Catalan’s identity, Cassini’s identity and d’Ocagne identity for the higher order Leonardo quaternions by using Binet formula.

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CHAPTER VII

Uniqueness of Coupled Common Fixed Point under a Symmetric Contraction in Ordered GV-FMS

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1. Introduction

In 1965, Zadeh [1] innovated the notion of fuzzy sets that lead the beginning of a new era providing quick headways into different branches of mathematics and its areas of applications. Particularly, metric space has been fuzzified in several inequivalent ways resulting into different definitions of fuzzy metric space [2 - 6].

The fuzzy version of the most celebrated “Banach contraction principle” in fuzzy metric spaces in the sense of Kramosi and Michalek (in short, KM) was presented by Grabiec [7]. The results proved by Fang [8] improved, generalized and unified the works of Edelstein [9], Istratescu [10], Sehgal and Bharucha-Reid [11].

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George and Veeramani [6, 12] modified the concept of fuzzy metric space due to Kramosil and Michalek [5] and thereby obtained Hausdorff topology in fuzzy metric space. After wards, various authors established several fixed point results in complete fuzzy metric spaces in the sense of George and Veeramani (GV) [6, 12], one can see ([13] – [14]). In the present work, we consider the definition of the fuzzy metric space introduced by George and Veeramani [6].

The work of Bhaskar and Lakshmikantham [15] is worth mentioning, as they introduced the new notion of fixed points for the mappings having domain the product space $X \times X$, which they called coupled fixed points, and thereby proved some coupled fixed point theorems for mappings satisfying the mixed monotone property in partially ordered metric spaces. As an application, they discussed the existence and uniqueness of a solution for a periodic boundary value problem. Lakshmikantham and Ćirić [16] extended the notion of the mixed monotone property to the mixed g -monotone property and generalized the results of Bhaskar and Lakshmikantham [15] by establishing the existence of coupled coincidence points, using a pair of commutative maps. This proved to be a foundation stone in the development of fixed point theory with applications to partially ordered sets. Since then much work has been done in this direction by different authors. For more details the reader may consult ([17-28]). Recently, the problems concerning the computation of coupled fixed points in metric space has been fuzzified, see [29–34].

In this paper, we establish coupled coincidence point and coupled fixed point results for a pair of mappings under the assumption of a new inequality in the setting of fuzzy metric space having Hadžić type t -norm. As an application, a corresponding result in the metric space is also obtained. As a matter of fact, in our results we do not require the pair of the mappings to be commuting or compatible nor we have assumed the self mapping $g: X \rightarrow X$ to be monotonic, further, the completeness of the space X has also been

replaced by the completeness of the range subspaces of any one of the mappings F or g , hence our results generalize a number of existing results present in the literature in general. Also, various existing results on coupled coincidence points and coupled fixed points are extended, for details one can refer to [23, 38, 39].

2. Preliminaries

Definition 2.1 [6]. “The 3-tuple $(X, M, *)$ is called a GV-FMS, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(FMJ-1) $M(x, y, 0) > 0$,

(FMJ-2) $M(x, y, t) = 1$ iff $x = y$,

(FMJ-3) $M(x, y, t) = M(y, x, t)$,

(FMJ-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FMJ-5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y, z \in X$ and $s, t > 0$ ”.

Remark 2.2. (i) In present work, we consider $(X, M, *)$ be a GV-FMS with a Hadžić type t-norm, $M(x, y, t) \rightarrow 1$ as $t \rightarrow \infty$ for all $x, y \in X$ and $\preceq$ be a partial order defined on X , then, we write it as $(X, M, *, \preceq)$ and in short, we call it as PO-GV-FMS.

(ii) Unless otherwise stated we consider the mappings $\mathcal{P}: X \times X \rightarrow X$, $g: X \rightarrow X$.

Definition 2.3[6]. Let $(X, M, *)$ be a GV-FMS, then

(i) “a sequence $\{x_n\}$ in X is said to be Convergent to a point $x \in X$, if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$, for all $t > 0$ ”;

(ii) “a sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$ ”;

(iii) “a fuzzy metric space in which every Cauchy sequence is convergent is said to be complete”.

Definition 2.4 ([16]). “Let $(X, \preceq)$ be a partially ordered set (POS) and $\mathcal{P}, g$ be the mappings. We say $\mathcal{P}$ has the mixed g -monotone property (shortly, MgMP) if $\mathcal{P}(x, y)$ is monotone g -non-decreasing in its first argument and is monotone g -non-increasing in its second argument”.

Taking g to be identity mapping in above definition, we can obtain the definition of mixed monotone property (shortly, MMP).

Definition 2.5 ([16]). “An element $(x, y) \in X \times X$, is called a coupled coincidence point (CCP) of the mappings $\mathcal{P}$ and g if $\mathcal{P}(x, y) = gx$ and $\mathcal{P}(y, x) = gy$ ”.

Taking g to be the identity mapping in the above mapping, we can obtain the definition of coupled fixed point (CFP).

Definition 2.6 ([16]). “An element $(x, y) \in X \times X$, is called a coupled common fixed point (CCFP) of the mappings $\mathcal{P}$ and g if $x = gx = \mathcal{P}(x, y)$ and $y = gy = \mathcal{P}(y, x)$ ”.

Lemma 2.7 [31]. “Let $(X, M, *)$ be a fuzzy metric space with a Hadžić type t -norm $*$ such that $M(x, y, t) \rightarrow 1$ as $t \rightarrow \infty$ for all $x, y \in X$. If the sequences $\{x_n\}$ and $\{y_n\}$ in X are such that, for all $n \geq 1$, $t > 0$, $M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t) \geq M(x_{n-1}, x_n, t/k) * M(y_{n-1}, y_n, t/k)$ where $0 < k < 1$, then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences”.

Definition 2.8. Let (X, d) be a metric space endowed with a partial ordering $(\preceq)$. We say that X is regular if it satisfies the following property:

- (i) “if a non-decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \preceq x$ for all $n \geq 0$ ”; (1.1)
- (ii) “if a non-decreasing sequence $\{y_n\} \rightarrow y$, then $y \preceq y_n$ for all $n \geq 0$ ”. (1.2)

Remark 2.9. Note that if $(X, M, *)$ be a GV-FMS endowed with a partial ordering $(\leq)$. We say that X is regular if it satisfies properties (1.1) and (1.2) with respect to the convergence in a GV-FMS.

3. Main Results

This section focuses on main coupled coincidence point results in fuzzy metric spaces.

Theorem 3.1. Let $(X, M, *, \leq)$ be a PO-GV-FMS. Let (P, g) be pair of mappings such that P has MgMP and satisfies the following conditions:

(X - 1) $P(X \times X) \subseteq g(X)$;

(X - 2) one of the range subspaces $P(X \times X)$ or $g(X)$ is complete;

(X - 3) there exists k in $(0, 1)$ such that

$$\begin{aligned} M(P(x, y), P(\mu, \nu), kt) * M(P(y, x), P(\nu, \mu), kt) \\ \geq M(g(x), g(\mu), t) * M(g(y), g(\nu), t), \end{aligned}$$

for all x, y, μ, ν in X and $t > 0$ with $g(x) \leq g(\mu)$ and $g(y) \geq g(\nu)$ (or, $g(x) \geq g(\mu)$ and $g(y) \leq g(\nu)$);

(X - 4) there exist two elements x_0, y_0 in X such that the following satisfies:

$$g(x_0) \leq P(x_0, y_0) \text{ and } g(y_0) \geq P(y_0, x_0) \quad (\text{or, } g(x_0) \geq P(x_0, y_0) \text{ and } g(y_0) \leq P(y_0, x_0));$$

Further, suppose either

- (a) both the mappings P and g are continuous, or
- (b) X is regular;

then, the (P, g) has a CCP in X .

Proof. Without loss of generality, by (X - 4) suppose that x_0, y_0 in X be such that $g(x_0) \leq P(x_0, y_0)$ and $g(y_0) \geq P(y_0, x_0)$. Since $P(X \times X) \subseteq g(X)$ and P has MgMP so inductively, the sequences $\{x_n\}$ and $\{y_n\}$ can be constructed in X such that

$$g(x_{n+1}) = P(x_n, y_n) \text{ and } g(y_{n+1}) = P(y_n, x_n) \text{ for all } n \geq 0. \quad (3.1)$$

Further, by using (3.1) and since P has $MgMP$, inductively, it can be shown that

$$g(x_n) \leq g(x_{n+1}) \text{ and } g(y_n) \geq g(y_{n+1}) \text{ for all } n \geq 0. \quad (3.2)$$

We suppose either $g(x_{n+1}) = P(x_n, y_n) \neq g(x_n)$ or $g(y_{n+1}) = P(y_n, x_n) \neq g(y_n)$ for all $n \geq 0$, otherwise, then we can obtain directly the CCP of the pair (P, g) .

Using (3.1) and (3.2), from $(X - 3)$, for $t > 0, n > 0$, we have

$$\begin{aligned} & M(g(x_n), g(x_{n+1}), kt) * M(g(y_n), g(y_{n+1}), kt) \\ &= M(P(x_{n-1}, y_{n-1}), P(x_n, y_n), kt) * M(P(y_{n-1}, x_{n-1}), P(y_n, x_n), kt) \\ &\geq M(g(x_{n-1}), g(x_n), t) * M(g(y_{n-1}), g(y_n), t) \\ &\geq M(g(x_{n-1}), g(x_n), t) * M(g(y_{n-1}), g(y_n), t). \end{aligned} \quad (3.3)$$

Using (3.3), by applying Lemma 7, the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ behaves like Cauchy sequences. Without loss of generality, suppose $g(X)$ is complete, so there exist x, y in X such that for all $t > 0$, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} M(g(x), g(x_n), t) &= \lim_{n \rightarrow \infty} M(g(x), P(x_n, y_n), t) = 1, \\ \lim_{n \rightarrow \infty} M(g(y), g(y_n), t) &= \lim_{n \rightarrow \infty} M(g(y), P(y_n, x_n), t) = 1. \end{aligned} \quad (3.4)$$

Suppose, condition (a) holds.

Define a multifunction $G: g(X) \rightarrow 2^X$ by $G(y) = \{x \in X: g(x) = y\}$. Making use of Axiom of Choice, a function $h: g(X) \rightarrow X$, so that $h(y)$ belongs to $G(y)$ for all y in $g(X)$ can be constructed, yielding $g(h(y)) = y$ for all y in $g(X)$. Consider $E = \{h(y): y \in g(X)\} \subseteq X$. Clearly, the map $g: E \rightarrow X$ is injective with $g(E) = G(X)$.

Define another map $H: g(E) \times g(E) \rightarrow X$ by

$$H(g(e), g(f)) = P(e, f) \text{ for } g(e), g(f) \text{ in } g(E) (=g(X)). \quad (3.5)$$

Since $g: E \rightarrow X$ is injective, the map H is well-defined. By (3.4) and (3.5),

$$\begin{aligned}\lim_{n \rightarrow \infty} H(g(x_n), g(y_n)) &= \lim_{n \rightarrow \infty} P(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = g(x), \\ \lim_{n \rightarrow \infty} H(g(y_n), g(x_n)) &= \lim_{n \rightarrow \infty} P(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = g(y).\end{aligned}\quad (3.6)$$

Using the continuity of P and g , the continuity of the map H can be achieved, so by (3.6) we have

$$H(g(x), g(y)) = g(x) \quad \text{and} \quad H(g(y), g(x)) = g(y). \quad (3.7)$$

On combining (3.5) with (3.7), we see that $P(x, y) = g(x)$ and $P(y, x) = g(y)$.

Next, we show the result for the condition (b).

By (3.2) and (3.4), we have

$$g(x_n) \leq g(x) \quad \text{and} \quad g(y) \leq g(y_n) \quad \text{for } n \geq 0. \quad (3.8)$$

Suppose that $(g(x_n), g(y_n)) \neq (g(x), g(y))$ for $n \geq 0$ otherwise (x_n, y_n) is a CCP of the pair (P, g) . Now, for $t > 0$, $n \geq 0$, we obtain

$$M(P(x, y), g(x), t) \geq M(P(x, y), P(x_n, y_n), kt) * M(P(x_n, y_n), g(x), t - kt), \quad (3.9)$$

$$M(P(y, x), g(y), t) \geq M(P(y, x), P(y_n, x_n), kt) * M(P(y_n, x_n), g(y), t - kt). \quad (3.10)$$

From (3.9) and (3.10), for all $t > 0$, we obtain that

$$\begin{aligned}& M(P(x, y), g(x), t) * M(P(y, x), g(y), t) \\& \geq [M(P(x, y), P(x_n, y_n), kt) * M(P(x_n, y_n), g(x), t - kt)] \\& * [M(P(y, x), P(y_n, x_n), kt) * M(P(y_n, x_n), g(y), t - kt)] \\& = [M(P(x, y), P(x_n, y_n), kt) * M(P(y, x), P(y_n, x_n), kt)] \\& * [M(P(x_n, y_n), g(x), t - kt) * M(P(y_n, x_n), g(y), t - kt)].\end{aligned}\quad (3.11)$$

Since $g(x_n) \leq g(x)$ and $g(y) \leq g(y_n)$ for all $n \geq 0$, using (X - 3), we obtain that

$$\begin{aligned}& M(P(x, y), P(x_n, y_n), kt) * M(P(y, x), P(y_n, x_n), kt) \\& \geq M(g(x), g(x_n), t) * M(g(y), g(y_n), t).\end{aligned}\quad (3.12)$$

By (3.11) and (3.12), we obtain that

$$M(P(x, y), g(x), t) * M(P(y, x), g(y), t)$$

$$\geq (M(g(x), g(x_n), t) * M(F(g(y), g(y_n), t)) \\ * [M(P(x_n, y_n), g(x), t - kt) * M(P(y_n, x_n), g(y), t - kt)]). \quad (3.13)$$

Letting limit as $n \rightarrow \infty$ in (3.13), for $t > 0$, we obtain that

$$M(P(x, y), g(x), t) * M(P(y, x), g(y), t) \\ \geq \lim_{n \rightarrow \infty} [M(g(x), g(x_n), t) * M(F(g(y), g(y_n), t))] \\ * \lim_{n \rightarrow \infty} [M(P(x_n, y_n), g(x), t - kt) * M(P(y_n, x_n), g(y), t - kt)].$$

By (3.4),

$$M(P(x, y), g(x), t) * M(P(y, x), g(y), t) \\ \geq (1 * 1) * (1 * 1) = 1.$$

That is, $M(P(x, y), g(x), t) * M(P(y, x), g(y), t) \geq 1$.

Therefore, $M(P(x, y), g(x), t) = 1$ and $M(P(y, x), g(y), t) = 1$, which implies $g(x) = P(x, y)$ and $g(y) = P(y, x)$.

Hence, we obtained the result.

Next, we give some examples in support of Theorem 3.1.

Example 3.2. Let $(X, \leq)$ is the POS with $X = [0, 1)$ with $\leq$ being PO $\leq$, the natural ordering of real numbers. Define $M(x, y, t) = \frac{t}{t + |x - y|}$ for x, y in X and $t > 0$ and $x * y = \min\{x, y\}$ for all x, y in $[0, 1]$. Then $(X, M, *, \leq)$ be a PO-GV-FMS, which is not complete. Also, X is regular. Define the mappings $P: X \times X \rightarrow X$ and $g: X \rightarrow X$ respectively by

$$P(x, y) = 0.4 \text{ for all } (x, y) \text{ in } X \times X$$

and

$$g(x) = \begin{cases} 0.7 & \text{if } 0 \leq x < 0.5, \\ x - 0.3 & \text{if } 0.5 \leq x < 1. \end{cases}$$

Since $g(P(x, y)) = g(0.4) = 0.7 \neq 0.4 = P(g(x), g(y))$ for all x, y in X , the mappings P and g are neither commutative nor compatible.

The following observations are immediate:

$$(i) \quad P(X \times X) \subseteq g(X);$$

- (ii) $g(X)$ is complete;
- (iii) the map g is not continuous;
- (iv) $\mathcal{P}$ has $MgMP$;
- (v) the map g is not monotonic.

Further, there exist $x_0 = 0.3$ and $y_0 = 0.5$ such that $g(x_0) = g(0.3) = 0.7 \geq 0.4 = \mathcal{P}(0.3, 0.5) = \mathcal{P}(x_0, y_0)$ and $g(y_0) = g(0.5) = 0.2 \leq 0.4 = \mathcal{P}(0.5, 0.3) = \mathcal{P}(y_0, x_0)$.

Also, the inequality $(X - 3)$ holds, since for any k in $(0, 1)$, $t > 0$

$$M(\mathcal{P}(x, y), \mathcal{P}(u, v), kt) * M(\mathcal{P}(y, x), \mathcal{P}(v, u), kt) = 1 * 1$$

$$\geq M(g(x), g(u), t) * M(g(y), g(v), t),$$

for all x, y, u, v in X , with $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Hence, all hypotheses of Theorem 3.1 hold, thus, the pair $(\mathcal{P}, g)$ has a CCP in $X \times X$. Thus on applying Theorem 3.1, we get that the point $(0.7, 0.7)$ is a CCP of the pair $(\mathcal{P}, g)$.

Corollary 3.3. Let $(X, M, *, \leq)$ be a PO-GV-FMS which is complete. Let the mapping $\mathcal{P}$ has MMP and subjected to following:

$(X - 5)$ there exists k in $(0, 1)$ such that

$$M(\mathcal{P}(x, y), \mathcal{P}(\mu, v), kt) * M(\mathcal{P}(y, x), \mathcal{P}(v, \mu), kt)$$

$$\geq M(x, \mu, t) * M(y, v, t),$$

for all x, y, μ, v in X , $t > 0$ with $x \leq \mu$ and $y \geq v$ (or, $x \geq \mu$ and $y \leq v$).

Further, suppose either

(a) the mapping $\mathcal{P}$ is continuous, or

(b) X is regular.

If there exist x_0, y_0 in X such that $x_0 \leq \mathcal{P}(x_0, y_0)$ and $y_0 \geq \mathcal{P}(y_0, x_0)$ (or, $x_0 \geq \mathcal{P}(x_0, y_0)$ and $y_0 \leq \mathcal{P}(y_0, x_0)$), then the map $\mathcal{P}$ has a CFP in X .

Proof. Proof follows by considering $g = I_X$ (identity map on X), in Theorem 3.1.

4. Existence and Uniqueness of CCFP

Very recently, in order to obtain CCFP for nonlinear contractive mappings in cone metric spaces, Abbas et al. [41] gave the notion of w -compatible mappings. In this section, utilizing the notion of w -compatible mappings we extend Theorem 3.1 to ensure the existence and uniqueness of the CCFP in fuzzy metric space.

Definition 4.1 ([41]). The mappings P and g are called w -compatible if $g(P(x, y)) = P(gx, gy)$ whenever $g(x) = P(x, y)$ and $g(y) = P(y, x)$ for $x, y \in X$.

Now, we give our result as follows:

Theorem 4.2. If in Theorem 3.1 an additional hypothesis that: “for every $(x, y), (x', y')$ in $X \times X$, there exists a (μ, ν) in $X \times X$ such that $(P(\mu, \nu), P(\nu, \mu))$ is comparable to $(P(x, y), P(y, x))$ and $(P(x', y'), P(y', x'))$. Let the pair (P, g) is w -compatible, then it has a unique CCFP.

Proof. Using Theorem 3.1, set of coupled coincidences of pair (P, g) is not empty. For obtaining the result, we first show: if (x, y) and (x', y') are CCP, then

$$g(x) = g(x') \text{ and } g(y) = g(y'). \quad (4.1)$$

By assumption there is (μ, ν) in $X \times X$ such that $(P(\mu, \nu), P(\nu, \mu))$ is comparable with $(P(x, y), P(y, x))$ and $(P(x', y'), P(y', x'))$. Put $\mu_0 = \mu, \nu_0 = \nu$ and choose μ_1, ν_1 in X so that $g(\mu_1) = P(\mu_0, \nu_0), g(\nu_1) = P(\nu_0, \mu_0)$.

Then, like in Theorem 3.1, sequences $\{g(\mu_n)\}$ and $\{g(\nu_n)\}$ can be defined such that $g(\mu_{n+1}) = P(\mu_n, \nu_n)$ and $g(\nu_{n+1}) = P(\nu_n, \mu_n)$.

Further, set $x_0 = x, y_0 = y, x'_0 = x', y'_0 = y'$, we construct sequences $\{g(x_n)\}, \{g(y_n)\}$ and $\{g(x'_n)\}, \{g(y'_n)\}$ following the same track.

Then, it can be easily obtained that

$$g(x_{n+1}) = P(x_n, y_n), g(y_{n+1}) = P(y_n, x_n)$$

and

$$g(x'_{n+1}) = P(x'_n, y'_n), g(y'_{n+1}) = P(y'_n, x'_n), \text{ for } n \geq 0.$$

Since $(P(\mu, \nu), P(\nu, \mu)) = (g(\mu_1), g(\nu_1))$ and $(g(x), g(y)) = (P(x, y), P(y, x)) = (g(x_1), g(y_1))$ are comparable, then $g(\mu_1) \geq g(x)$ and $g(\nu_1) \leq g(y)$. Easily, it can be shown $(g(\mu_n), g(\nu_n))$ and $(g(x), g(y))$ are comparable; that is, $g(\mu_n) \geq g(x)$ and $g(\nu_n) \leq g(y)$ for all $n > 0$.

Thus, from (X - 3), for $n > 0$,

$$\begin{aligned} & M(g(\mu_{n+1}), g(x), kt) * M(g(\nu_{n+1}), g(y), kt) \\ &= M(P(\mu_n, \nu_n), P(x, y), kt) * M(P(\nu_n, \mu_n), P(y, x), kt) \\ &\geq M(g(\mu_n), g(x), t) * M(g(\nu_n), g(y), t), \end{aligned}$$

that is, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned} & M(g(\mu_{n+1}), g(x), kt) * M(g(\nu_{n+1}), g(y), kt) \\ &\geq M(g(\mu_n), g(x), t) * M(g(\nu_n), g(y), t). \end{aligned} \tag{4.2}$$

We claim that

$$\lim_{n \rightarrow \infty} g(\mu_n) = g(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} g(\nu_n) = g(y). \tag{4.3}$$

If $\lim_{n \rightarrow \infty} g(\mu_n) = \mu$ and $\lim_{n \rightarrow \infty} g(\nu_n) = \nu$ for some μ, ν in X , it suffices to obtain that $\mu = g(x)$ and $\nu = g(y)$. Letting limit as $n \rightarrow \infty$ in (4.2) and using the properties of γ , we obtain that

$$\begin{aligned} & M(\mu, g(x), kt) * M(\nu, g(y), kt) \geq M(\mu, g(x), t) * M(\nu, g(y), t) \\ &\geq M(\mu, g(x), t) * M(\nu, g(y), t). \end{aligned} \tag{4.4}$$

Thus,

$M(\mu, g(x), t) * M(\nu, g(y), t) \geq M(\mu, g(x), t/k^n) * M(\nu, g(y), t/k^n)$, for all $n \in \mathbb{N}$, implying that $M(\mu, g(x), t) * M(\nu, g(y), t) = 1$ for $t > 0$. It follows $M(\mu, g(x), t) = M(\nu, g(y), t) = 1$ for all $t > 0$, therefore we can obtain that $\mu = g(x)$ and $\nu = g(y)$, and hence (4.3) holds.

Similarly, we obtain that

$$\lim_{n \rightarrow \infty} g(\mu_n) = g(x') \quad \text{and} \quad \lim_{n \rightarrow \infty} g(\nu_n) = g(y). \tag{4.5}$$

Using uniqueness of limit, it is obtainable that $g(x) = g(x')$ and $g(y) = g(y')$. Thus, we have proved (4.1).

As $g(x) = P(x, y)$, $g(y) = P(y, x)$ and the pair (P, g) is w -compatible, it is obtainable that

$$g(g(x)) = g(P(x, y)) = P(g(x), g(y)) \text{ and } g(g(y)) = g(P(y, x)) = P(g(y), g(x)). \quad (4.6)$$

Denote $g(x) = z$, $g(y) = w$. Then, using (4.6),

$$g(z) = P(z, w) \text{ and } g(w) = P(w, z). \quad (4.7)$$

Thus, (z, w) is a CCP.

Then, by (4.1) with $x' = z$ and $y' = w$, we get $g(z) = g(x)$ and $g(w) = g(y)$;

$$\text{that is, } g(z) = z, g(w) = w. \quad (4.8)$$

Using (4.7) and (4.8), we obtain

$$z = g(z) = P(z, w) \text{ and } w = g(w) = P(w, z).$$

Hence, (z, w) is CCFP of P and g .

For uniqueness, let (p, q) be any CCFP. Then, by (4.1), we get $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$.

5. Application in metric space

In this section, as application of the results proved in the earlier sections of this paper, we obtain CCP results in the framework of ordered metric spaces.

Theorem 5.1. Let $(X, \preceq)$ be POS and d a metric on X so that (X, d) is a metric space, the pair (P, g) of maps be such that P has MgmJP and subjected to following:

$(X - 6)$ there exists some k in $(0, 1)$ such that

$$\max\{d(P(x, y), P(\mu, \nu)), d(P(y, x), P(\nu, \mu))\} \leq \frac{k}{2} [d(g(x), g(\mu)) + d(g(y), g(\nu))],$$

for all x, y, μ, ν in X for which $g(x) \preceq g(\mu)$, $g(y) \succeq g(\nu)$ (or, $g(x) \succeq g(\mu)$ and $g(y) \preceq g(\nu)$).

Suppose that (X - 1), (X - 2) and (X - 4) holds. Also, assume that

- (a) both the maps $\mathbb{P}$ and g are continuous, or
- (b) X is regular,

then the pair $(\mathbb{P}, g)$ has a CCP.

Proof. For all x, y in X and $t > 0$, define $M(x, y, t) = \frac{t}{t+d(x, y)}$ and $p * q = \min\{p, q\}$ with p, q in $[0, 1]$. Then, $(X, M, *)$ is FMS, $*$ as the Hadžić type t -norm. Also, it can be easily seen that $M(x, y, t) = \frac{t}{t+d(x, y)} \rightarrow 1$ as $t \rightarrow \infty$, for all x, y in X . We next verify that the inequality (X - 6) implies (X - 3). If otherwise, for $\gamma(t) = t$, from (X - 3), for some $t > 0$ and x, y, μ, v in X with $g(x) \leq g(\mu)$, $g(y) \geq g(v)$, we have

$$\min\{(t/(t + \frac{1}{k} d(\mathbb{P}(x, y), \mathbb{P}(\mu, v))))), (t/(t + \frac{1}{k} d(\mathbb{P}(y, x), \mathbb{P}(v, \mu))))\} \\ < \min\{(t/(t + d(g(x), g(\mu))))), (t/(t + d(g(y), g(v))))\},$$

then, we have either,

$$(t/(t + \frac{1}{k} d(\mathbb{P}(x, y), \mathbb{P}(\mu, v)))) < \min\{(t/(t + d(g(x), g(\mu))))), (t/(t + d(g(y), g(v))))\}, \quad (5.1)$$

or,

$$(t/(t + \frac{1}{k} d(\mathbb{P}(y, x), \mathbb{P}(v, \mu)))) < \min\{(t/(t + d(g(x), g(\mu))))), (t/(t + d(g(y), g(v))))\}. \quad (5.2)$$

From (5.1), it follows

$$t + \frac{1}{k} d(\mathbb{P}(x, y), \mathbb{P}(\mu, v)) > t + d(g(x), g(\mu)). \quad (5.3)$$

$$t + \frac{1}{k} d(\mathbb{P}(x, y), \mathbb{P}(\mu, v)) > t + d(g(y), g(v)). \quad (5.4)$$

Combining (5.3) and (5.4), we attain

$$d(\mathbb{P}(x, y), \mathbb{P}(\mu, v)) > \frac{k}{2} [d(g(x), g(\mu)) + d(g(y), g(v))]. \quad (5.5)$$

Similarly, from (5.2), we obtain that

$$d(\mathbb{P}(y, x), \mathbb{P}(v, \mu)) > \frac{k}{2} [d(g(x), g(\mu)) + d(g(y), g(v))]. \quad (5.6)$$

Using (5.5) and (5.6), we have that

$$\max\{d(P(x, y), P(\mu, \nu)), d(P(y, x), P(\nu, \mu))\} > \frac{k}{2} [d(g(x), g(\mu)) + d(g(y), g(\nu))],$$

which is a contradiction to (X - 6). Now, on applying Theorem 3.1, we get desired result.

Theorem 5.2. Let $(X, \leq)$ be POS and d a metric on X so that (X, d) is a metric space. Let the pair (f, g) of maps be such that P has M_gM_P and subjected to following:

(X - 7) there exists some k in $(0, 1)$ such that

$$d(P(x, y), P(\mu, \nu)) + d(P(y, x), P(\nu, \mu)) \leq k [d(g(x), g(\mu)) + d(g(y), g(\nu))],$$

for all x, y, μ, ν in X for which $g(x) \leq g(\mu)$, $g(y) \geq g(\nu)$ (or, $g(x) \geq g(\mu)$ and $g(y) \leq g(\nu)$).

Suppose that (X - 1), (X - 2) and (X - 4) holds. Also, assume that

- (a) both the maps P and g are continuous, or
- (b) X is regular,

then, the pair (P, g) has a CCP.

Proof. Result follows immediately by considering the well known fact that, $\frac{x+y}{2} \leq \max\{x, y\}$ in Theorem 5.1.

Remark 5.3. (i) Theorem 5.1 provides an extension and generalization of a result of Berinde [Corollary 1, 38] for a pair of mappings.

(ii) Theorem 5.2 improves the recent result of Jain et al. [Corollary 2.3, 23]. By taking $g = I_X$ (the identity mapping on X) in Theorem 5.2, we obtain the result of Berinde [Theorem 3, 39].

Remark 5.4. Unique coupled common fixed point for the pair (P, g) under the hypotheses of Theorem 5.1 (or Theorem 5.2) can be obtained by assuming the additional assumptions as in Theorem 4.2.

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CHAPTER VIII

Mathematical Analysis of an SVEIHR Epidemic Model for the Measles Transmission Dynamics

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Introduction

Measles is an infectious respiratory disease and the disease is caused by a virus from the Paramyxoviridae family (Yanagi et al., 2006; Griffin, 2016). The incubation period for measles is usually 10-14 days. People who are exposed and infected usually recover within three weeks without any complications. However, some people exposed to the disease die or suffer from serious illness and lifelong complications (Beay, 2004; Abad and Safdar, 2015).

Annual reports on the estimated number of measles cases and deaths caused by measles worldwide are announced by the World

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Health Organization (WHO), by evaluating the reports of member countries (WHO, 2019).

Although measles is a vaccine-preventable disease, it still has alarming data globally. Measles has become a leading cause of illness and death among young children under five years of age worldwide, especially in underdeveloped countries. The spread of the virus occurs through the coughing or sneezing of an infected person. Therefore, the spread rate of the virus may be quite high. Clinical symptoms of the disease are high fever, runny nose, cough, conjunctivitis, rhinitis, small white spots, and rashes on the body of infected people.

In recent years, research on measles has become one of the most important research areas in epidemiology due to the impact of mortality rates. Many scientists are focused on finding the best ways to prevent and control the disease in the first place. Researchers who generate ideas using different models have presented their mathematical, experimental and theoretical works to the scientific world. Some of these are stated below.

Momoh et al. (2013) took into account the impact of asymptomatic individuals on measles dynamics in their study. Adewale et al. (2014) proved in their studies that the connection between people infected and uninfected with measles is effective in controlling the spread of the disease. In examining the dynamics on the effect of vaccination on measles, Smith et al. (2016) and Peter et al. (2018) conducted various research and studies.

Garba et al. (2017) created a model system that examines the dynamics of disease treatment along with the vaccination period. The effect of treatment and quarantine stages on the spread of measles was examined by Beay (2018). Zhang et al. (2010) studied the status of the epidemic versus vaccination dynamics on random graphs and scale-free networks. Mossong and Muller (2003) conducted a study on modeling the reemergence of measles within vaccinated populations.

Beay (2004) proposed a SIQR epidemic model for the disease in his study. This study also performed numerical analysis of the model to investigate the impact of treatment and quarantine on measles dynamics. Data from the study showed that quarantine and treatment combined were more effective in controlling and preventing measles. In addition, it was observed with the help of this study that the spread of measles decreased due to the treatment and quarantine of infected individuals.

The article that motivated our study is Peter et al. (2022). In this study, the literature focused on the deterministic modeling of measles disease in Nigeria. This study is based on an SVEIHR (S-susceptible, V-vaccinated, E-exposed, I-infectious, H-hospitalized, and R-recovered) model. The SVEIHR model is based on the assumption of continuous vaccination.

The findings of the current study may assist government and public health authorities in creating strategic vaccination plans to address vaccination gaps and thus prevent measles outbreaks. In our study, the numerical analysis of this type of model was examined to help prevent and control the disease. Thus, the effect of the system obtained in the stability analysis on people at different stages can be easily interpreted.

In our study, the nonstandard finite difference method (NSFD) was used as the numerical method. The method developed by Mickens (1994) can easily eliminate instabilities, unlike other numerical methods. It is very important to eliminate instabilities by choosing the denominator function (Mickens, 2002).

A lot of work has been done on the NSFD scheme. Ongun and Turhan (2012) made a numerical comparison for discrete HIV infection. For this comparison, they used the NSFD scheme in the disclosure part. Kocabıyık and Ongun (2022) also used this scheme in their distributed-order smoking model, adding one of the rare studies in this field to the literature. In their study, Kocabıyık and Ongun (2023) used the NSFD scheme in nonlinear equation systems and obtained productive results. For more detailed information, you

can refer to the sources (Kocabıyık et al., 2020; Çetinkaya et al., 2021; Özdoğan and Ongun,2022).

This paper is organized as follows. In the second section, the basic information necessary for the analysis is given. This information is about the Nonstandard finite difference scheme (NSFD) scheme and stability analysis. In the third section, the system of Measles disease is defined. Again in this section, the discretization of the system was found with the NSFD method. In the fourth section, the equilibrium points of the discretized system are examined. The stability analysis of the equilibrium points was obtained in this section by finding the eigenvalues. The fifth chapter includes the results and discussion section based on the discretization and stability analysis data.

1.Basic Definitions About Nonstandard Finite Difference Scheme and Stability Analysis

In this section, information about the non-standard finite difference method and stability analysis is given.

The nonstandard finite difference method was defined by Mickens (1994). It was also applied to differential equations by Mickens. With this method, Mickens tried to solve the instabilities encountered during numerical analysis. The denominator function is used in the method to remove instabilities. With the different selection of the denominator function, instabilities are eliminated and analysis operations under different effects are made easier.

The method described by Mickens is as follows.

Definition 2.1: (Mickens, 1994) The Nonstandard finite difference scheme is defined as follows:

$$\frac{dk}{dt} \rightarrow \frac{k_{n+1} - k_n}{\phi}, t \rightarrow t_n, F(z) \rightarrow F(k_n), k(t) \rightarrow k(t_n),$$

where ϕ is a denominator function. The denominator function depends on the variable p , which can be found by the equilibrium point, and the variable h , which is the step interval.

Definition 2.2: (Richter, 2002) Let the equilibrium point of the difference equation be Eq , in which case the following statements are true.

i. If the absolute values of all roots of the characteristic polynomial are less than one, the equilibrium point is locally asymptotically stable.

ii. If at least one of the absolute values of the roots of the characteristic polynomial is greater than one, the equilibrium point is unstable.

2. Discretization of Measles Transmission Dynamics

There are 6 different classifications in the population equation of the measles system. In this system, $S(t)$: the susceptible population, $V(t)$: vaccinated individuals, $E(t)$: the population exposed to the disease, $I(t)$: infected individuals, $H(t)$: the population hospitalized due to the disease, and the last $R(t)$: refers to the population that survived the disease and recovered. The ordinary form of the SVEIHR model can be given as (Peter et al., 2022):

$$\begin{aligned}\frac{dS}{dt} &= \phi - aS(t)I(t) + wV(t) - (\tau + \mu)S(t), \\ \frac{dV}{dt} &= \tau S(t) - (\mu + w)V(t), \\ \frac{dE}{dt} &= aS(t)I(t) - (\beta + \mu)E(t), \\ \frac{dI}{dt} &= \beta E(t) - (\mu + p + \delta)I(t), \\ \frac{dH}{dt} &= pI(t) - (\gamma + \delta + \mu)H(t),\end{aligned}$$

$$\frac{dR}{dt} = \gamma H(t) - \mu R(t),$$

where, the daily intake of the susceptible population is at the rate ϕ . The transmission rate for the susceptible population is α , loss of immunity at the rate of vaccine decline is w and individuals in the susceptible population are vaccinated at a rate τ . Natural mortality occurs at a rate μ in all populations and the progression from the exposed class to the infected class is at the rate β . Infected people visit the hospital at a rate of ρ due to complications from measles. The measles-related mortality rate is denoted by δ and the recovery rate from the effects of measles infection following treatment is γ . The discretization processes of the SVEIHR model are as follows:

$$\frac{S_{n+1} - S_n}{\phi_1(h)} = \phi - aS_nI_n + wV_n - (\tau + \mu)S_{n+1},$$

$$\frac{V_{n+1} - V_n}{\phi_2(h)} = \tau S_n - (\mu + w)V_{n+1},$$

$$\frac{E_{n+1} - E_n}{\phi_3(h)} = aS_nI_n - (\beta + \mu)E_{n+1},$$

$$\frac{I_{n+1} - I_n}{\phi_4(h)} = \beta E_n - (\mu + p + \delta)I_{n+1},$$

$$\frac{H_{n+1} - H_n}{\phi_5(h)} = pI_n - (\gamma + \delta + \mu)H_{n+1},$$

$$\frac{R_{n+1} - R_n}{\phi_6(h)} = \gamma H_n - \mu R_{n+1},$$

where $\phi_i, i = 1, 2, \dots, 6$ are denominator functions and are chosen as follows:

$$\begin{aligned}
\phi_1(h) &= \frac{e^{(\tau+\mu)h} - 1}{(\tau + \mu)}, \phi_2(h) = \frac{e^{(\mu+w)h} - 1}{(\mu + w)}, \phi_3(h) \\
&= \frac{e^{(\beta+\mu)h} - 1}{(\beta + \mu)}, \\
\phi_4(h) &= \frac{e^{(\mu+p+\delta)h} - 1}{(\mu + p + \delta)}, \phi_5(h) = \frac{e^{(\gamma+\delta+\mu)h} - 1}{(\gamma + \delta + \mu)}, \phi_6(h) \\
&= \frac{e^{\mu h} - 1}{\mu}.
\end{aligned}$$

If the necessary arrangements are made in the following order:

$$\begin{aligned}
S_{n+1} - S_n &= \phi_1(h)(\varphi - aS_nI_n + wV_n - (\tau + \mu)S_{n+1}), \\
V_{n+1} - V_n &= \phi_2(h)(\tau S_n - (\mu + w)V_{n+1}), \\
E_{n+1} - E_n &= \phi_3(h)(aS_nI_n - (\beta + \mu)E_{n+1}), \\
I_{n+1} - I_n &= \phi_4(h)(\beta E_n - (\mu + p + \delta)I_{n+1}), \\
H_{n+1} - H_n &= \phi_5(h)(pI_n - (\gamma + \delta + \mu)H_{n+1}), \\
R_{n+1} - R_n &= \phi_6(h)(\gamma H_n - \mu R_{n+1}),
\end{aligned}$$

and then,

$$\begin{aligned}
S_{n+1}(1 - \phi_1(h)(\tau + \mu)) &= S_n - \phi_1(h)(\varphi - aS_nI_n + wV_n), \\
V_{n+1}(1 - \phi_2(h)(\mu + w)) &= V_n - \phi_2(h)\tau S_n, \\
E_{n+1}(1 - \phi_3(h)(\beta + \mu)) &= E_n - \phi_3(h)aS_nI_n, \\
I_{n+1}(1 - \phi_4(h)(\mu + p + \delta)) &= I_n - \phi_4(h)\beta E_n, \\
H_{n+1}(1 - \phi_5(h)(\gamma + \delta + \mu)) &= H_n - \phi_5(h)pI_n, \\
R_{n+1}(1 - \phi_6(h)\mu) &= R_n - \phi_6(h)\gamma H_n,
\end{aligned}$$

The final version of the discretized form of the SVEIHR system is as follows:

$$\begin{aligned}
S_{n+1} &= \frac{S_n - \phi_1(h)(\varphi - aS_nI_n + wV_n)}{1 - \phi_1(h)(\tau + \mu)}, \\
V_{n+1} &= \frac{V_n - \phi_2(h)\tau S_n}{1 - \phi_2(h)(\mu + w)}, \\
E_{n+1} &= \frac{E_n - \phi_3(h)aS_nI_n}{1 - \phi_3(h)(\beta + \mu)}, \\
I_{n+1} &= \frac{I_n - \phi_4(h)\beta E_n}{1 - \phi_4(h)(\mu + p + \delta)}, \\
H_{n+1} &= \frac{H_n - \phi_5(h)pI_n}{1 - \phi_5(h)(\gamma + \delta + \mu)}, \\
R_{n+1} &= \frac{R_n - \phi_6(h)\gamma H_n}{1 - \phi_6(h)\mu}.
\end{aligned}$$

If the following system of equations is solved to find the equilibrium point of the system:

$$\begin{aligned}
S_n &= \frac{S_n - \phi_1(h)(\varphi - aS_nI_n + wV_n)}{1 - \phi_1(h)(\tau + \mu)}, \\
V_n &= \frac{V_n - \phi_2(h)\tau S_n}{1 - \phi_2(h)(\mu + w)}, \\
E_n &= \frac{E_n - \phi_3(h)aS_nI_n}{1 - \phi_3(h)(\beta + \mu)}, \\
I_n &= \frac{I_n - \phi_4(h)\beta E_n}{1 - \phi_4(h)(\mu + p + \delta)},
\end{aligned}$$

$$H_n = \frac{H_n - \phi_5(h)pI_n}{1 - \phi_5(h)(\gamma + \delta + \mu)},$$

$$R_n = \frac{R_n - \phi_6(h)\gamma H_n}{1 - \phi_6(h)\mu}.$$

Thus, the equilibrium points are found as:

$$Eq_1 = \left(\frac{\varphi(\mu + w)}{\mu(\tau + \mu) + w(\mu + 2\tau)}, \frac{\varphi\tau}{\mu(\tau + \mu) + w(\mu + 2\tau)}, 0, 0, 0, 0 \right),$$

$$Eq_2 = \left(\frac{(\beta + \mu)((\mu + p + \delta))}{a\beta}, \frac{\tau(\beta + \mu)((\mu + p + \delta))}{a\beta(\mu + w)}, eq_3, eq_4, eq_5, eq_6 \right).$$

Here, quite complex operations are required for the last 4 components of the Eq_2 equilibrium point. For this reason, in the stability analysis part, the components of this balance point were obtained with numerical values.

3.Stability Analysis of SVEIHR Epidemic Model

In this section, stability analysis was performed to better understand the impact of measles. Two different equilibrium points were considered for the stability analysis. The parameter values to be used in this section are expressed in the table below (Peter et al., 2022):

$\varphi: 680.27$	$\mu: 0.000309$	$\delta: 0.033720$	$\tau: 0.0000001$	$w: 0.003286$
$a: 10^{-9}$	$\beta: 0.5$	$p: 0.036246$	$\gamma: 0.062366$	$h: 0.01$

By substituting the parameters at the expressed equilibrium points, these points are obtained as follows:

$$Eq_1 = (2201459.799, 61.23671208, 0, 0, 0, 0),$$

Eq_2

$$= (70318429.96, 1956.006397, -42071.45835, -299334.4601, -112554.3$$

For stability analysis of equilibrium points, Jacobian matrix is required. The Jacobian matrix of the resulting discretized system is shown as follows:

$$\begin{pmatrix} \frac{1 + \phi_1(h)(aI_n)}{1 - \phi_1(h)(\tau + \mu)} & \frac{-\phi_1(h)(w)}{1 - \phi_1(h)(\tau + \mu)} & 0 & \frac{\phi_1(h)(aS_n)}{1 - \phi_1(h)(\tau + \mu)} & 0 & 0 \\ \frac{-\phi_2(h)\tau}{1 - \phi_2(h)(\mu + w)} & \frac{1}{1 - \phi_2(h)(\mu + w)} & 0 & 0 & 0 & 0 \\ \frac{-\phi_3(h)aI_n}{1 - \phi_3(h)(\beta + \mu)} & 0 & \frac{1}{1 - \phi_3(h)(\beta + \mu)} & \frac{-\phi_3(h)aS_n}{1 - \phi_3(h)(\beta + \mu)} & 0 & 0 \\ 0 & 0 & \frac{-\phi_4(h)\beta}{1 - \phi_4(h)(\mu + p + \delta)} & \frac{1}{1 - \phi_4(h)(\mu + p + \delta)} & 0 & 0 \\ 0 & 0 & 0 & \frac{-\phi_5(h)p}{1 - \phi_5(h)(\gamma + \delta + \mu)} & \frac{1}{1 - \phi_5(h)(\gamma + \delta + \mu)} & 0 \\ 0 & 0 & 0 & 0 & \frac{-\phi_6(h)\gamma}{1 - \phi_6(h)\mu} & \frac{1}{1 - \phi_6(h)\mu} \end{pmatrix}$$

By substituting the parameters and the obtained equilibrium point Eq_1 into the Jacobian matrix, the characteristic equation is obtained as:

$$P(\lambda) = \lambda^6 - 5.993301 \lambda^5 + 14.966515 \lambda^4 - 19.933049 \lambda^3 + 14.933067 \lambda^2 - 5.966543 \lambda + 0.993310$$

With the necessary calculations, the eigenvalues of this characteristic equation are in the following form:

$$\lambda_1 = 0.994983, \lambda_2 = 0.999036, \lambda_3 = 0.999322,$$

$$\lambda_4 = 0.999964, \lambda_5 = 0.999996, \lambda_6 = 0.999997.$$

Thus, with the help of Definition 2.2, the absolute values of all eigenvalues are less than 1, that is, the Eq_1 equilibrium point is locally asymptotically stable.

If the same operations are repeated for Eq_2 , the characteristic equation is:

$$P(\lambda) = \lambda^6 - 5.993304 \lambda^5 + 14.966527 \lambda^4 - 19.933065 \lambda^3 + 14.933077 \lambda^2 - 5.966544 \lambda + 0.993310$$

And the eigenvalues are,

$$\lambda_1 = 0.994307, \lambda_2 = 0.999036, \lambda_3 = 0.999956,$$

$$\lambda_4 = 0.999964, \lambda_5 = 0.999999, \lambda_6 = 1.000042.$$

As can be seen, the Eq_2 equilibrium point is unstable because the λ_6 eigenvalue does not meet the stability condition.

4. Conclusion

In this study, a stability analysis of an epidemic model of measles has been obtained. The effect of the vaccination component in the model has been examined. Discretization for numerical analysis has been obtained by the nonstandard finite difference method. Afterwards, equilibrium points were found again with this discretization method. Stability analysis of equilibrium points was examined with parameters. Thus, it has been seen that the method is suitable for this type of endemic equations. We have shown that analysis can be performed under different situations by selecting the appropriate denominator function.

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CHAPTER IX

Application of Kashuri Fundo Transform to Solve Logistic Growth Model, and Prey-Predator Model

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Introduction

Mathematical modeling is vital for resolving real-life problems in engineering, physics, statistics, economics, finance, chemistry, and other domains. Differential equations often serve as an important tool for modeling problems in these fields. Differential equations comprise quantities whose values fluctuate by one another and their respective rates of change. Modeling the relation between derivatives with differential equations, real-world issues become simpler to comprehend and easier to develop solutions for these

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models. The majority of real-world issues involve nonlinear differential equations. It is challenging to solve some of these problems analytically. As a result, new approaches and studies are being developed to find better and newer numerical solutions to nonlinear equations.

Thomas Malthus introduced one of the earliest mathematical models illustrating the dynamic change of populations. The Malthusian model posits that the rate of population growth in a country is directly proportional to its total population, denoted as $\mathcal{W}(t)$ at any given time t . According to this theory, the population growth at a specific moment in time is directly proportional to the projected population growth in the future. The mathematical representation of this model is described as a first-order ordinary linear differential equation,

$$\frac{d\mathcal{W}}{dt} = \kappa\mathcal{W}$$

with initial condition

$$\mathcal{W}(t_0) = \mathcal{W}_0$$

where, $\mathcal{W}$ represents the population at time t , $\mathcal{W}_0$ indicates the starting population at time t_0 and κ is a constant of proportionality. Based on this equation, it can be deduced that the population graph demonstrates exponential growth (Weigelhofer & Lindsay, 1999).

The Malthus growth model demonstrates that a population can increase unlimitedly over time. Nevertheless, each population has a carrying capacity. When a population achieves its carrying capacity, competition for limited resources such as food, space, and other factors occurs, resulting in a divergence from exponential population growth. As a result, Pierre Verhulst's logistics model, which describes growth in a confined area, replaces the Malthusian model. The nonlinear biological models encompass a logistic growth model within a population, represented by the equation

$$\frac{d\mathcal{W}(t)}{dt} = r\mathcal{W}(t) \left(1 - \frac{\mathcal{W}(t)}{\kappa}\right) \quad (1)$$

where r is growth rate and κ is the carrying capacity. The function $\mathcal{W}(t)$ denotes the population of the species at time t , while the expression $r\mathcal{W}(t) \left(1 - \frac{\mathcal{W}(t)}{\kappa}\right)$ represents the per capita growth rate. Non-dimensionalization of equation (1) is accomplished by

$$\varphi(\tau) = \frac{\mathcal{W}(t)}{\kappa}, \quad \tau = rt$$

which yields

$$\frac{d\varphi}{d\tau} = \varphi(1 - \varphi). \quad (2)$$

If the initial condition is given as $\mathcal{W}(0) = \mathcal{W}_0$, then $\varphi(0) = \frac{\mathcal{W}_0}{\kappa}$. Consequently, the analytical solution of equation (2) is achieved by

$$\varphi(\tau) = \frac{1}{1 + \left(\frac{\kappa}{\mathcal{W}_0} - 1\right) e^{-\tau}}. \quad (3)$$

The Lotka-Volterra equations were formulated to describe the dynamics of biological systems. This system of nonlinear differential equations focuses on predator-prey interactions. A predator-prey relationship refers to the dynamic between two species and the reciprocal influence they have on each other. At that point, one species is, in fact, consuming the other species for food. A predator is an organism that consumes or hunts other organisms for food, whereas a prey is an organism that another organism kills for food. Examples of predators with their prey are the fox and the rabbit, the lion, and the zebra. Predator-prey dynamics is a notion that applies not just to animals but also to plants. The relationship between the grasshopper and the leaf serves as an illustrative example in this context. The predator-prey models: Lotka–Volterra systems as an interacting species model denoted by

$$\frac{d\mathcal{W}}{dt} = \mathcal{W}(a - b\mathcal{V}) \quad (4)$$

$$\frac{d\mathcal{V}}{dt} = \mathcal{V}(c\mathcal{W} - d) \quad (5)$$

where a, b, c and d are constants (Murray, 1993). Here $\mathcal{W} = \mathcal{W}(t)$ represents the prey population, and $\mathcal{V} = \mathcal{V}(t)$ represents the population of predators at the time t . The non-dimensionalization of the system (4)-(5) is given as

$$\varphi(\tau) = \frac{c\mathcal{W}(t)}{d}, \quad \psi(\tau) = \frac{b\mathcal{V}(t)}{a}, \quad \tau = at, \quad \mu = d/a$$

and it turns into

$$\frac{d\varphi}{d\tau} = \varphi(1 - \psi) \quad (6)$$

$$\frac{d\psi}{d\tau} = \mu[g(\varphi, \psi) - \psi]. \quad (7)$$

Integral transforms are a valuable mathematical tool for solving a wide range of processes and phenomena in the fields of science, engineering, and real-life applications. These transforms allow us to express various complex problems in a mathematical framework, enabling their solution through rigorous mathematical techniques. In fields such as engineering, physics, and chemistry, integral transformations are employed to solve initial value problems, boundary value problems, differential equations, and integral equations. They convert the original domain of problems into another domain, simplifying intricate problems and enhancing comprehensibility. Subsequently, Inverse integral transforms are used to return the solution discovered by integral transforms to its original domain. Due to the diverse range of applications, many novel integral transforms have been defined. The most widely used and well-known integral transforms are the Laplace, Fourier, and Sumudu transforms. In this study, we deal with the Kashuri Fundo transform which is one of the integral transforms. The Kashuri

Fundo transform stands out as a convenient, efficient, easy-to-use, and dependable method, enabling the attainment of solutions without the need for intricate calculations. As a result of this, one may find several studies on the Kashuri Fundo transform in the literature. Kashuri and Fundo (2013) introduced the Kashuri Fundo transform to the literature. Numerous researchers, Kashuri and Fundo among them, developed various applications (Kashuri et al. 2013a, 2013b, 2015; Fundo et al. 2016) of this transform in subsequent stages. Shah et al. (2015a, 2015b) examine to solve a nonlinear differential-difference equation arising in nanotechnology and the concentration of the longitudinal dispersion phenomenon arising in fluid flow through porous media by mixture Kashuri Fundo transform and homotopy perturbation method. Güngör (2021) investigated the solution of Volterra integral equations via Kashuri Fundo transform. Johansyah et al. (2022) solved the economic growth acceleration model with memory effects for the quadratic cost function using Kashuri Fundo transformation method. Moreover, Peker et al. (2022, 2022a, 2022b, 2022c, 2023) utilized this transform to solve Abel's integral equation, steady heat transfer problem, decay problem, cardiovascular models.

Now, we will present the Kashuri Fundo integral transform, along with its necessary properties.

Definition 1. (Kashuri & Fundo, 2013) Let F be a function set defined by

$$F = \left\{ f(t) \mid \exists M, k_1, k_2 > 0, \text{ such that } |f(t)| \leq M e^{\frac{|t|}{k_2^2}}, \text{ if } t \in (-1)^i \times [0, \infty) \right\}.$$

where M must be finite number k_1, k_2 may be finite or infinite. A new integral transform denoted by the operator $\mathcal{K}(\cdot)$ is defined by

$$\mathcal{K}[f(t)](v) = A(v) = \frac{1}{v} \int_0^\infty e^{\frac{-t}{v^2}} f(t) dt, \quad t \geq 0, -k_1 < v < k_2.$$

Theorem 1 (Linearity property). (Kashuri & Fundo, 2013)

Let $f(t)$ and $g(t)$ be functions whose Kashuri Fundo integral transforms exist and λ_1, λ_2 be constants, then

$$\mathcal{K}[(\lambda_1 f + \lambda_2 g)(t)] = \lambda_1 \mathcal{K}[(f)(t)] + \lambda_2 \mathcal{K}[(g)(t)].$$

Theorem 2 (Kashuri Fundo transform of the derivative).

(Kashuri & Fundo, 2013) Let suppose $A(v)$ be Kashuri Fundo transform of $f(t)$. Then

$$\mathcal{K}[f^{(n)}(t)] = \frac{A(v)}{v^{2n}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2(n-k)-1}}.$$

Table 1. Kashuri Fundo Transform of Some Standart Functions

$f(t)$	$\mathcal{K}[f(t)] = A(v)$
1	v
t	v^3
t^n	$n! v^{2n+1}$
$e^{\lambda t}$	$\frac{v}{1 - \lambda v^2}$
$\sin(\lambda t)$	$\frac{\lambda v^3}{1 + \lambda^2 v^4}$
$\cos(\lambda t)$	$\frac{v}{1 + \lambda^2 v^4}$
$\sinh(\lambda t)$	$\frac{\lambda v^3}{1 - \lambda^2 v^4}$
$\cosh(\lambda t)$	$\frac{\lambda}{1 - \lambda^2 v^4}$

Main Results

This section presents the Kashuri Fundo transform as a technique that facilitates the solution of nonlinear differential equations through biological models, specifically those comprising a logistic growth model to study population dynamics and the prey-predator model to analyze ecological interactions.

Kashuri-Fundo Method for Logistic Growth Model

In this subsection, take into the model equation in the form of

$$\frac{d\varphi}{dt} = \varphi - g(\varphi), \quad \varphi(0) = \varphi_0 \quad (8)$$

where g represents a nonlinear function of φ . Therefore, we assume that the solution of φ satisfying (8) has a representation of the form of an infinite series

$$\varphi = \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \quad (9)$$

In addition, it fulfills the necessary criteria for the Kashuri Fundo transform to be in existence. Applying the Kashuri Fundo transform to the either side of the differential equation in (8), then we find

$$\frac{\mathcal{K}[\varphi(t)]}{v^2} - \frac{\varphi_0}{v} = \mathcal{K}[\varphi(t)] - \mathcal{K}[g(\varphi(t))].$$

Hence, we obtain

$$\mathcal{K}[\varphi(t)] = \varphi_0 \frac{v}{1-v^2} - \frac{v^2}{1-v^2} \mathcal{K}[g(\varphi(t))]. \quad (10)$$

Thus, assuming the existence of the inverse Kashuri Fundo transform $\mathcal{K}^{-1}$ exists and applying it to expression (10), the equation can be written as:

$$\varphi(t) = \varphi_0 e^t - \mathcal{K}^{-1} \left[\frac{v^2}{1-v^2} \mathcal{K}[g(\varphi(t))] \right].$$

Example 1. Let us consider the logistic growth model equation (8) where $\mathcal{W}_0 = 2$ and $\kappa = 1$. Hence φ_0 can be expressed as $\varphi_0 = \frac{\mathcal{W}_0}{\kappa} = 2$. We set $g(\varphi) = \varphi^2$ as in (2) so that one finds

$$\begin{aligned} g(\varphi) &= \left(\sum_{k=0}^{\infty} c_k t^k \right)^2 \\ &= (c_0 + c_1 t + c_2 t^2 + \dots + c_k t^k + \dots)^2 \\ &= c_0^2 + 2c_0 c_1 t + (2c_0 c_2 + c_1^2) t^2 \\ &\quad + (2c_0 c_3 + 2c_1 c_2) t^3 + (2c_0 c_4 + 2c_1 c_3 + c_2^2) t^4 + \dots \end{aligned}$$

Implementing Kashuri Fundo transform to each side of the equation

$$\begin{aligned} \mathcal{K}[g(\varphi)] &= G(v) \\ &= c_0^2 v + 2c_0 c_1 v^3 + (2c_0 c_2 + c_1^2) 2! v^5 \\ &\quad + (2c_0 c_3 + 2c_1 c_2) 3! v^7 \\ &\quad + (2c_0 c_4 + 2c_1 c_3 + c_2^2) 4! v^9 + \dots \end{aligned}$$

Using (10) one gets

$$\begin{aligned} \Phi(v) &= 2 \frac{v}{1-v^2} - \left[\frac{v^3 c_0^2}{1-v^2} + \frac{2c_0 c_1 v^5}{1-v^2} \right. \\ &\quad + 2! \frac{(2c_0 c_2 + c_1^2) v^7}{1-v^2} + 3! \frac{(2c_0 c_3 + 2c_1 c_2) v^9}{1-v^2} \\ &\quad \left. + 4! \frac{(2c_0 c_4 + 2c_1 c_3 + c_2^2) v^{11}}{1-v^2} + \dots \right] \\ &= 2 \frac{v}{1-v^2} - \left[c_0^2 \left(\frac{v}{1-v^2} - v \right) + 2c_0 c_1 \left(\frac{v}{1-v^2} - v - v^3 \right) \right. \\ &\quad \left. + 2! (2c_0 c_2 + c_1^2) \left(\frac{v}{1-v^2} - v - v^3 - v^5 \right) \right. \end{aligned}$$

$$\begin{aligned}
& +3!(2c_0c_3 + 2c_1c_2) \left(\frac{v}{1-v^2} - v - v^3 - v^5 - v^7 \right) \\
& +4!(2c_0c_4 + 2c_1c_3 + c_2^2) \left(\frac{v}{1-v^2} - v - v^3 - v^5 - v^7 - v^9 \right) \\
& + \dots]
\end{aligned}$$

Upon application of the inverse Kashuri Fundo transform to this equation yields

$$\begin{aligned}
c_0 + c_1t + c_2t^2 + c_3t^3 + \dots &= 2 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) \\
&- (c_0^2 + 2c_0c_1 + 4c_0c_2 + 2c_1^2 + 12c_0c_3 + 12c_1c_2 + 48c_0c_4 + 48c_1c_3 \\
&+ 24c_2^2 + \dots) \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) \\
&+ (c_0^2 + 2c_0c_1 + 4c_0c_2 + 2c_1^2 + 12c_0c_3 + 12c_1c_2 + 48c_0c_4 \\
&+ 48c_1c_3 + 24c_2^2 + \dots) \\
&+ (2c_0c_1 + 4c_0c_2 + 2c_1^2 + 12c_0c_3 + 12c_1c_2 + 48c_0c_4 \\
&+ 48c_1c_3 + 24c_2^2 + \dots)t \\
&+ (2c_0c_1 + c_1^2 + 6c_0c_3 + 6c_1c_2 + 24c_0c_4 + 24c_1c_3 + 12c_2^2 + \dots)t^2 \\
&+ (2c_0c_3 + 2c_1c_2 + 8c_0c_4 + 8c_1c_3 + 4c_2^2 + \dots)t^3 \\
&+ (2c_0c_4 + 2c_1c_3 + c_2^2 + \dots)t^4 + \dots \\
&= 2 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) - c_0^2t - \left(\frac{c_0^2}{2} + c_0c_1 \right) t^2
\end{aligned}$$

$$\begin{aligned}
& -\left(\frac{c_0^2}{6} + \frac{c_0c_1}{3} + \frac{2c_0c_2}{3} + \frac{c_1^2}{3}\right)t^3 \\
& -\left(\frac{c_0^2}{24} + \frac{c_0c_1}{12} + \frac{c_0c_2}{6} + \frac{c_1^2}{12} + \frac{c_0c_3}{2} + \frac{c_1c_2}{2}\right)t^4 - \dots \\
& = 2 + (2 - c_0^2)t + \left(1 - \frac{c_0^2}{2} - c_0c_1\right)t^2 \\
& + \left(\frac{1}{3} - \frac{c_0^2}{6} - \frac{c_0c_1}{3} - \frac{2c_0c_2}{3} - \frac{c_1^2}{3}\right)t^3 \\
& + \left(\frac{1}{12} - \frac{c_0^2}{24} - \frac{c_0c_1}{12} - \frac{c_0c_2}{6} - \frac{c_1^2}{12} - \frac{c_0c_3}{2} - \frac{c_1c_2}{2}\right)t^4 + \dots
\end{aligned}$$

from (9). When the coefficients of power t are equated, the result is

$$c_0 = 2,$$

$$c_1 = 2 - c_0^2 \Rightarrow c_1 = -2,$$

$$c_2 = 1 - \frac{c_0^2}{2} - c_0c_1 \Rightarrow c_2 = 3,$$

$$c_3 = \frac{1}{3} - \frac{c_0^2}{6} - \frac{c_0c_1}{3} - \frac{2c_0c_2}{3} - \frac{c_1^2}{3} \Rightarrow c_3 = -\frac{13}{3},$$

$$c_4 = \frac{1}{12} - \frac{c_0^2}{24} - \frac{c_0c_1}{12} - \frac{c_0c_2}{6} - \frac{c_1^2}{12} - \frac{c_0c_3}{2} - \frac{c_1c_2}{2} \Rightarrow c_4 = \frac{25}{4},$$

$\vdots$

and so on. Consequently, the solution $\varphi(t)$ is obtained from (9) as follows

$$\varphi(t) = 2 - 2t + 3t^2 - \frac{13}{3}t^3 + \frac{25}{4}t^4 + \dots$$

that is the closed form exact solution obtained in (3). This solution is identical to the one discovered in (Pamuk & Soylu, 2020; Pamuk, 2005).

$s_n(t)$ represents the n th partial sums of the series (9) which is equivalent to

$$s_n(t) = \sum_{k=0}^n c_k t^k. \quad (11)$$

Based on the observation of Figure 1, it is evident that a highly accurate approximation of the exact solution for the logistic growth model within the time interval $[0,0.25]$ has been achieved by computing only five terms of the series in (9). This indicates that the rate of convergence of the Kashuri Fundo transform method is highly rapid. Furthermore, it is possible to minimize the overall errors and obtain a reasonably accurate estimation of the exact solution for $t \geq 0.25$ by incorporating new terms into the series.

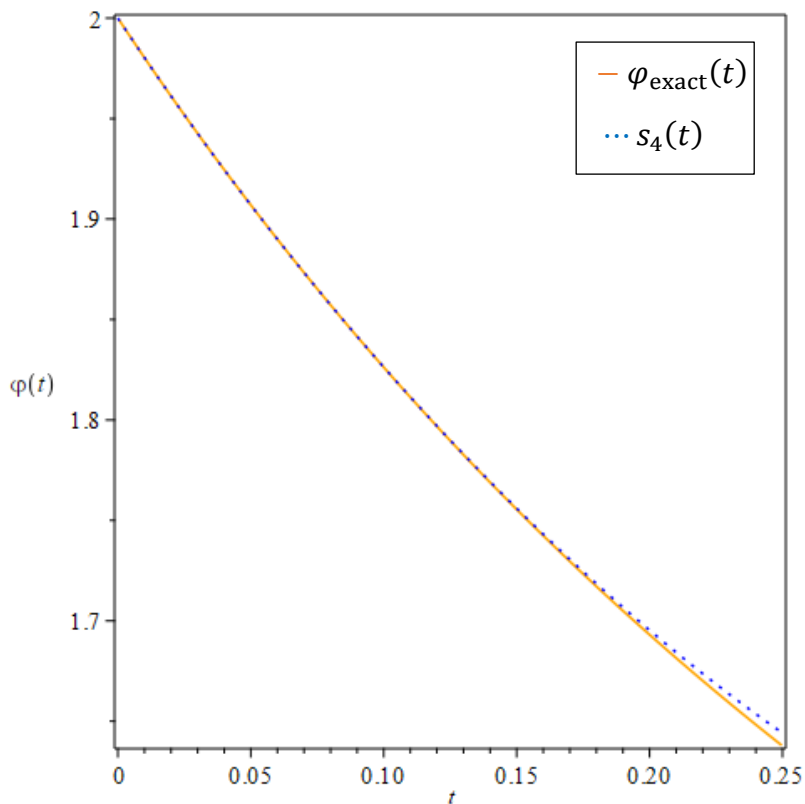


Figure 1. The resolution of the logistic growth model in population dynamics

Applying of Kashuri Fundo Transform for Prey-Predator Model

Let us consider the system of non-linear differential equations that determines the predator-prey model.

$$\frac{d\varphi}{dt} = \varphi - g(\varphi, \psi) \quad (12)$$

$$\frac{d\psi}{dt} = \alpha[h(\varphi, \psi) - \psi] \quad (13)$$

with initial conditions

$$\varphi(0) = \varphi_0, \psi(0) = \psi_0 \quad (14)$$

where g and h are nonlinear functions of φ and ψ and also α be a positive constant. It is assumed that the solutions φ and ψ of the system (12)-(13) possess infinite series expansions in the following form:

$$\varphi(t) = \sum_{k=0}^{\infty} c_k t^k, \quad \psi(t) = \sum_{k=0}^{\infty} d_k t^k. \quad (15)$$

Furthermore, the necessary criteria for the existence of their Kashuri Fundo transforms are satisfied by them. By utilizing the Kashuri Fundo for the equations (12)-(13) and utilizing (14), we obtain

$$\frac{\Phi(v)}{v^2} - \frac{\varphi_0}{v} = \Phi(v) - G(v) \quad (16)$$

$$\frac{\Psi(v)}{v^2} - \frac{\psi_0}{v} = \alpha[H(v) - \Psi(v)] \quad (17)$$

where $\mathcal{K}[\varphi(t)] = \Phi(v)$, $\mathcal{K}[g(\varphi(t), \psi(t))] = G(v)$, $\mathcal{K}[\psi(t)] = \Psi(v)$, $\mathcal{K}[h(\varphi(t), \psi(t))] = H(v)$ are the Kashuri Fundo transforms of the functions $\varphi(t)$, $g(\varphi(t), \psi(t))$, $\psi(t)$ and

$h(\varphi(t), \psi(t))$, respectively. By solving the equations (16)-(17) for $\Phi(v)$ and $\Psi(v)$, one gets

$$\Phi(v) = \frac{v}{1-v^2} \varphi_0 - \frac{v^2}{1-v^2} G(v)$$

$$\Psi(v) = \frac{v}{1+\alpha v^2} \psi_0 + \alpha \frac{v^2}{1+\alpha v^2} H(v)$$

Assuming inverse Kashuri Fundo transforms exist and utilizing them to the system, we obtain

$$\varphi(t) = \varphi_0 e^t - \mathcal{K}^{-1} \left[\frac{v^2}{1-v^2} G(v) \right]$$

$$\psi(t) = \psi_0 e^{-\alpha t} + \alpha \mathcal{K}^{-1} \left[\frac{v^2}{1+\alpha v^2} H(v) \right]$$

desired solutions to the initial value problem (12)-(14).

Example 2. Consider the differential equation system that governs the predator and prey model

$$\frac{d\varphi}{dt} = \varphi - \varphi\psi \tag{18}$$

$$\frac{d\psi}{dt} = \varphi\psi - \psi \tag{19}$$

with initial data $\varphi(0) = 1.3$, $\psi(0) = 0.6$.

Suppose that $\varphi(t) = \sum_{k=0}^{\infty} c_k t^k$, $\psi(t) = \sum_{k=0}^{\infty} d_k t^k$ be solutions of the system of (18)-(19). Hence, we find

$$\begin{aligned}
g(\varphi, \psi) &= h(\varphi, \psi) = \varphi\psi = \left(\sum_{k=0}^{\infty} c_k t^k \right) \left(\sum_{k=0}^{\infty} d_k t^k \right) \\
&= c_0 d_0 + (c_0 d_1 + c_1 d_0) t + (c_0 d_2 + c_1 d_1 + c_2 d_0) t^2 \\
&\quad + (c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) t^3 + \dots
\end{aligned}$$

The corresponding Kashuri Fundo transforms of these functions become

$$\begin{aligned}
G(v) &= H(v) = \mathcal{K}[\varphi\psi] \\
&= c_0 d_0 \mathcal{K}[1] + (c_0 d_1 + c_1 d_0) \mathcal{K}[t] \\
&\quad + (c_0 d_2 + c_1 d_1 + c_2 d_0) \mathcal{K}[t^2] \\
&\quad + (c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) \mathcal{K}[t^3] + \dots \\
&= c_0 d_0 v + (c_0 d_1 + c_1 d_0) v^3 \\
&\quad + (c_0 d_2 + c_1 d_1 + c_2 d_0) 2! v^5 \\
&\quad + (c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) 3! v^7 + \dots
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
\Phi(v) &= \frac{v}{1-v^2} 1.3 - \frac{v^3}{1-v^2} c_0 d_0 - \frac{v^5}{1-v^2} (c_0 d_1 + c_1 d_0) \\
&\quad - \frac{2! v^7}{1-v^2} (c_0 d_2 + c_1 d_1 + c_2 d_0) \\
&\quad - \frac{3! v^9}{1-v^2} (c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) - \dots \\
&= \frac{v}{1-v^2} 1.3 - \left(\frac{v}{1-v^2} - v \right) c_0 d_0
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{v}{1-v^2} - v - v^3 \right) (c_0 d_1 + c_1 d_0) \\
& - 2! \left(\frac{v}{1-v^2} - v - v^3 - v^5 \right) (c_0 d_2 + c_1 d_1 + c_2 d_0) \\
& - 3! \left(\frac{v}{1-v^2} - v - v^3 - v^5 - v^7 \right) (c_0 d_3 + c_1 d_2 \\
& \quad + c_2 d_1 + c_3 d_0) - \dots
\end{aligned}$$

and

$$\begin{aligned}
\Psi(v) &= \frac{v}{1+v^2} 0.6 + \frac{v^3}{1+v^2} c_0 d_0 + \frac{v^5}{1+v^2} (c_0 d_1 + c_1 d_0) \\
&+ \frac{2! v^7}{1+v^2} (c_0 d_2 + c_1 d_1 + c_2 d_0) \\
&+ \frac{3! v^9}{1+v^2} (c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) + \dots \\
&= \frac{v}{1+v^2} 0.6 + \left(v - \frac{v}{1+v^2} \right) c_0 d_0 \\
&+ \left(v^3 - v + \frac{v}{1+v^2} \right) (c_0 d_1 + c_1 d_0) \\
&+ 2! \left(v^5 - v^3 + v - \frac{v}{1+v^2} \right) (c_0 d_2 + c_1 d_1 + c_2 d_0) \\
&+ 3! \left(v^7 - v^5 + v^3 - v + \frac{v}{1+v^2} \right) (c_0 d_3 + c_1 d_2 \\
&\quad + c_2 d_1 + c_3 d_0) + \dots
\end{aligned}$$

by using (15). By utilizing inverse Kashuri Fundo transform to these equations, we obtain

$$\begin{aligned}
& c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots \\
& = 1.3 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) - c_0 d_0 t \\
& \quad - (c_0 d_0 + c_0 d_1 + c_1 d_0) \frac{t^2}{2!} \\
& \quad - (c_0 d_0 + c_0 d_1 + c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 + 2c_2 d_0) \frac{t^3}{3!} - \dots \\
& = 1.3 + (1.3 - c_0 d_0) t + (1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0) \frac{t^2}{2!} \\
& \quad + (1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0 - 2c_0 d_2 - 2c_1 d_1 - 2c_2 d_0) \frac{t^3}{3!} \\
& \quad + \dots
\end{aligned}$$

and

$$\begin{aligned}
& d_0 + d_1 t + d_2 t^2 + d_3 t^3 + \dots \\
& = 0.6 \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right) \\
& \quad - c_0 d_0 t - (c_0 d_0 + c_0 d_1 + c_1 d_0) \frac{t^2}{2!} \\
& \quad - (c_0 d_0 + c_0 d_1 + c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 + 2c_2 d_0) \frac{t^3}{3!} - \dots \\
& = 0.6 + (c_0 d_0 - 0.6) t + (0.6 - c_0 d_0 + c_0 d_1 + c_1 d_0) \frac{t^2}{2!} \\
& \quad + (-0.6 + c_0 d_0 - c_0 d_1 - c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 \\
& \quad \quad + 2c_2 d_0) \frac{t^3}{3!} + \dots .
\end{aligned}$$

If the coefficients are equalized to powers of t , it is found as

$$c_0 = 1.3$$

$$d_0 = 0.6$$

$$c_1 = 1.3 - c_0 d_0$$

$$d_1 = c_0 d_0 - 0.6$$

$$c_1 = 0.52$$

$$d_1 = 0.18$$

$$c_2 = \frac{1}{2!} (1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0)$$

$$d_2 = \frac{1}{2!} (0.6 + c_0 d_1 + c_1 d_0 - c_0 d_0)$$

$$c_2 = -0.013$$

$$d_2 = 0.183$$

$$c_3 = \frac{1}{3!} (1.3 - c_0 d_0 - c_0 d_1 - c_1 d_0 - 2c_0 d_2 - 2c_1 d_0 - 2c_2 d_0)$$

$$d_3 = \frac{1}{3!} (-0.6 + c_0 d_0 - c_0 d_1 - c_1 d_0 + 2c_0 d_2 + 2c_1 d_1 + 2c_2 d_0)$$

$$c_4 = -0.1122$$

$$d_4 = 0.0469$$

$\vdots$

$\vdots$

The subsequent terms of the series can be obtained using this method. By substituting these terms into equation (15), we obtain the approximate solutions for the problem described by equations (18)-(19):

$$\varphi(t) = 1.3 + 0.52t - 0.013t^2 - 0.1122t^3 - \dots$$

$$\psi(t) = 0.6 + 0.18t + 0.183t^2 + 0.0469t^3 + \dots$$

The current results we have obtained align with the findings from the referenced research. in reference (Pamuk & Soylu, 2020; Pamuk, 2005).

The approximate solutions to systems (18)-(19) derived through the Kashuri Fundo transform using solely four elements of the series (15) are illustrated in Figure 2. The numerical solutions for this system appear in Figure 3. The system's numerical solutions are computed with Ode45, an integrated ordinary differential equation solver in MATLAB.

Within the time period of $[0,1.5]$, the two solutions for φ (prey population) and ψ (predator population) are found to be quite near when comparing the two figures. Adding more terms to the series provides an even closer approximation to the numerical answer for $t \geq 1.5$, as previously mentioned in the context of the logistic growth model.

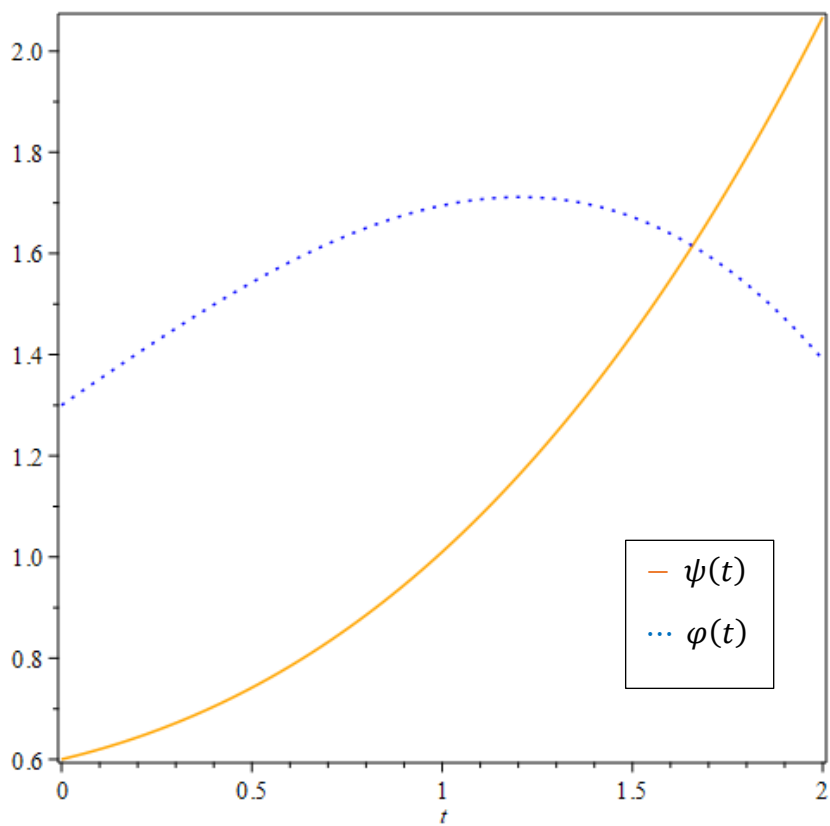


Figure 2. Approximate solutions to the system (18)-(19) by Kashuri Fundo transform method

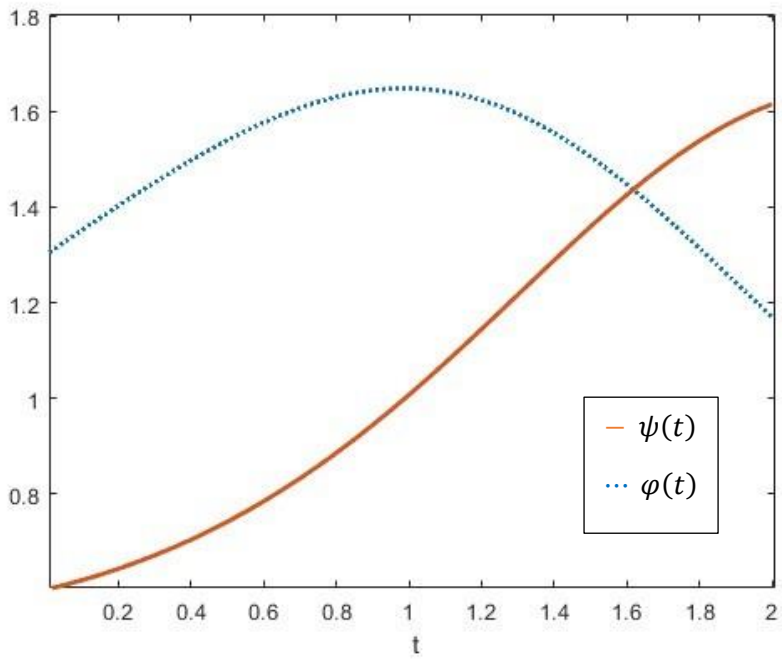


Figure 3. Numerical solutions to the system (18)-(19)

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CHAPTER X

The Representations and Fnite Sums of the Mersenne-Padovan Numbers

Özgür ERDAĞ

Introduction and Preliminaries

A Mersenne number, by M_n , is a number of the form $M_n = 2^n - 1$. The Mersenne sequence $\{M_n\}_{n \geq 0}$ can also be defined recursively by

$$M_{n+2} = 3M_{n+1} - 2M_n$$

with initial values $M_0 = 0$ and $M_1 = 1$. It is worth noting that Mersenne numbers belong to the same family as Fermat numbers, and thus, they share the same properties. (Catarino, Campos & Vasco, 2016)

The Padovan sequence is the sequence of the integer $\{P(n)\}$ defined by the following recurrence relation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$ and with initial values $P(0) = P(1) = P(2) = 1$.

It is easy to see that the characteristic polynomials of the Mersenne sequence and Padovan sequence are $k_1(x) = x^2 - 3x + 2$ and $k_2(x) = x^3 - x - 1$, respectively.

Erdağ (Erdağ, 2023) defined the Mersenne-Padovan sequence $\{M_n^{Pa}\}$ by the following homogeneous linear recurrence relation:

$$M_{n+5}^{Pa} = 3M_{n+4}^{Pa} - M_{n+3}^{Pa} - 2M_{n+2}^{Pa} - M_{n+1}^{Pa} + 2M_n^{Pa} \quad (1)$$

for $n \geq 0$ and with initial conditions $M_0^{Pa} = \dots = M_3^{Pa} = 0$ and $M_4^{Pa} = 1$.

Also in (Erdağ, 2023), by the recurrence relation (1), they have

$$\begin{bmatrix} M_{n+5}^{Pa} \\ M_{n+4}^{Pa} \\ M_{n+3}^{Pa} \\ M_{n+2}^{Pa} \\ M_{n+1}^{Pa} \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M_{n+4}^{Pa} \\ M_{n+3}^{Pa} \\ M_{n+2}^{Pa} \\ M_{n+1}^{Pa} \\ M_n^{Pa} \end{bmatrix}$$

for the Mersenne-Padovan sequence $\{M_n^{Pa}\}$ and they gave the generating matrix of the Mersenne-Padovan sequence $\{M_n^{Pa}\}$ as follows:

$$E_{Pa}^M = \begin{bmatrix} 3 & -1 & -2 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The matrix $E_{Pa}^M = [e_{ij}]_{5 \times 5}$ is said to be the Mersenne-Padovan matrix. Then, they obtained that

$$(E_{Pa}^M)^n = \begin{bmatrix} M_{n+4}^{Pa} & -M_{n+4}^{Pa} + P(n+4)-1 & -M_{n+4}^{Pa} + P(n) & -P(n+2)+1 & 2M_{n+3}^{Pa} \\ M_{n+3}^{Pa} & -M_{n+3}^{Pa} + P(n+3)-1 & -M_{n+3}^{Pa} + P(n-1) & -P(n+1)+1 & 2M_{n+2}^{Pa} \\ M_{n+2}^{Pa} & -M_{n+2}^{Pa} + P(n+2)-1 & -M_{n+2}^{Pa} + P(n-2) & -P(n)+1 & 2M_{n+1}^{Pa} \\ M_{n+1}^{Pa} & -M_{n+1}^{Pa} + P(n+1)-1 & -M_{n+1}^{Pa} + P(n-3) & -P(n-1)+1 & 2M_n^{Pa} \\ M_n^{Pa} & -M_n^{Pa} + P(n)-1 & -M_n^{Pa} + P(n-4) & -P(n-2)+1 & 2M_{n-1}^{Pa} \end{bmatrix}$$

for $n \geq 4$, which can be readily established by mathematical induction.

Many authors have recently investigated the characteristics of number theory, such as those derived from homogeneous linear recurrence relations, which are pertinent to this study; see for example: (Bradie, 2010; Horadam, 1994; Shannon, Horadam & Anderson, 2006; Taşçı & Firengiz, 2010; Tuğlu, Koçer & Stakhov, 2011). In (Aküzüm, 2020; Aküzüm & Deveci, 2021; Deveci & Aküzüm, 2022; Deveci, Aküzüm & Rashedi, 2022; Kilic, 2008; Kılıc, 2009; Kilic & Tasci, 2006; Stakhov, 1999), the authors defined some linear recurrence sequences and provided their various properties using matrix methods. Obtaining new sequences by multiplication of the characteristic polynomials of the sequences was first started in (Deveci, 2021). Later the concept was expanded by authors to different linear recurrence sequences; see for example: (Deveci & Shannon, 2021; Erdağ & Deveci, 2020a, 2020b; Erdağ & Deveci, 2021; Erdağ & Deveci, 2022; Erdağ, Deveci & Shannon, 2020; Shannon, Erdağ & Deveci, 2021). In this study, we consider

the Mersenne-Padovan sequence. Also, we derive the permanental and the determinantal representations of the Mersenne-Padovan numbers by using certain matrices that are obtained from the generating matrix of the Mersenne-Padovan sequence. Finally, we obtain the combinatorial and exponential representations and the finite sums of the Mersenne-Padovan numbers by the aid of the generating function and the generating matrix of the Mersenne-Padovan sequence.

Main Results

Definition 1. A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k th column (resp. row.) if the k th column (resp. row.) contains exactly two non-zero entries.

Suppose that $x_1, x_2, \dots, x_u$ are row vectors of the matrix M . If M is contractible in the k th column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ is obtained from M by replacing the i th row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j th row. The k th column is called the contraction in the k th column relative to the i th row and the j th row.

In (Brualdi & Gibson, 1997), Brualdi and Gibson obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M .

Now we concentrate on finding relationships among the Mersenne-Padovan numbers and the permanents of certain matrices which are obtained by using the generating matrix of the Mersenne-Padovan sequences.

Let $F_\eta^{M,Pa} = [f_{ij}]$ be the $\eta \times \eta$ super-diagonal matrix, defined by

$$f_{ij} = \begin{cases} 3, & \text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \leq \tau \leq \eta, \\ 2, & \text{if } i = \tau \text{ and } j = \tau + 4 \text{ for } 1 \leq \tau \leq \eta - 4, \\ 1, & \text{if } i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \leq \tau \leq \eta - 1, \\ & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq \eta - 1 \\ -1, & \text{and} \\ & i = \tau \text{ and } j = \tau + 3 \text{ for } 1 \leq \tau \leq \eta - 3, \\ -2, & \text{if } i = \tau \text{ and } j = \tau + 2 \text{ for } 1 \leq \tau \leq \eta - 2, \\ 0, & \text{otherwise.} \end{cases}$$

for $\eta \geq 5$. Then we have the following Theorem.

Theorem 1. For $\eta \geq 5$,

$$\text{per}F_{\eta}^{M,Pa} = M_{\eta+4}^{Pa}.$$

Proof. Let us consider matrix $F_{\eta}^{M,Pa}$ and the equation be hold for $\eta \geq 5$. Then we show that the equation holds for $\eta + 1$. If we expand the $\text{per}F_{\eta}^{M,Pa}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\text{per}F_{\eta+1}^{M,Pa} = 3\text{per}F_{\eta}^{M,Pa} - \text{per}F_{\eta-1}^{M,Pa} - 2\text{per}F_{\eta-2}^{M,Pa} - \text{per}F_{\eta-3}^{M,Pa} + 2\text{per}F_{\eta-4}^{M,Pa}.$$

Since

$$\text{per}F_{\eta}^{M,Pa} = M_{\eta+4}^{Pa},$$

$$\text{per}F_{\eta-1}^{M,Pa} = M_{\eta+3}^{Pa},$$

$$\text{per}F_{\eta-2}^{M,Pa} = M_{\eta+2}^{Pa},$$

$$\text{per}F_{\eta-3}^{M,Pa} = M_{\eta+1}^{Pa}$$

and

$$perF_{\eta-4}^{M,Pa} = M_{\eta}^{Pa}$$

we easily obtain that $perF_{\eta+1}^{M,Pa} = M_{\eta+5}^{Pa}$. So the proof is complete.

Let $G_{\eta}^{M,Pa} = [g_{ij}]$ be the $\eta \times \eta$ matrix, defined by

$$g_{ij} = \begin{cases} 3, & \text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \leq \tau \leq \eta - 1, \\ 2, & \text{if } i = \tau \text{ and } j = \tau + 4 \text{ for } 1 \leq \tau \leq \eta - 4, \\ & \text{if } i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \leq \tau \leq \eta - 2 \\ 1, & \text{and} \\ & i = \tau \text{ and } j = \tau \text{ for } \tau = \eta, \\ & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq \eta - 2 \\ -1, & \text{and} \\ & i = \tau \text{ and } j = \tau + 3 \text{ for } 1 \leq \tau \leq \eta - 3, \\ -2, & \text{if } i = \tau \text{ and } j = \tau + 2 \text{ for } 1 \leq \tau \leq \eta - 3, \\ 0, & \text{otherwise.} \end{cases}$$

for $\eta \geq 5$. Then we have the following Theorem.

Theorem 2. For $\eta \geq 5$,

$$perG_{\eta}^{M,Pa} = M_{\eta+3}^{Pa}.$$

Proof. Let us consider matrix $G_{\eta}^{M,Pa}$ and the equation be hold for $\eta \geq 5$. Then we show that the equation holds for $\eta + 1$. If we expand the $perG_{\eta}^{M,Pa}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perG_{\eta+1}^{M,Pa} = 3perG_{\eta}^{M,Pa} - perG_{\eta-1}^{M,Pa} - 2perG_{\eta-2}^{M,Pa} - perG_{\eta-3}^{M,Pa} + 2perG_{\eta-4}^{M,Pa}$$

.

Since

$$\text{per}G_{\eta}^{M,Pa} = M_{\eta+3}^{Pa},$$

$$\text{per}G_{\eta-1}^{M,Pa} = M_{\eta+2}^{Pa},$$

$$\text{per}G_{\eta-2}^{M,Pa} = M_{\eta+1}^{Pa},$$

$$\text{per}G_{\eta-3}^{M,Pa} = M_{\eta}^{Pa}$$

and

$$\text{per}G_{\eta-4}^{M,Pa} = M_{\eta-1}^{Pa}$$

we easily obtain that $\text{per}G_{\eta+1}^{M,Pa} = M_{\eta+4}^{Pa}$. So the proof is complete.

Assume that $H_{\eta}^{M,Pa} = [h_{ij}]$ is the $\eta \times \eta$ matrix, defined by

$$H_{\eta}^{M,Pa} = \begin{bmatrix} 1 & \cdots & 1 & 0 \\ 1 & & & \\ 0 & & G_{\eta-1}^{M,Pa} & \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

$(\eta-1)\text{th}$
 $\downarrow$

for $\eta > 5$, then we have the following results:

Theorem 3. For $\eta > 5$,

$$\text{per}H_{\eta}^{M,Pa} = \sum_{i=0}^{\eta+2} M_i^{Pa}.$$

Proof. If we extend $\text{per}G_{\alpha}^{M,F}$ with respect to the first row, we write

$$\text{per}H_{\eta}^{M,Pa} = \text{per}H_{\eta-1}^{M,Pa} + \text{per}G_{\eta-1}^{M,Pa}$$

Thus, by the results and an inductive argument, the proof is easily seen. $\square$

A matrix K is called convertible if there is an $n \times n$ $(1, -1)$ -matrix L such that $\text{per}K = \det(K * L)$, where $K * L$ denotes the Hadamard product of K and L .

Now we give relationships among the Mersenne-Padovan numbers and the determinants of certain matrices which are obtained by using the matrices $F_\eta^{M, Pa}$, $G_\eta^{M, Pa}$ and $H_\eta^{M, Pa}$. Let $\eta > 5$ and let R be the $\eta \times \eta$ matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 1. For $\eta > 5$,

$$\det(F_\eta^{M, Pa} * R) = M_{\eta+4}^{Pa},$$

$$\det(G_\eta^{M, Pa} * R) = M_{\eta+3}^{Pa}$$

and

$$\det(H_\eta^{M, Pa} * R) = \sum_{i=0}^{\eta+2} M_i^{Pa}.$$

Proof. Since $perF_{\eta}^{M,Pa} = \det(F_{\eta}^{M,Pa} * R)$,
 $perG_{\eta}^{M,Pa} = \det(G_{\eta}^{M,Pa} * R)$ and $perH_{\eta}^{M,Pa} = \det(H_{\eta}^{M,Pa} * R)$ for
 $\eta > 5$ by Theorem 1., Theorem 2. and Theorem 3., we have the
conclusion.

□

Let $K(k_1, k_2, \dots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \dots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

For more details on the companion type matrices, see (Lancaster & Tismenetsky, 1985; Lidl & Niederreiter, 1986).

Theorem 4. (Chen and Louck (Chen & Louck, 1996)) The (i, j)
entry $k_{i,j}^{(n)}(k_1, k_2, \dots, k_v)$ in the matrix $K^n(k_1, k_2, \dots, k_v)$ is given by
the following formula:

$$k_{i,j}^{(n)}(k_1, k_2, \dots, k_v) = \sum_{(t_1, t_2, \dots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \dots, t_v} k_1^{t_1} \cdots k_v^{t_v} \quad (2)$$

where the summation is over nonnegative integers satisfying
 $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1 + \cdots + t_v}{t_1, \dots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$ is a
multinomial coefficient, and the coefficients in (2) are defined to be
1 if $n = i - j$.

Then we can give other combinatorial representations than for the Mersenne-Padovan numbers by the following Corollary.

Corollary 2. Let M_n^{Pa} be the n th Mersenne-Padovan number for $n \geq 4$. Then

i.

$$M_n^{Pa} = \sum_{(t_1, t_2, t_3, t_4, t_5)} \binom{t_1 + t_2 + t_3 + t_4 + t_5}{t_1, t_2, t_3, t_4, t_5} 3^{t_1} 2^{t_5} (-1)^{t_2 + t_4} (-2)^{t_3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = n - 4$.

ii.

$$M_n^F = \frac{1}{2} \sum_{(t_1, t_2, t_3, t_4, t_5)} \frac{t_5}{t_1 + t_2 + t_3 + t_4 + t_5} \times \binom{t_1 + t_2 + t_3 + t_4 + t_5}{t_1, t_2, t_3, t_4, t_5} 3^{t_1} 2^{t_5} (-1)^{t_2 + t_4} (-2)^{t_3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 = n + 1$.

Proof. If we take $i = 5$, $j = 1$ for the case i. and $i = 4$, $j = 5$ for the case ii. in Theorem 4., then we can directly see the conclusions from $(E_{Pa}^M)^n$. $\square$

It is easy to see that the generating function of the Mersenne-Padovan sequence $\{M_n^{Pa}\}$ is as follows:

$$q(x) = \frac{x^4}{1 - 3x + x^2 + 2x^3 + x^4 - 2x^5}, \quad (0 \leq 3x - x^2 - 2x^3 - x^4 + 2x^5 < 1)$$

Now considering the function $q(x)$, we can give an exponential representation for the Mersenne-Padovan sequence by the following Theorem.

Theorem 5. Let $q(x)$ be generating function for the Mersenne-Padovan sequence. The exponential representation for the Mersenne-Padovan sequence is as follows:

$$q(x) = x^4 \exp \left(\sum_{i=1}^{\infty} \frac{(x)^i}{i} (3 - x - 2x^2 - x^3 + 2x^4)^i \right).$$

Proof. Since

$$\ln \frac{q(x)}{x^4} = -\ln(1 - 3x + x^2 + 2x^3 + x^4 - 2x^5)$$

and

$$\begin{aligned} \ln(1 - 3x + x^2 + 2x^3 + x^4 - 2x^5) = & - \left[x(3 - x - 2x^2 - x^3 + 2x^4) + \right. \\ & \frac{1}{2} x^2 (3 - x - 2x^2 - x^3 + 2x^4)^2 + \\ & \left. \dots + \frac{1}{i} x^i (3 - x - 2x^2 - x^3 + 2x^4)^i \right] \end{aligned}$$

by a simple calculation, we obtain the conclusion.
□

Now we consider the sums of the Mersenne-Padovan numbers. Let

$$S_n = \sum_{j=0}^n M_j^{Pa}$$

for $n \geq 4$ and let Q_{Pa}^M and $(Q_{Pa}^M)^n$ be the 6×6 matrix such that

$$Q_{Pa}^M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & & & & \\ 0 & E_{Pa}^M & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}.$$

If we use induction on n , then we obtain

$$\left(Q_{Pa}^M\right)^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ S_{n+3} & & & & & \\ S_{n+2} & & & & & \\ S_{n+1} & (E_{Pa}^M)^n & & & & \\ S_n & & & & & \\ S_{n-1} & & & & & \end{bmatrix}.$$

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CHAPTER XI

Parametric Version Of Modified Bernstein Operators

Emine GÜVEN

The approximation theory developed under the leadership of Karl Weierstrass is an important step towards a deep understanding of the analysis of functions in mathematics. Weierstrass developed this theory by addressing the problems that arose regarding differentiable and continuous functions in the mid-19th century.

Approximation theory can be broadly defined as the field that studies how closely a function can be imitated by another sequence or series of functions. Addressing some particularly challenging situations in mathematics, Weierstrass showed that a function could be expressed as an infinite sum with a series of other functions. This means that any function can be emulated to the desired precision, despite the complexity of previously determined functions.

Approximation theory was developed because it is an important tool in the mathematical world in efforts to understand the complexity of a function and reach more general conclusions.

Russian mathematician Sergei Natanovich Korovkin, who worked in this field, made important contributions to this field. This theorem, known as the Korovkin Theorem, defines the property of function sequences to approach a certain class under certain conditions and opens the door to many important results in this field.

One of the mathematicians who made significant contributions to Korovkin's approximation theory is David Emmanuel Bernstein. Bernstein defined a class of polynomials that was particularly effective in the approximation of bounded functions and their derivatives. These polynomials increase the applicability to Korovkin's approximation theory by characterizing the general behavior of functions and providing certain approximation properties. Bernstein's works are considered important steps towards building more firmly the foundations of Korovkin's approximation theory in mathematical analysis. These works have been a source of inspiration for many mathematicians in the fields of function theory and approximation theory.

Parametric generalizations of operators within approximation theory is a field that studies the parametric approximation of one class of functions by another class of functions. These generalizations attempt to understand the approximation properties of a function in a broader context, usually by examining parametric families of operators. These studies aim to develop new methods for better approximation of functions in mathematical analysis and applied mathematics.

Parametric generalizations of Bernstein operators are given in [Chen et al., 2017]. Parametric generalizations of operators [Aral et al., 2019], [Cai et al., 2018], [Cai et al., 2021], [Cai et al., 2022], [Çekim et al., 2022], [Kadak et al., 2021], [Kajla et al., 2020], [Mohiuddine et al., 2021], [Mohiuddine et al., 2020], [Özger, 2019], [Srivastava et al., 2019] and [Srivastava et al., 2021].

In this study, the proof of the Korovkin type theorem will be given by defining the parametric modified Bernstein operator for the

modified Bernstein operator on the symmetric interval, which was defined and studied by Izgi and Cilo in 2012.

Definition 1.1

$t \in [-1,1]$ and $f \in C[-1,1]$,

$$\mathbf{C}_\eta(f; t) = \frac{1}{2^\eta} \sum_{k=0}^{\eta} \binom{\eta}{k} (1+t)^\kappa (1-t)^{\eta-k} f\left(2\frac{\kappa}{\eta} - 1\right) \quad (1)$$

is called the operator $\mathbf{C}_\eta(f; t)$.

Definition 1.2

$$\wp_{\eta,\kappa} = \binom{\eta}{\kappa} (1-t)^{\eta-\kappa} (1+t)^\kappa$$

Parametric generalization of the modified Bernstein operator for each $f \in C[-1,1]$, $\eta \in \mathbb{N}$ and $t \in [-1,1]$

$$\mathfrak{C}_{\eta,\gamma}(f; t) = \sum_{k=0}^{\eta} \wp_{\eta,\kappa}^{(\gamma)} f\left(2\frac{\kappa}{\eta} - 1\right).$$

Where $\eta \geq 1$, $0 \leq \varpi \leq 1$, $t \in [-1,1]$ and

$$\wp_{1,0}^\gamma(t) = 1 - t, \quad \wp_{1,1}^{(\gamma)}(t) = 1 + t \text{ and}$$

to be

$$\begin{aligned} \wp_{\eta,\kappa}^{(\gamma)}(t) = & \left\{ \frac{(1-\gamma)}{2^{\eta-1}} \binom{\eta-2}{\kappa} (1+t) + \frac{(1-\gamma)}{2^{\eta-1}} \binom{\eta-2}{\kappa-2} (1-t) \right. \\ & \left. + \frac{\gamma}{2^\eta} \binom{\eta}{\kappa} (1+t)(1-t) \right\} (1+t)^{\kappa-1} (1-t)^{\eta-\kappa-1}, \quad \eta \geq 2 \end{aligned}$$

and

$$\binom{\eta}{\kappa} = \begin{cases} \frac{\eta!}{(\eta - \kappa)! \kappa!} & , \quad 0 \leq \kappa \leq \eta \\ 0 & , \quad \text{other} \end{cases}$$

is. Here

$$\binom{\eta - 2}{-2} = \binom{\eta - 2}{-1} = 0$$

is.

$$\binom{\eta - 2}{\kappa} = \left(1 - \frac{\kappa}{\eta - 1}\right) \binom{\eta - 1}{\kappa} \quad \text{and} \quad \binom{\eta - 2}{\kappa - 2} = \frac{\kappa}{\eta - 1} \binom{\eta - 1}{\kappa}$$

it is clear that it is.

Lemma 1.1

For every $f \in C[-1,1]$, $\eta \in \mathbb{N}$ and $t \in [-1,1]$

$$g\left(2\frac{\kappa}{\eta} - 1\right) = f\left(2\frac{\kappa}{\eta} - 1\right) \left(1 - \frac{\kappa}{\eta - 1}\right) + f\left(2\frac{\kappa + 1}{\eta} - 1\right) \frac{\kappa}{\eta - 1}$$

So,

$$\begin{aligned} \mathfrak{E}_{\eta,\gamma}(f; t) &= (1 - \gamma) \sum_{\kappa=0}^{\eta-1} g\left(2\frac{\kappa}{\eta} - 1\right) (1 - t)^{\eta-\kappa-1} \binom{\eta - 1}{\kappa} (1 + t)^{\kappa} \\ &+ \gamma \sum_{\kappa=0}^{\eta} f\left(2\frac{\kappa}{\eta} - 1\right) (1 - t)^{\eta-\kappa} \binom{\eta}{\kappa} (1 + t)^{\kappa} \end{aligned}$$

is.

Proof:

$$\mathfrak{E}_{\eta,\gamma}(f; t) = \sum_{\kappa=0}^{\eta} \left\{ \frac{(1 - \gamma)}{2^{\eta-1}} \binom{\eta - 2}{\kappa} (1 + t) + \frac{(1 - \gamma)}{2^{\eta-1}} \binom{\eta - 2}{\kappa - 2} (1 - t) \right\}$$

$$\begin{aligned}
& + \frac{\gamma}{2^\eta} \binom{\eta}{\kappa} (1 + t)(1 - t) \} (1 - t)^{\eta - \kappa - 1} (1 + t)^{\kappa - 1} f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& = \sum_{\kappa=0}^{\eta} \frac{(1 - \gamma)}{2^{\eta - 1}} \binom{\eta - 2}{\kappa} (1 + t)(1 - t)^{\eta - \kappa - 1} (1 + t)^{\kappa - 1} f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& + \sum_{\kappa=0}^{\eta} \frac{(1 - \gamma)}{2^{\eta - 1}} \binom{\eta - 2}{\kappa - 2} (1 - t)(1 - t)^{\eta - \kappa - 1} (1 + t)^{\kappa - 1} f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& + \sum_{\kappa=0}^{\eta} \frac{\gamma}{2^\eta} \binom{\eta}{\kappa} (1 + t)(1 - t)(1 - t)^{\eta - \kappa - 1} (1 + t)^{\kappa - 1} f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& = \sum_{\kappa=0}^{\eta} \frac{(1 - \gamma)}{2^{\eta - 1}} \binom{\eta - 2}{\kappa} (1 - t)^{\eta - \kappa - 1} (1 + t)^\kappa f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& + \sum_{\kappa=0}^{\eta} \frac{(1 - \gamma)}{2^{\eta - 1}} \binom{\eta - 2}{\kappa - 2} (1 - t)^{\eta - \kappa} (1 + t)^{\kappa - 1} f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& + \sum_{\kappa=0}^{\eta} \frac{\gamma}{2^\eta} \binom{\eta}{\kappa} (1 - t)^{\eta - \kappa} (1 + t)^\kappa f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& = (1 - \gamma) \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta - 1}} (1 - t)^{\eta - \kappa - 1} \binom{\eta - 2}{\kappa} (1 + t)^\kappa f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& + (1 - \gamma) \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta - 1}} (1 - t)^{\eta - \kappa} \binom{\eta - 2}{\kappa - 2} (1 + t)^{\kappa - 1} f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^\eta} (1 - t)^{\eta - \kappa} \binom{\eta}{\kappa} (1 + t)^\kappa f \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& = (1 - \gamma)(c_1 + c_2) + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^\eta} (1 - t)^{\eta - \kappa} \binom{\eta}{\kappa} (1 + t)^\kappa f \left(2 \frac{\kappa}{\eta} - 1 \right)
\end{aligned}$$

$$c_1 = \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right)$$

$$c_2 = \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa} \binom{\eta-2}{\kappa-2} (1+t)^{\kappa-1} f\left(2\frac{\kappa}{\eta}-1\right)$$

c_1 is zero for $\eta = \kappa$, c_2 , is zero for $\kappa = 0$. Let's

$$c_1 = \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right)$$

$$c_2 = \sum_{\kappa=1}^{\eta} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa} \binom{\eta-2}{\kappa-2} (1+t)^{\kappa-1} f\left(2\frac{\kappa}{\eta}-1\right)$$

is getting.

$$\begin{aligned} c_1 &= \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right) \\ &= \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} \left(1 - \frac{\kappa}{\eta-1}\right) \binom{\eta-1}{\kappa} (1-t)^{\eta-\kappa-1} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right) \end{aligned}$$

and

$$c_2 = \frac{1}{2^{\eta-1}} \sum_{\kappa=1}^{\eta} (1-t)^{\eta-\kappa} \binom{\eta-2}{\kappa-2} (1+t)^{\kappa-1} f\left(2\frac{\kappa}{\eta}-1\right)$$

$$\begin{aligned}
&= \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \binom{\eta-2}{\kappa-1} (1+t)^{\kappa} f\left(2\frac{\kappa+1}{\eta}-1\right) \\
&= \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} \frac{\kappa}{\eta-1} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa+1}{\eta}-1\right)
\end{aligned}$$

$$\begin{aligned}
c_1 + c_2 &= \frac{1}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} \left\{ f\left(2\frac{\kappa}{\eta}-1\right) \left(1-\frac{\kappa}{\eta-1}\right) + f\left(2\frac{\kappa+1}{\eta}-1\right) \frac{\kappa}{\eta-1} \right\} \\
&\times (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa}
\end{aligned}$$

is found. Here if we write,

$$g\left(2\frac{\kappa}{\eta}-1\right) = f\left(2\frac{\kappa}{\eta}-1\right) \left(1-\frac{\kappa}{\eta-1}\right) + f\left(2\frac{\kappa+1}{\eta}-1\right) \frac{\kappa}{\eta-1}$$

then, it is called modified Bernstein operators parametric version

$$\begin{aligned}
\mathfrak{E}_{\eta,\gamma}(f; t) &= (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} g\left(2\frac{\kappa}{\eta}-1\right) \\
&+ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta}-1\right).
\end{aligned}$$

Lemma 1.2

For $\mathfrak{E}_{\eta,\gamma}(f; t)$

$$i) \mathfrak{E}_{\eta,\gamma}(1; t) = 1,$$

$$ii) \mathfrak{E}_{\eta,\gamma}(t; t) = t,$$

$$iii) \mathfrak{C}_{\eta,\gamma}(t^2; t) = \frac{\eta(\eta-1) + 2(\gamma-1)}{\eta^2} t^2 - \frac{\eta + 2(1-\gamma)}{\eta^2}$$

Proof:

$$i) \mathfrak{C}_{\eta,\gamma}(1; t) = (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa$$

$$+ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^\eta} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^\kappa$$

$$= (1-\gamma) \frac{1}{2^{\eta-1}} 2^{\eta-1} + \gamma = 1.$$

$$ii) \mathfrak{C}_{\eta,\gamma}(t; t) = (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa g\left(2\frac{\kappa}{\eta} - 1\right)$$

$$+ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^\eta} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^\kappa f\left(2\frac{\kappa}{\eta} - 1\right)$$

$$= (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa$$

$$\times \left[f\left(2\frac{\kappa}{\eta} - 1\right) \left(1 - \frac{\kappa}{\eta-1}\right) + f\left(2\frac{\kappa+1}{\eta} - 1\right) \frac{\kappa}{\eta-1} \right]$$

$$+ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^\eta} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^\kappa f\left(2\frac{\kappa}{\eta} - 1\right)$$

$$= (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa$$

$$\begin{aligned}
& \times \left[\left(2 \frac{\kappa}{\eta} - 1 \right) \left(1 - \frac{\kappa}{\eta - 1} \right) + \left(2 \frac{\kappa + 1}{\eta} - 1 \right) \frac{\kappa}{\eta - 1} \right] \\
& + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1 - t)^{\eta - \kappa} \binom{\eta}{\kappa} (1 + t)^{\kappa} \left(2 \frac{\kappa}{\eta} - 1 \right) \\
& = (1 - \gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1 - t)^{\eta - \kappa - 1} \binom{\eta - 1}{\kappa} (1 + t)^{\kappa} \\
& \times \left[\left(\frac{2\kappa - \eta}{\eta} \right) \left(\frac{\eta - 1 - \kappa}{\eta - 1} \right) + \left(\frac{2\kappa + 2 - \eta}{\eta} \right) \frac{\kappa}{\eta - 1} \right] \\
& + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1 - t)^{\eta - \kappa} \binom{\eta}{\kappa} (1 + t)^{\kappa} \left(\frac{2\kappa}{\eta} - 1 \right) \\
& = (1 - \gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1 - t)^{\eta - \kappa - 1} \binom{\eta - 1}{\kappa} (1 + t)^{\kappa} \\
& \times \left[\frac{2\kappa\eta - 2\kappa - 2\kappa^2 - \eta^2 + \eta + \eta\kappa + 2\kappa^2 + 2\kappa - \eta\kappa}{\eta(\eta - 1)} \right] \\
& + \gamma \frac{2}{2^{\eta}\eta} \sum_{\kappa=0}^{\eta} \kappa (1 - t)^{\eta - \kappa} \binom{\eta}{\kappa} (1 + t)^{\kappa} \\
& \quad - \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1 - t)^{\eta - \kappa} \binom{\eta}{\kappa} (1 + t)^{\kappa} \\
& = (1 - \gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1 - t)^{\eta - \kappa - 1} \binom{\eta - 1}{\kappa} (1 + t)^{\kappa} \left[\frac{2\kappa\eta - \eta^2 + \eta}{\eta(\eta - 1)} \right]
\end{aligned}$$

$$\begin{aligned}
& + \gamma \frac{2}{2^\eta \eta} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^\kappa \\
& \quad - \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^\eta} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^\kappa \\
& = (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa \left[\frac{\eta(2\kappa-\eta+1)}{\eta(\eta-1)} \right] \\
& + \gamma \frac{2}{2^\eta \eta} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^\kappa - \gamma \\
& = (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa \left[\frac{2\kappa-\eta+1}{\eta-1} \right] \\
& + \gamma \frac{2}{2^\eta \eta} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^\kappa - \gamma \\
& = (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa \left[\frac{2\kappa}{\eta-1} - 1 \right] \\
& + \gamma \frac{2}{2^\eta \eta} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)! \kappa!} (1+t)^\kappa - \gamma \\
& = \frac{2(1-\gamma)}{(\eta-1)2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} \kappa (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa \\
& - \frac{(1-\gamma)}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^\kappa
\end{aligned}$$

$$\begin{aligned}
& + \gamma \frac{2}{2^\eta \eta} \sum_{\kappa=1}^{\eta} \kappa (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)! \kappa!} (1+t)^\kappa - \gamma \\
& = \frac{2(1-\gamma)}{(\eta-1)2^{\eta-1}} \sum_{\kappa=1}^{\eta-1} \kappa (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)! \kappa!} (1+t)^\kappa \\
& \quad - \frac{(1-\gamma)}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)! \kappa!} (1+t)^\kappa \\
& \quad + \gamma \frac{2}{2^\eta \eta} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{\eta!}{(\eta-\kappa-1)! \kappa!} (1+t)^{\kappa+1} - \gamma \\
& = \frac{2(1-\gamma)}{(\eta-1)2^{\eta-1}} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{(\eta-1)!}{(\eta-\kappa-2)! \kappa!} (1+t)^{\kappa+1} \\
& \quad - \frac{(1-\gamma)}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)! \kappa!} (1+t)^\kappa \\
& \quad + \gamma \frac{2}{2^\eta \eta} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{\eta!}{(\eta-\kappa-1)! \kappa!} (1+t)^{\kappa+1} - \gamma \\
& = \frac{2(1-\gamma)(1+t)(\eta-1)}{(\eta-1)2^{\eta-1}} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{(\eta-2)!}{(\eta-\kappa-2)! \kappa!} (1+t)^\kappa \\
& \quad - \frac{(1-\gamma)}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)! \kappa!} (1+t)^\kappa \\
& \quad + \gamma \frac{2(1+t)\eta}{2^\eta \eta} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)! \kappa!} (1+t)^\kappa - \gamma
\end{aligned}$$

$$\begin{aligned}
&= \frac{2(1-\gamma)(1+t)}{2^{\eta-1}} 2^{\eta-2} - \frac{(1-\gamma)}{2^{\eta-1}} 2^{\eta-1} + \gamma \frac{2(1+t)\eta}{2^{\eta}\eta} 2^{\eta-1} - \gamma \\
&= (1-\gamma)(1+t) - (1-\gamma) + \gamma(1+t) - \gamma \\
&= (1-\gamma)(1+t-1) + \gamma(1+t-1) \\
&= (1-\gamma)t + \gamma t
\end{aligned}$$

$$\mathfrak{E}_{\eta,\gamma}(t; t) = t$$

is obtained.

iii) If $\mathfrak{E}_{\eta,\gamma}(t^2; t)$ is calculated,

$$\begin{aligned}
\mathfrak{E}_{\eta,\gamma}(t^2; t) &= (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} g\left(2\frac{\kappa}{\eta} - 1\right) \\
&\quad + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} f\left(2\frac{\kappa}{\eta} - 1\right) \\
&= (1-\gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} \\
&\quad \times \left[\left(2\frac{\kappa}{\eta} - 1\right)^2 \left(1 - \frac{\kappa}{\eta-1}\right) + \left(2\frac{\kappa+1}{\eta} - 1\right)^2 \frac{\kappa}{\eta-1} \right] \\
&\quad + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} \left(2\frac{\kappa}{\eta} - 1\right)^2
\end{aligned}$$

Here,

$$\begin{aligned}
& \left[\left(\frac{2\kappa}{\eta} - 1 \right)^2 \left(1 - \frac{\kappa}{\eta - 1} \right) + \left(\frac{2\kappa + 2}{\eta} - 1 \right)^2 \frac{\kappa}{\eta - 1} \right] \\
&= \left(\frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1 \right) \left(1 - \frac{\kappa}{\eta - 1} \right) + \left(\frac{4\kappa^2 + 8\kappa + 4}{\eta^2} - \frac{4\kappa + 4}{\eta} + 1 \right) \frac{\kappa}{\eta - 1} \\
&= \frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1 - \left(\frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1 \right) \frac{\kappa}{\eta - 1} + \frac{4\kappa^2 + 8\kappa + 4}{\eta^2} \frac{\kappa}{\eta - 1} \\
&\quad - \frac{4\kappa + 4}{\eta} \frac{\kappa}{\eta - 1} + \frac{\kappa}{\eta - 1} \\
&= \frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1 \\
&\quad + \left[-\frac{4\kappa^2}{\eta^2} + \frac{4\kappa}{\eta} - 1 + \frac{4\kappa^2 + 8\kappa + 4}{\eta^2} - \frac{4\kappa + 4}{\eta} \right. \\
&\quad \left. + 1 \right] \frac{\kappa}{\eta - 1} \\
&= \frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1 \\
&\quad + \left[-\frac{4\kappa^2}{\eta^2} + \frac{4\kappa}{\eta} - 1 + \frac{4\kappa^2}{\eta^2} + \frac{8\kappa}{\eta^2} + \frac{4}{\eta^2} - \frac{4\kappa}{\eta} - \frac{4}{\eta} \right. \\
&\quad \left. + 1 \right] \frac{\kappa}{\eta - 1} \\
&= \frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1 + \left[\frac{8\kappa}{\eta^2} + \frac{4}{\eta^2} - \frac{4}{\eta} \right] \frac{\kappa}{\eta - 1} \\
&= \frac{4\kappa^2(\eta - 1)}{\eta^2(\eta - 1)} - \frac{4\kappa\eta(\eta - 1)}{\eta^2(\eta - 1)} + \frac{8\kappa^2}{\eta^2(\eta - 1)} + \frac{4\kappa}{\eta^2(\eta - 1)} - \frac{4\kappa\eta}{\eta^2(\eta - 1)} + 1 \\
&= \frac{4\kappa^2\eta - 4\kappa^2 - 4\kappa\eta^2 + 4\kappa\eta + 8\kappa^2 + 4\kappa - 4\kappa\eta}{\eta^2(\eta - 1)} + 1 \\
&= \frac{4\kappa^2\eta + 4\kappa\eta^2 - 4\kappa\eta^2 + 4\kappa}{\eta^2(\eta - 1)} + 1
\end{aligned}$$

$$\begin{aligned}
&= \frac{4\kappa^2(\eta + 1) - 4\kappa(\eta^2 - 1)}{\eta^2(\eta - 1)} + 1 \\
&= \frac{4\kappa^2(\eta + 1)}{\eta^2(\eta - 1)} - \frac{4\kappa(\eta - 1)(\eta + 1)}{\eta^2(\eta - 1)} + 1 \\
&= \frac{4\kappa^2(\eta + 1)}{\eta^2(\eta - 1)} - \frac{4\kappa(\eta + 1)}{\eta^2} + 1
\end{aligned}$$

if we substitute in $\mathfrak{C}_{n,\gamma}(t^2; t)$

$$\begin{aligned}
&\mathfrak{C}_{\eta,\gamma}(t^2; t) = \\
&= (1 - \gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1 - t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1 + t)^\kappa \left[\frac{4(\eta+1)}{\eta^2(\eta-1)} \kappa^2 \right. \\
&\quad \left. - \frac{4(\eta+1)}{\eta^2} \kappa + 1 \right] \\
&+ \gamma \sum_{\kappa=0}^{\eta} \frac{1}{\eta} (1 - t)^{\eta-\kappa} \binom{\eta}{\kappa} (1 + t)^\kappa \left(\frac{4\kappa^2}{\eta^2} - \frac{4\kappa}{\eta} + 1 \right) \\
&= (1 - \gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1 - t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1 + t)^\kappa \frac{4(\eta+1)}{\eta^2(\eta-1)} \kappa^2 \\
&- (1 - \gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1 - t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1 + t)^\kappa \frac{4(\eta+1)}{\eta^2} \kappa \\
&+ (1 - \gamma) \sum_{\kappa=0}^{\eta-1} \frac{1}{2^{\eta-1}} (1 - t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1 + t)^\kappa \\
&\quad + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^\eta} (1 - t)^{\eta-\kappa} \binom{\eta}{\kappa} (1 + t)^\kappa \frac{4\kappa^2}{\eta^2}
\end{aligned}$$

$$\begin{aligned}
& -\gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} \frac{4\kappa}{\eta} \\
& \quad + \gamma \sum_{\kappa=0}^{\eta} \frac{1}{2^{\eta}} (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} \\
& = \frac{4(\eta+1)(1-\gamma)}{2^{\eta-1}\eta^2(\eta-1)} \sum_{\kappa=0}^{\eta-1} \kappa^2 (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} \\
& \quad - \frac{4(1-\gamma)(\eta+1)}{2^{\eta-1}\eta^2} \sum_{\kappa=0}^{\eta-1} \kappa (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} \\
& \quad + \frac{(1-\gamma)}{2^{\eta-1}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} \\
& \quad \quad + \frac{4\gamma}{\eta^2 2^{\eta}} \sum_{\kappa=0}^{\eta} \kappa^2 (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} \\
& \quad - \frac{4\gamma}{\eta^2 2^{\eta}} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} + \frac{\gamma}{2^{\eta}} 2^{\eta} \\
& = \frac{(\eta+1)(1-\gamma)}{2^{n-3}\eta^2(\eta-1)} \sum_{\kappa=0}^{\eta-1} \kappa^2 (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} \\
& \quad + \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=0}^{\eta} \kappa^2 (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} \\
& \quad - \frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^2} \sum_{\kappa=0}^{\eta-1} \kappa (1-t)^{\eta-\kappa-1} \binom{\eta-1}{\kappa} (1+t)^{\kappa} \\
& \quad - \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=0}^{\eta} \kappa (1-t)^{\eta-\kappa} \binom{\eta}{\kappa} (1+t)^{\kappa} + \frac{(1-\gamma)}{2^{\eta-1}} 2^{\eta-1} + \gamma
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=2}^{\eta-1} \kappa(\kappa-1) (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\kappa!} (1+t)^\kappa \\
&+ \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=1}^{\eta-1} \kappa (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\kappa!} (1+t)^\kappa \\
&+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=2}^{\eta} \kappa(\kappa-1)(1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\kappa!} (1+t)^\kappa \\
&+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=1}^{\eta} \kappa (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\kappa!} (1+t)^\kappa \\
&- \frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^2} \sum_{\kappa=1}^{\eta-1} \kappa(1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!\kappa!} (1+t)^\kappa \\
&- \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=1}^{\eta} \kappa (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!\kappa!} (1+t)^\kappa + 1 \\
&= \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=2}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!(\kappa-2)!} (1+t)^\kappa \\
&+ \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=1}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!(\kappa-1)!} (1+t)^\kappa \\
&+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=2}^{\eta} (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!(\kappa-2)!} (1+t)^\kappa \\
&+ \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=1}^{\eta} (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!(\kappa-1)!} (1+t)^\kappa
\end{aligned}$$

$$\begin{aligned}
& -\frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^2} \sum_{\kappa=1}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)!(\kappa-1)!} (1+t)^\kappa \\
& -\frac{\gamma}{\eta 2^{\eta-2}} \sum_{\kappa=1}^{\eta} (1-t)^{\eta-\kappa} \frac{\eta!}{(\eta-\kappa)!(\kappa-1)!} (1+t)^\kappa + 1 \\
& = \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=0}^{\eta-3} (1-t)^{\eta-\kappa-3} \frac{(\eta-1)!}{(\eta-\kappa-3)!\kappa!} (1+t)^{\kappa+2} \\
& + \frac{(\eta+1)(1-\gamma)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{(\eta-1)!}{(\eta-\kappa-2)!\kappa!} (1+t)^{\kappa+1} \\
& + \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{\eta!}{(\eta-\kappa-2)!\kappa!} (1+t)^{\kappa+2} \\
& + \frac{\gamma}{\eta^2 2^{\eta-2}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{\eta!}{(\eta-\kappa-1)!\kappa!} (1+t)^{\kappa+1} \\
& - \frac{(1-\gamma)(\eta+1)}{2^{\eta-3}\eta^2} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{(\eta-1)!}{(\eta-\kappa-2)!\kappa!} (1+t)^{\kappa+1} \\
& - \frac{\gamma}{\eta 2^{\eta-2}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{\eta!}{(\eta-\kappa-1)!\kappa!} (1+t)^{\kappa+1} + 1 \\
& = \frac{(\eta+1)(1-\gamma)(\eta-1)(\eta-2)(1+t)^2}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=0}^{\eta-3} \left(1 \right. \\
& \quad \left. - t \right)^{\eta-\kappa-3} \frac{(\eta-3)!}{(\eta-\kappa-3)!\kappa!} (1+t)^\kappa
\end{aligned}$$

$$\begin{aligned}
& + \frac{(\eta+1)(1-\gamma)(\eta-1)(1+t)}{2^{\eta-3}\eta^2(\eta-1)} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{(\eta-2)!}{(\eta-\kappa-2)! \kappa!} (1+t)^{\kappa} \\
& + \frac{\gamma\eta(\eta-1)(1+t)^2}{\eta^2 2^{\eta-2}} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{(\eta-2)!}{(\eta-\kappa-2)! \kappa!} (1+t)^{\kappa} \\
& + \frac{\gamma\eta(1+t)}{\eta^2 2^{\eta-2}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)! \kappa!} (1+t)^{\kappa} \\
& - \frac{(1-\gamma)(\eta+1)(\eta-1)(1+t)}{2^{\eta-3}\eta^2} \sum_{\kappa=0}^{\eta-2} (1-t)^{\eta-\kappa-2} \frac{(\eta-2)!}{(\eta-\kappa-2)! \kappa!} (1+t)^{\kappa} \\
& - \frac{\gamma\eta(1+t)}{\eta^2 2^{\eta-2}} \sum_{\kappa=0}^{\eta-1} (1-t)^{\eta-\kappa-1} \frac{(\eta-1)!}{(\eta-\kappa-1)! \kappa!} (1+t)^{\kappa} + 1 \\
& = \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2}{2^{\eta-3}\eta^2} 2^{\eta-3} + \frac{(\eta+1)(1-\gamma)(1+t)}{2^{\eta-3}\eta^2} 2^{\eta-2} \\
& + \frac{\gamma(\eta-1)(1+t)^2}{\eta^2 2^{\eta-2}} 2^{\eta-2} + \frac{\gamma(1+t)}{\eta^2 2^{\eta-2}} 2^{\eta-1} \\
& \quad - \frac{(1-\gamma)(\eta+1)(\eta-1)(1+t)}{2^{\eta-3}\eta^2} 2^{\eta-2} \\
& - \frac{\gamma(1+t)}{2^{\eta-2}} 2^{\eta-1} + 1 \\
& = \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2}{\eta^2} + \frac{2(\eta+1)(1-\gamma)(1+t)}{\eta^2} \\
& + \frac{\gamma(\eta-1)(1+t)^2}{\eta} + \frac{2\gamma(1+t)}{\eta} - \frac{2(1-\gamma)(\eta+1)(\eta-1)(1+t)}{\eta^2} \\
& - 2\gamma(1+t) + 1
\end{aligned}$$

$$\begin{aligned}
&= \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2 + 2(\eta+1)(1-\gamma)(1+t)}{\eta^2} \\
&\quad - \frac{2(1-\gamma)(\eta+1)(\eta-1)(1+t)}{\eta^2} \\
&\quad + \frac{\gamma(\eta-1)(1+t)^2 + 2\gamma(1+t)}{\eta} - 2\gamma(1+t) + 1 \\
&= \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2 + [1-(\eta-1)]2(\eta+1)(1-\gamma)(1+t)}{\eta^2} \\
&\quad + \frac{\gamma(\eta-1)\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2} \\
&= \frac{(1-\gamma)(\eta+1)(\eta-2)(1+t)^2 + (2-\eta)2(\eta+1)(1-\gamma)(1+t)}{\eta^2} \\
&\quad + \frac{\gamma(\eta-1)\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2} \\
&= \frac{(1-\gamma)(\eta+1)(\eta-2)[(1+t)^2 - 2(1+t)]}{\eta^2} \\
&\quad + \frac{\gamma\eta^2(1+t)^2 - \gamma\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2} \\
&= \frac{(1-\gamma)(\eta+1)(\eta-2)[t^2 + 2t + 1 - 2 - 2t]}{\eta^2} \\
&\quad + \frac{\gamma\eta^2(1+t)^2 - \gamma\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2} \\
&= \frac{(1-\gamma)(\eta+1)(\eta-2)(t^2 - 1)}{\eta^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma\eta^2(1+t)^2 - \gamma\eta(1+t)^2 + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2} \\
& = \frac{(1-\gamma)(\eta+1)(\eta-2)t^2}{\eta^2} - \frac{(1-\gamma)(\eta+1)(\eta-2)}{\eta^2} \\
& + \frac{\gamma\eta^2(1+2t+t^2) - \gamma\eta(1+2t+t^2) + 2\gamma\eta(1+t) - 2\gamma(1+t)\eta^2 + \eta^2}{\eta^2} \\
& = \frac{(1-\gamma)(\eta+1)(\eta-2)t^2}{\eta^2} - \frac{(1-\gamma)(\eta+1)(\eta-2)}{\eta^2} \\
& + \frac{\gamma\eta^2 + 2\gamma\eta^2t + \gamma\eta^2t^2 - \gamma\eta - 2\gamma\eta t - \gamma\eta t^2 + 2\gamma\eta + 2\gamma\eta t - 2\gamma\eta^2 - 2\gamma\eta^2t + \eta^2}{n\eta^2} \\
& = \frac{(1-\gamma)(\eta+1)(\eta-2)t^2}{\eta^2} - \frac{(1-\gamma)(\eta+1)(\eta-2)}{\eta^2} \\
& + \frac{\gamma\eta^2t^2 - \gamma\eta t^2 + \gamma\eta - \gamma\eta^2 + \eta^2}{\eta^2} \\
& = \frac{(1-\gamma)(\eta+1)(\eta-2)t^2 + \gamma\eta^2t^2 - \gamma\eta t^2}{\eta^2} \\
& \quad - \frac{(1-\gamma)(\eta+1)(\eta-2) - \gamma\eta + \gamma\eta^2 - \eta^2}{\eta^2} \\
& = \frac{[(1-\gamma)(\eta^2 - \eta - 2) + \gamma\eta^2 - \gamma\eta]t^2}{\eta^2} \\
& \quad - \frac{(1-\gamma)(\eta^2 - \eta - 2) - \gamma\eta + \gamma\eta^2 - \eta^2}{\eta^2} \\
& = \frac{[n^2 - n - 2 - n^2\gamma + n\gamma + 2\gamma + \gamma n^2 - \gamma n]t^2}{\eta^2} \\
& + \frac{\eta^2 - \eta - 2 - \eta^2\gamma + \eta\gamma + 2\gamma - \gamma\eta + \gamma\eta^2 - \eta^2}{\eta^2}
\end{aligned}$$

$$= \frac{[\eta^2 - \eta - 2 + 2\gamma]t^2}{\eta^2} - \frac{\eta + 2 - 2\gamma}{\eta^2}$$

$$C_{n,\gamma}(t^2; t) = \frac{\eta(\eta - 1) + 2(\gamma - 1)}{\eta^2} t^2 - \frac{\eta + 2(1 - \gamma)}{\eta^2}$$

is obtained.

Lemma 1.3

The central moments of the operator $\mathfrak{C}_{n,\gamma}(f; t)$ are as follows.

$$i) \mathfrak{C}_{n,\gamma}((t - t)^0, x) = 1$$

$$ii) \mathfrak{C}_{n,\gamma}(t - t, t) = 0$$

$$iii) \mathfrak{C}_{n,\gamma}((t - t)^2, t) = \left[\frac{-\eta - 2 + 2\gamma}{\eta^2} \right] x^2 - \frac{\eta + 2 - 2\gamma}{\eta^2}$$

Proof:

$$i) \mathfrak{C}_{n,\gamma}((t - t)^0, t) = \mathfrak{C}_{n,\gamma}(1, t) = 1$$

$$ii) \mathfrak{C}_{n,\gamma}(t - t, t) = t - t = 0$$

$$iii) \mathfrak{C}_{n,\gamma}((t - t)^2, t) = \mathfrak{C}_{n,\gamma}(t^2, t) - 2t\mathfrak{C}_{n,\gamma}(t, t) + t^2 \mathfrak{C}_{n,\gamma}(1, t)$$

$$= \frac{\eta(\eta - 1) + 2(\gamma - 1)}{\eta^2} t^2 - \frac{\eta + 2(1 - \gamma)}{\eta^2} - 2tt + t^2$$

$$= \frac{\eta(\eta - 1) + 2(\gamma - 1)}{\eta^2} t^2 - \frac{\eta + 2(1 - \gamma)}{\eta^2} - t^2$$

$$= \left[\frac{\eta(\eta - 1) + 2(\gamma - 1)}{\eta^2} - 1 \right] t^2 - \frac{\eta + 2(1 - \gamma)}{\eta^2}$$

$$\mathfrak{C}_{n,\gamma}((t - t)^2, t) = \left[\frac{-\eta + 2(\gamma - 1)}{\eta^2} \right] t^2 - \frac{\eta + 2(1 - \gamma)}{\eta^2}$$

is having.

Theorem 1.1

$f \in \mathfrak{C}[-1,1]$, $n \in \mathbb{N}$ and $t \in [-1,1]$. In this case,

$$\lim_{\eta \rightarrow \infty} \|\mathfrak{C}_{\eta,\gamma}(f; t) - f(t)\|_{C[-1,1]} = 0$$

Proof:

For $f(t) = 1$ and $f(t) = t$ the proof is clearly. Now, for $f(t) = t^2$,

$$\lim_{\eta \rightarrow \infty} \|\mathfrak{C}_{\eta,\gamma}(f; t) - t^2\|_{C[-1,1]} =$$

$$\begin{aligned} & \lim_{\eta \rightarrow \infty} \max_{x \in [-1,1]} |\mathfrak{C}_{\eta,\gamma}(t^2; t) - t^2| \\ &= \lim_{\eta \rightarrow \infty} \max_{x \in [-1,1]} \left| \frac{\eta(\eta-1) + 2(\gamma-1)}{\eta^2} t^2 - \frac{\eta + 2(1-\gamma)}{\eta^2} \right. \\ & \quad \left. - t^2 \right| \end{aligned}$$

$$= \lim_{\eta \rightarrow \infty} \max_{x \in [-1,1]} \left| \frac{\eta(\eta-1) + 2(\gamma-1)}{\eta^2} t^2 - \frac{\eta + 2(1-\gamma)}{\eta^2} - t^2 \right|$$

$$= \lim_{\eta \rightarrow \infty} \max_{x \in [-1,1]} \left| \left[\frac{\eta(\eta-1) + 2(\gamma-1)}{\eta^2} - 1 \right] t^2 - \frac{\eta + 2(1-\gamma)}{\eta^2} \right|$$

$$= \lim_{\eta \rightarrow \infty} \max_{x \in [-1,1]} \left| \left[\frac{-\eta + 2(\gamma-1)}{\eta^2} \right] t^2 + \lim_{\eta \rightarrow \infty} \max_{x \in [-1,1]} \left| \frac{\eta + 2(1-\gamma)}{\eta^2} \right| \right|$$

$$= 0$$

is getting. So, we have next form

$$\lim_{\eta \rightarrow \infty} \|\mathfrak{C}_{\eta,\gamma}(f; t) - f(t)\|_{C[-1,1]} = 0.$$

Conclusion

Firstly, important equations that will be used to examine the operator's approximation results have been obtained. Then, the

central moment of the operator was calculated. Finally, with the help of Korovkin type theorem, it was shown that every function in space can be approached with the defined operator. In fact, many different features of the defined operator can be examined. For example convexity, monotonicity of operators. In fact, many different features of the defined operator can be examined. for example convexity, monotonicity of operators. Using modulus of continuity we could be calculated of rate of convergence and Vornovskaya type theorem. On the other hand, with this operator, some special sequences and polynomials can be selected and the approximation can be shown visually and numerically. See (Yılmaz & Soykan, 2023), (Aktas, & Soykan, 2023). Also, we On the other hand, this method we use can be applied to different operators in the literature. For example: (Bilgin and Eren 2023), (Bozma and Bilgin, 2023).

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CHAPTER XII

Approximation with Gadjiev-Ibragimov Operator on a Mobile Interval

Gurel BOZMA¹

INTRODUCTION

Studies in the field of approximation theory gained momentum after Weierstrass, who proved the existence of a polynomial converging to every continuous function defined in a finite interval, and Bohman's study investigating the approximation conditions for linear positive operators in the interval $[0,1]$ and the field is largely shaped by showing that in Korovkin's study, it is shown that if the conditions for the test functions are met, the convergence of the operator will be obtained for all functions in space.

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After this important theorem of Korovkin, operators that could be the application of the theorem were constructed and new operators were designed by combining several operators by different researchers and the field was developed rapidly. Here are a few examples of these operators: (Bernstein, 1915), (Dogru, 1997), (Coskun, 2011), (Gonul & Coskun, 2012, 2013), (Kaya & Gonul, 2013), (Deniz & Aral, 2015), (Acar, 2017), (Bilgin & Cetinkaya, 2018), (Gonul Bilgin & Ozgur, 2019), (Bilgin & Eren, 2021), (Bozma & Bars, 2022) (Herdem & Buyukyazici, 2020).

The Gadjiev-Ibragimov operator, which is one of the preferred operators in the studies in the literature, was first defined in (Ibragimov & Gadjiev, 1970) and has been studied from different perspectives by many researchers, especially the students trained by Gadjiev.

In the study, the generalized Gadjiev-Ibragimov operator, which is well known in the literature, is modified to provide suitable properties for test functions over a general interval whose limit is variable-dependent. Important properties of the approximation to continuous functions are introduced with a sequences of linear positive operators defined on a mobile interval. Also, the rate of convergent is calculated numerically by obtaining significant equations for the defined our operators.

The approach speed was calculated with the help of the continuity module by obtaining important equations for the operator. The approximation results obtained with graphical and numerical calculations have been put forward in practice.

METHOD

In this section, the operator to be studied will be defined and important approximation properties will be given.

Definition 2.1 Let ϕ_n and g_n be sequences satisfying the conditions $\lim_{n \rightarrow \infty} g_n = \infty$, $\lim_{n \rightarrow \infty} \frac{\phi_n}{g_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\phi_n}{g_n + n + \psi} n = 1$, $\lim_{n \rightarrow \infty} \frac{n + \mu}{g_n + n + \psi} = 0$.

Let $R_{n,e}(x)$ be a function that satisfies the following conditions depending on the parameters e and n and let $\mu, \psi \in \mathbb{R}^+$.

1-) For all $n \in \mathbb{N}$, $e \in \mathbb{N}_0$ and $\mu < \psi$, also for $x \in \left[0, \frac{n + \mu}{n + \psi}\right]$,

$$(-1)^e R_{n,e}(x) \geq 0,$$

2-) For every $x \in \left[0, \frac{n + \mu}{n + \psi}\right]$

$$\sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} = 1,$$

3-) For every $x \in \left[0, \frac{n + \mu}{n + \psi}\right]$, there is an integer z such that the equality

$$R_{n,e}(x) = -nx R_{n+z,e-1}(x),$$

will be satisfied and $(n + z) \in \mathbb{N}_0$. In this case, the operator, which is a generalization of the Gadjiev-Ibragimov operator, for every $f \in C \left[0, \frac{n + \mu}{n + \psi}\right]$

$$\tilde{L}_n(f, x) = \sum_{e=0}^{\infty} f\left(\frac{e + n + \mu}{g_n + n + \psi}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

will be in the form. Obviously this operator is linear and positive.

Lemma 2. 1 Let $f \in C \left[0, \frac{n+\mu}{n+\psi}\right]$ The following equations are valid for

$$\tilde{L}_n(f, x) = \sum_{e=0}^{\infty} f\left(\frac{e+n+\mu}{g_n+n+\psi}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}.$$

$$\text{i)} \tilde{L}_n(1, x) = 1,$$

$$\text{ii)} \tilde{L}_n(h, x) = \frac{n\phi_n}{g_n+n+\psi} x + \frac{n+\mu}{g_n+n+\psi},$$

$$\begin{aligned} \text{iii)} \tilde{L}_n(h^2, x) &= \frac{(\phi_n)^2 n(n+z)}{(g_n+n+\psi)^2} x^2 + \frac{((2n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2} x \\ &+ \frac{(n+\mu)^2}{(g_n+n+\psi)^2}. \end{aligned}$$

Proof According to the 2nd property of the operator, since

$$\sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} = 1$$

obviously

$$\tilde{L}_n(1, x) = \sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} = 1$$

is found. Using the 3rd property of the operator; Since $(n + z) \in \mathbb{N}_0$

$$\begin{aligned}
\tilde{L}_n(h, x) &= \sum_{e=0}^{\infty} \left(\frac{e + n + \mu}{g_n + n + \psi} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&= \sum_{e=1}^{\infty} \left(\frac{e}{g_n + n + \psi} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&\quad + \sum_{e=0}^{\infty} \left(\frac{n + \mu}{g_n + n + \psi} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&= \sum_{e=1}^{\infty} \left(\frac{1}{g_n + n + \psi} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{(e-1)!} \\
&\quad + \sum_{e=0}^{\infty} \left(\frac{n + \mu}{g_n + n + \psi} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&= \sum_{e=1}^{\infty} \left(\frac{-nx}{g_n + n + \psi} \right) R_{n+z,e-1}(x) \frac{(-\phi_n)^e}{(e-1)!} + \frac{n + \mu}{g_n + n + \psi} \\
&= \frac{-nx}{g_n + n + \psi} \sum_{e=1}^{\infty} R_{n+z,e-1}(x) \frac{(-\phi_n)^e}{(e-1)!} + \frac{n + \mu}{g_n + n + \psi} \\
&= \frac{-nx}{g_n + n + \psi} \sum_{e=1}^{\infty} (-\phi_n) R_{n+z,e-1}(x) \frac{(-\phi_n)^{e-1}}{(e-1)!} + \frac{n + \mu}{g_n + n + \psi}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{n\phi_n x}{g_n + n + \psi} \right) \sum_{e=0}^{\infty} R_{n+z,e}(x) \frac{(-\phi_n)^e}{e!} \\
&+ \frac{n + \mu}{g_n + n + \psi} \quad [(e \rightarrow e + 1) (n + z) \in \mathbb{N}]
\end{aligned}$$

$$= \frac{n\phi_n x}{g_n + n + \psi} + \frac{n + \mu}{g_n + n + \psi}$$

equality is reached. Similarly

$$\begin{aligned}
\tilde{L}_n(h^2, x) &= \sum_{e=0}^{\infty} \left(\frac{e + n + \mu}{g_n + n + \psi} \right)^2 R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&= \sum_{e=0}^{\infty} \left(\frac{e^2 + 2e(n + \mu) + (n + \mu)^2}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&= \sum_{e=0}^{\infty} \left(\frac{e^2}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&+ \sum_{e=0}^{\infty} \left(\frac{2e(n + \mu)}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&+ \sum_{e=0}^{\infty} \left(\frac{(n + \mu)^2}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
&= \sum_{e=2}^{\infty} \left(\frac{e(e - 1)}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{e=1}^{\infty} \left(\frac{e}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
& + \sum_{e=1}^{\infty} \left(\frac{2(n + \mu)}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{(e-1)!} \\
& + \sum_{e=0}^{\infty} \left(\frac{(n + \mu)^2}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\
& = \sum_{e=2}^{\infty} \left(\frac{\phi_n^2}{(g_n + n + \psi)^2} \right) R_{n,e}(x) \frac{(-\phi_n)^{e-2}}{(e-2)!} \\
& + \left(\frac{1}{g_n + n + \psi} \right) \left(\frac{n\phi_n x}{g_n + n + \psi} \right) \\
& + \sum_{e=1}^{\infty} \left(\frac{2(n + \mu)(-nx)}{(g_n + n + \psi)^2} \right) R_{n+z,e-1}(x) \frac{(-\phi_n)^e}{(e-1)!} + \frac{(n + \mu)^2}{(g_n + n + \psi)^2} \\
& = \sum_{e=2}^{\infty} \left(\frac{\phi_n^2 n(n + z)x^2}{(g_n + n + \psi)^2} \right) R_{n+2z,e-2}(x) \frac{(-\phi_n)^{e-2}}{(e-2)!} \\
& + \left(\frac{1}{g_n + n + \psi} \right) \left(\frac{n\phi_n x}{g_n + n + \psi} \right) \\
& + \sum_{e=1}^{\infty} \left(\frac{2(n + \mu)\phi_n nx}{(g_n + n + \psi)^2} \right) R_{n+z,e-1}(x) \frac{(-\phi_n)^{e-1}}{(e-1)!} + \frac{(n + \mu)^2}{(g_n + n + \psi)^2}.
\end{aligned}$$

$$\begin{aligned}
\tilde{L}_n(h^2, x) &= \sum_{e=0}^{\infty} \left(\frac{\phi_n^2 n(n+z)x^2}{(g_n + n + \psi)^2} \right) R_{n+2z,e}(x) \frac{(-\phi_n)^e}{e!} \\
&+ \left(\frac{1}{g_n + n + \psi} \right) \left(\frac{n\phi_n x}{g_n + n + \psi} \right) \\
&+ \sum_{e=0}^{\infty} \left(\frac{2(n+\mu)\phi_n n x}{(g_n + n + \psi)^2} \right) R_{n+z,e}(x) \frac{(-\phi_n)^e}{e!} + \frac{(n+\mu)^2}{(g_n + n + \psi)^2} \\
&= \frac{\phi_n^2 n(n+z)x^2}{(g_n + n + \psi)^2} + \frac{n\phi_n x}{(g_n + n + \psi)^2} + \frac{2(n+\mu)\phi_n n x}{(g_n + n + \psi)^2} \\
&+ \frac{(n+\mu)^2}{(g_n + n + \psi)^2} \\
&= \frac{\phi_n^2 n(n+z)}{(g_n + n + \psi)^2} x^2 + \frac{n\phi_n(2(n+\mu) + 1)}{(g_n + n + \psi)^2} x + \frac{(n+\mu)^2}{(g_n + n + \psi)^2}
\end{aligned}$$

using $(n+z) \in \mathbb{N}_0$ and $(n+2z) \in \mathbb{N}_0$

$$\begin{aligned}
\tilde{L}_n(h^2, x) &= \frac{(\phi_n)^2 n(n+z)}{(g_n + n + \psi)^2} x^2 + \frac{(2(n+\mu) + 1)n\phi_n}{(g_n + n + \psi)^2} x \\
&+ \frac{(n+\mu)^2}{(g_n + n + \psi)^2}
\end{aligned}$$

is found. This completes the proof.

Theorem 2.1 Let

$$\tilde{L}_n(f, x) = \sum_{e=0}^{\infty} f\left(\frac{e+n+\mu}{g_n+n+\psi}\right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!}.$$

Then, for all $f \in C\left[0, \frac{n+\mu}{n+\psi}\right]$,

$$\lim_{n \rightarrow \infty} \|\tilde{L}_n(f, x) - f(x)\|_{C\left[0, \frac{n+\mu}{n+\psi}\right]} = 0.$$

Proof It is sufficient to clearly show that the conditions of the Korovkin's Theorem are satisfied. According to Lemma 2.1, since the equation

$$|\tilde{L}_n(1, x) - 1| = \left| \sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} - 1 \right| = 0$$

is valid, according to the norm definition in $C\left[0, \frac{n+\mu}{n+\psi}\right]$;

$$\lim_{n \rightarrow \infty} \|\tilde{L}_n(1, x) - 1\|_{C\left[0, \frac{n+\mu}{n+\psi}\right]} = 0.$$

is found. Similarly since the equality

$$\begin{aligned} |\tilde{L}_n(h, x) - x| &= \left| \frac{n\phi_n x}{g_n + n + \psi} + \frac{n + \mu}{g_n + n + \psi} - x \right| \\ &= \left| x \left(\frac{\phi_n}{g_n + n + \psi} n - 1 \right) + \frac{n + \mu}{g_n + n + \psi} \right| \end{aligned}$$

is valid. For $x \in \left[0, \frac{n+\mu}{n+\psi}\right]$ and $\lim_{n \rightarrow \infty} \frac{\phi_n}{g_n + n + \psi} n = 1$, the inequality

$$\begin{aligned}
& \max_{x \in \left[0, \frac{n+\mu}{n+\psi}\right]} |\tilde{L}_n(h, x) - x| \\
&= \max_{x \in \left[0, \frac{n+\mu}{n+\psi}\right]} \left| x \left(\frac{\phi_n}{g_n + n + \psi} n - 1 \right) + \frac{n + \mu}{g_n + n + \psi} \right| \\
&\leq \left| \frac{n + \mu}{n + \psi} \right| \left| \frac{\phi_n}{g_n + n + \psi} n - 1 \right| + \left| \frac{n + \mu}{g_n + n + \psi} \right|.
\end{aligned}$$

can be written. If the limit of both sides is taken

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left\| \tilde{L}_n(h, x) - x \right\|_{C\left[0, \frac{n+\mu}{n+\psi}\right]} \\
&\leq \lim_{n \rightarrow \infty} \left| \frac{n + \mu}{n + \psi} \right| \lim_{n \rightarrow \infty} \left| \frac{n\phi_n}{g_n + n + \psi} - 1 \right| + \lim_{n \rightarrow \infty} \left| \frac{n + \mu}{g_n + n + \psi} \right| = 0
\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \left\| \tilde{L}_n(h, x) - x \right\|_{C\left[0, \frac{n+\mu}{n+\psi}\right]} = 0.$$

is found. Finally

$$\lim_{n \rightarrow \infty} \left\| \tilde{L}_n(h^2, x) - x^2 \right\|_{C\left[0, \frac{n+\mu}{n+\psi}\right]} = 0.$$

will be shown.

$$\begin{aligned}
& |\tilde{L}_n(h^2, x) - x^2| \\
&= \left| \frac{(\phi_n)^2 n(n + \psi)}{(g_n + n + \psi)^2} x^2 + \frac{(2(n + \mu) + 1)n\phi_n}{(g_n + n + \psi)^2} x \right|
\end{aligned}$$

$$\begin{aligned}
& + \left| \frac{(n + \mu)^2}{(g_n + n + \psi)^2} - x^2 \right| \\
& = \left| x^2 \left(\left(\frac{\phi_n}{g_n + n + \psi} \right)^2 n(n + z) - 1 \right) \right. \\
& \quad \left. + x \left(\frac{(2(n + \mu) + 1)n\phi_n}{(g_n + n + \psi)^2} \right) + \frac{(n + \mu)^2}{(g_n + n + \psi)^2} \right|
\end{aligned}$$

and then

$$\begin{aligned}
& \max_{x \in \left[0, \frac{n+\mu}{n+\psi}\right]} |\tilde{L}_n(h^2, x) - x^2| \\
& = \max_{x \in \left[0, \frac{n+\mu}{n+\psi}\right]} \left| x^2 \left(\left(\frac{\phi_n}{g_n + n + \psi} \right)^2 n(n + z) - 1 \right) \right. \\
& \quad \left. + x \left(\frac{(2(n + \mu) + 1)n\phi_n}{(g_n + n + \psi)^2} \right) + \frac{(n + \mu)^2}{(g_n + n + \psi)^2} \right| \\
& \leq \left(\frac{n + \mu}{n + \psi} \right)^2 \left| \left(\left(\frac{\phi_n}{g_n + n + \psi} \right)^2 n(n + z) - 1 \right) \right| \\
& \quad + \left(\frac{n + \mu}{n + \psi} \right) \left| \left(\frac{(2(n + \mu) + 1)n\phi_n}{(g_n + n + \psi)^2} \right) \right| + \left| \frac{(n + \mu)^2}{(g_n + n + \psi)^2} \right|
\end{aligned}$$

is found. Since the definitions of (ϕ_n) , (g_n) and using $(n + z) \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \left(\frac{\phi_n}{g_n + n + \psi} \right)^2 n(n + z)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{\phi_n}{g_n + n + \psi} \right) n \lim_{n \rightarrow \infty} \left(\frac{\phi_n}{g_n + n + \psi} \right) (n + z) = 1,$$

$$\lim_{n \rightarrow \infty} \left(\frac{(2(n + \mu) + 1)n\phi_n}{(g_n + n + \psi)^2} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{(2(n + \mu) + 1)}{(g_n + n + \psi)} \right) \lim_{n \rightarrow \infty} \left(\frac{n\phi_n}{g_n + n + \psi} \right) = 0$$

equations are valid. So

$$\lim_{n \rightarrow \infty} \|\tilde{L}_n(h^2, x) - x^2\|_{C[0, \frac{n+\mu}{n+\psi}]}$$

$$\leq \lim_{n \rightarrow \infty} \left(\frac{n + \mu}{n + \psi} \right)^2 \left(\left(\frac{\phi_n}{g_n + n + \psi} \right)^2 n(n + z) - 1 \right)$$

$$+ \lim_{n \rightarrow \infty} \left(\frac{n + \mu}{n + \psi} \right) \left(\frac{(2(n + \mu) + 1)n\phi_n}{(g_n + n + \psi)^2} \right) + \lim_{n \rightarrow \infty} \frac{(n + \mu)^2}{(g_n + n + \psi)^2}$$

Since the inequality is valid

$$\lim_{n \rightarrow \infty} \|\tilde{L}_n(h^2, x) - x^2\|_{C[0, \frac{n+\mu}{n+\psi}]} = 0$$

will be shown. Then, since all the conditions of Korovkin's Theorem

are valid, for every $f \in C\left[0, \frac{n+\mu}{n+\psi}\right]$

$$\lim_{n \rightarrow \infty} \|\tilde{L}_n(f, x) - f(x)\|_{C[0, \frac{n+\mu}{n+\psi}]} = 0$$

is achieved.

RESULTS

In this section, to analyze the results of the method is shown in this study, the approximation rate of the generalized Gadjiev Ibragimov operator in the space of continuous functions on $\left[0, \frac{n+\mu}{n+\psi}\right]$ will be calculated with the help of the modulus of continuity. First of all, the definition of modulus of continuity will be reminded.

Let $K = \left[0, \frac{n+\mu}{n+\psi}\right]$, for all $f \in C(K)$, $\delta > 0$ modulus of continuity of function f as defined

$$\omega(f, \delta) = \sup_{\substack{|t-x| < \delta \\ x \in K}} |f(t) - f(x)|.$$

Here, the following known features of the modulus of continuity will be used.

Let $f \in C(K)$.

i) Let $\delta \geq 0$. $\omega(f, \delta)$ is a monotonically increasing function with respect to δ .

ii) For all $f \in C(K)$, $\lim_{\delta \rightarrow 0} \omega(f, \delta) = 0$.

iii) For $\lambda > 0$, $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$.

iv) Let $x, t \in B$ and for all $f \in C(K)$,

$$|f(t) - f(x)| \leq \omega(f, \delta) \left(1 + \frac{|t - x|}{\delta}\right).$$

Theorem 3.1 Let $f \in C(K)$ and $(\phi_n), (g_n)$ be sequences defined in Definition 2.1. In this case, for n is large enough and D is a constant where independent from $(\phi_n), (g_n)$;

$$\begin{aligned} & \|\tilde{L}_n(f, x) - f(x)\|_{C\left[0, \frac{n+\mu}{n+\psi}\right]} \\ & \leq D\omega\left(f; \sqrt{\left(n\frac{\phi_n}{g_n + n + \psi} - 1\right)^2 + \frac{2\phi_n}{g_n + n + \psi}z + 5}\right). \end{aligned}$$

Proof

Using

$$|\tilde{L}_n(f, x) - f(x)| \leq \tilde{L}_n(|f(t) - f(x)|, x),$$

we get

$$|\tilde{L}_n(f, x) - f(x)| \leq \sum_{e=0}^{\infty} \left| f\left(\frac{e + n + \mu}{g_n + n + \psi}\right) - f(x) \right| R_{n,e}(x) \frac{(-\phi_n)^e}{e!}$$

If $h = \frac{e+n+\mu}{g_n+n+\psi}$ is selected in properties of modulus of continuity; for all $\delta_n > 0$

$$\left| f\left(\frac{e+n+\mu}{g_n+n+\psi}\right) - f(x) \right| \leq \omega(f, \delta_n) \left(1 + \frac{\left| \frac{e+n+\mu}{g_n+n+\psi} - x \right|}{\delta_n} \right)$$

is written. Using linearity and positivity, it is found as

$$\begin{aligned} & \left| \tilde{L}_n(f, x) - f(x) \right| \\ & \leq \sum_{e=0}^{\infty} \omega(f, \delta_n) \left(1 + \frac{\left| \frac{e+n+\mu}{g_n+n+\psi} - x \right|}{\delta_n} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\ & = \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \sum_{e=0}^{\infty} \left| \left(\frac{e+n+\mu}{g_n+n+\psi} \right) - x \right| R_{n,e}(x) \frac{(-\phi_n)^e}{e!} + 1 \right\} \end{aligned}$$

Here, if M is defined as

$$\begin{aligned} M &= \sum_{e=0}^{\infty} \left| \left(\frac{e+n+\mu}{g_n+n+\psi} \right) - x \right| R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \\ &= \sum_{e=0}^{\infty} \left(\left| \left(\frac{e+n+\mu}{g_n+n+\psi} \right) - x \right|^2 \right)^{\frac{1}{2}} \left[R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right]^{\frac{1}{2}} \\ &\quad \times \left[R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right]^{1/2}. \end{aligned}$$

From the Cauchy Schwarz inequality, it can be written as

$$\begin{aligned}
M &\leq \left[\sum_{e=0}^{\infty} \left| \left(\frac{e+n+\mu}{g_n+n+\psi} \right) - x \right|^2 R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right]^{\frac{1}{2}} \\
&\times \left[R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right]^{1/2} \\
&= \left[\sum_{e=0}^{\infty} \left| \left(\frac{e+n+\mu}{g_n+n+\psi} \right) - x \right|^2 R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right]^{1/2}
\end{aligned}$$

Therefore, since

$$\begin{aligned}
&|\tilde{L}_n(f, x) - f(x)| \\
&\leq \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left[\sum_{e=0}^{\infty} \left| \frac{e+n+\mu}{g_n+n+\psi} - x \right|^2 R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right]^{1/2} + 1 \right\}
\end{aligned}$$

here using

$$\begin{aligned}
&\left(\left(\frac{e+n+\mu}{g_n+n+\psi} \right) - x \right)^2 = \left(\frac{e+n+\mu}{g_n+n+\psi} \right)^2 - 2x \left(\frac{e+n+\mu}{g_n+n+\psi} \right) + x^2, \\
&|\tilde{L}_n(f, x) - f(x)| \leq \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left[\sum_{e=0}^{\infty} \left(\frac{e+n+\mu}{g_n+n+\psi} \right)^2 R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right. \right. \\
&\quad \left. \left. - 2x \sum_{e=0}^{\infty} \left(\frac{e+n+\mu}{g_n+n+\psi} \right) R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \right] \right\}
\end{aligned}$$

$$+x^2 \sum_{e=0}^{\infty} R_{n,e}(x) \frac{(-\phi_n)^e}{e!} \Big]^{1/2} + 1 \Big\}$$

$$= \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left(\tilde{L}_n(h^2, x) - 2x\tilde{L}_n(h, x) + x^2\tilde{L}_n(1, x) \right)^{1/2} + 1 \right\}$$

inequality is written. For $x \in \left[0, \frac{n+\mu}{n+\psi}\right]$ If

$\tilde{L}_n(h^2, x), \tilde{L}_n(h, x), \tilde{L}_n(1, x)$ are written in this inequality.

Considering definition of $(\phi_n), (g_n)$, for n large enough $\frac{\phi_n}{g_n+n+\psi} \leq$

$1, \frac{\phi_n}{g_n+n+\psi} n \leq 2$ and using $\frac{2\mu n(\phi_n)^2}{(g_n+n+\psi)^2} < \frac{4\mu}{g_n+n+\psi} < \frac{4\mu}{n+\psi}$ and

$\frac{n(\phi_n)^2}{(g_n+n+\psi)^2} < \frac{2}{g_n+n+\psi} < \frac{2}{n+\psi}$ we get

$$|\tilde{L}_n(f, x) - f(x)|$$

$$\leq \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left(\tilde{L}_n(h^2, x) - 2x\tilde{L}_n(h, x) + x^2\tilde{L}_n(1, x) \right)^{1/2} + 1 \right\}$$

$$\leq \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left[\left(\frac{n+\mu}{n+\psi} \right)^2 \left(\frac{\phi_n}{g_n+n+\psi} \right)^2 n(n+\psi) \right. \right.$$

$$+ \left(\frac{n+\mu}{n+\psi} \right) \left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2} \right) + \frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

$$\left. - 2 \left(\frac{n+\mu}{n+\psi} \right) \left(\frac{n\phi_n}{g_n+n+\psi} \left(\frac{n+\mu}{n+\psi} \right) + \frac{n+\mu}{g_n+n+\psi} \right) \right\}$$

$$\begin{aligned}
& + \left(\frac{n + \mu}{n + \psi} \right)^2 \Big]^{1/2} + 1 \Big\} \\
& = \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left[\left(\frac{n + \mu}{n + \psi} \right)^2 \left(\left(\frac{\phi_n}{g_n + n + \psi} \right)^2 n(n + z) \right. \right. \right. \\
& \quad \left. \left. - 2 \frac{n\phi_n}{g_n + n + \psi} + 1 \right) \right. \right. \\
& \quad \left. + \left(\frac{n + \mu}{n + \psi} \right) \left(\frac{(2(n + \mu) + 1)n\phi_n}{(g_n + n + \psi)^2} - 2 \frac{n + \mu}{g_n + n + \psi} \right) \right. \\
& \quad \left. + \left(\frac{n + \mu}{g_n + n + \psi} \right)^2 \Big]^{1/2} + 1 \right\} \\
& = \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left[\left(\frac{n + \mu}{n + \psi} \right)^2 \left(\left(\frac{n\phi_n}{g_n + n + \psi} \right)^2 - 2 \frac{n\phi_n}{g_n + n + \psi} + 1 \right) \right. \right. \\
& \quad \left. + \left(\frac{n + \mu}{n + \psi} \right)^2 \left(\left(\frac{\phi_n}{g_n + n + \psi} \right)^2 nz \right. \right. \\
& \quad \left. \left. + \frac{1}{\left(\frac{n + \mu}{n + \psi} \right)} \left(\frac{(2(n + \mu) + 1)\phi_n n}{(g_n + n + \psi)^2} - 2 \frac{n + \mu}{g_n + n + \psi} \right) \right) \right. \\
& \quad \left. + \left(\frac{n + \mu}{g_n + n + \psi} \right)^2 \Big]^{1/2} + 1 \right\}
\end{aligned}$$

$$\begin{aligned}
&= \omega(f, \delta_n) \left\{ \frac{1}{\delta_n} \left[\left(\frac{n+\mu}{n+\psi} \right)^2 \left(\left(\frac{n\phi_n}{g_n+n+\psi} - 1 \right)^2 + 2 \frac{\phi_n}{g_n+n+\psi} z \right. \right. \right. \\
&\quad \left. \left. + \frac{1}{\left(\frac{n+\mu}{n+\psi} \right)} \left(\frac{2n^2\phi_n}{(g_n+n+\psi)^2} + \frac{2\mu n\phi_n}{(g_n+n+\psi)^2} + \frac{n\phi_n}{(g_n+n+\psi)^2} \right. \right. \right. \\
&\quad \left. \left. - \frac{2n}{g_n+n+\psi} - \frac{2\mu}{g_n+n+\psi} \right) \right] + \left(\frac{n+\mu}{g_n+n+\psi} \right)^2 \right]^{1/2} + 1 \Big\} \\
&\leq \omega(f, \delta_n) \left\{ \frac{\left(\frac{n+\mu}{n+\psi} \right)}{\delta_n} \left[\left(n \frac{\phi_n}{g_n+n+\psi} - 1 \right)^2 + 2 \frac{\phi_n}{g_n+n+\psi} z \right. \right. \\
&\quad \left. \left. + \frac{1}{\left(\frac{n+\mu}{n+\psi} \right)} \left(\frac{4\mu}{n+\psi} + \frac{4}{n+\psi} \right) + \left(\frac{n+\mu}{g_n+n+\psi} \right)^2 \right]^{1/2} + 1 \right\} \\
&\leq \omega(f, \delta_n) \left\{ \frac{\left(\frac{n+\mu}{n+\psi} \right)}{\delta_n} \left[\left(n \frac{\phi_n}{g_n+n+\psi} - 1 \right)^2 \right. \right. \\
&\quad \left. \left. + 2 \frac{\phi_n}{g_n+n+\psi} z + 5 \right]^{\frac{1}{2}} + 1 \right\}.
\end{aligned}$$

Then, with selection

$$\delta_n := \sqrt{\left(n \frac{\phi_n}{g_n + n + \psi} - 1\right)^2 + \frac{2\phi_n}{g_n + n + \psi} z + 5}$$

for independent constant D from n , we have

$$\begin{aligned} & \|\tilde{L}_n(f, x) - f(x)\|_{C\left[0, \frac{n+\mu}{n+\psi}\right]} \\ & \leq D\omega\left(f; \sqrt{\left(n \frac{\phi_n}{g_n + n + \psi} - 1\right)^2 + \frac{2\phi_n}{g_n + n + \psi} z + 5}\right). \end{aligned}$$

It has been shown by this theorem that the rate of approximation is

$\sqrt{\left(n \frac{\phi_n}{g_n + n + \psi} - 1\right)^2 + \frac{2\phi_n}{g_n + n + \psi} z + 5}$ and this rate can be increased according to the selection of ϕ_n and g_n .

Lemma 3.1 The first three central moments for the defined Gadjiev Ibragimov type operator are as follows

i) $\varphi_{n,0}(x) = 1,$

ii) $\varphi_{n,1}(x) = x \left[n \frac{\phi_n}{g_n + n + \psi} - 1 \right] + \frac{n+\mu}{g_n + n + \psi},$

iii) $\varphi_{n,2}(x) = \left(\frac{(\phi_n)^2 n(n+\psi)}{(g_n + n + \psi)^2} - 2 \frac{n\phi_n}{g_n + n + \psi} + 1 \right) x^2 + \left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n + n + \psi)^2} \right. \\ \left. - 2 \frac{n+\mu}{g_n + n + \psi} \right) x + \left(\frac{n+\mu}{g_n + n + \psi} \right)^2.$

Proof.

$$\text{i) } \varphi_{n,0}(x) = \tilde{L}_n((h-x)^0, x) = \tilde{L}_n(1, x) = 1.$$

$$\text{ii) } \varphi_{n,1}(x) = \tilde{L}_n(h-x, x) = \tilde{L}_n(h, x) - x\tilde{L}_n(1, x)$$

$$= \frac{n\phi_n}{g_n+n+\psi}x + \frac{n+\mu}{g_n+n+\psi} - x.$$

$$\text{iii) } \varphi_{n,2}(x) = \tilde{L}_n((h-x)^2, x)$$

$$= \tilde{L}_n(h^2, x) - 2x\tilde{L}_n(h, x) + x^2\tilde{L}_n(1, x)$$

$$= \frac{(\phi_n)^2 n(n+z)}{(g_n+n+\psi)^2} x^2 + \left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2} \right) x + \frac{(n+\mu)^2}{(g_n+n+\psi)^2}$$

$$- 2x \left(\frac{n\phi_n}{g_n+n+\psi} x + \frac{n+\mu}{g_n+n+\psi} \right) + x^2$$

$$= \left(\frac{(\phi_n)^2 n(n+z)}{(g_n+n+\psi)^2} - 2 \frac{n\phi_n}{g_n+n+\psi} + 1 \right) x^2$$

$$+ \left(\frac{(2(n+\mu)+1)n\phi_n}{(g_n+n+\psi)^2} - 2 \frac{n+\mu}{g_n+n+\psi} \right) x + \frac{(n+\mu)^2}{(g_n+n+\psi)^2}.$$

Definition 3.2 Let $P \in]0,1[$, functions that satisfy the condition

$$|f(t) - f(x)| \leq N|t - x|^P$$

are called Lipschitz class functions. N is called the Lipschitz constant. The class of Lipschitz functions is denoted by $f \in Lip_N \left(P, C \left[0, \frac{n+\mu}{n+\psi} \right] \right)$.

Theorem 3.2 Let $x \in \left[0, \frac{n+\mu}{n+\psi} \right]$. For bounded f defined on $\mathbb{R}$, $f \in Lip_N \left(P, C \left[0, \frac{n+\mu}{n+\psi} \right] \right)$ and $0 < P \leq 1$. Then

$$|\tilde{L}_n(f, x) - f(x)| \leq PN \left(\left(n \frac{\phi_n}{g_n + n + \psi} - 1 \right)^2 + \frac{2\phi_n}{g_n + n + \psi} z + 5 \right)^{\frac{P}{2}}.$$

Proof

Let $f \in Lip_N(P)$. Using modulus of continuity we get $\omega(f, \delta) \leq N\delta^P$. From Theorem 3.1

$$\begin{aligned} & |\tilde{L}_n(f, x) - f(x)| \\ & \leq P\omega \left(f, \sqrt{\left(n \frac{\phi_n}{g_n + n + \psi} - 1 \right)^2 + \frac{2\phi_n}{g_n + n + \psi} z + 5} \right) \\ & \leq PN \left[\left(n \frac{\phi_n}{g_n + n + \psi} - 1 \right)^2 + \frac{2\phi_n}{g_n + n + \psi} z + 5 \right]^{P/2} \end{aligned}$$

is written. This completes the proof.

Example 3.1 Below is the graph of the approximation of the function $f(x) = x^2$ (dark blue) with the operator $\tilde{L}_n(f, x)$ in Fig.1. Here $(\phi_n) = n$, $(g_n) = n^2$, $\mu = 3$, $\psi = 5$, $\tilde{L}_5(f, x)$ with red, $\tilde{L}_{10}(f, x)$ with black, $\tilde{L}_{12}(f, x)$ with cyan, $\tilde{L}_{13}(f, x)$ with magenta drawing is made.

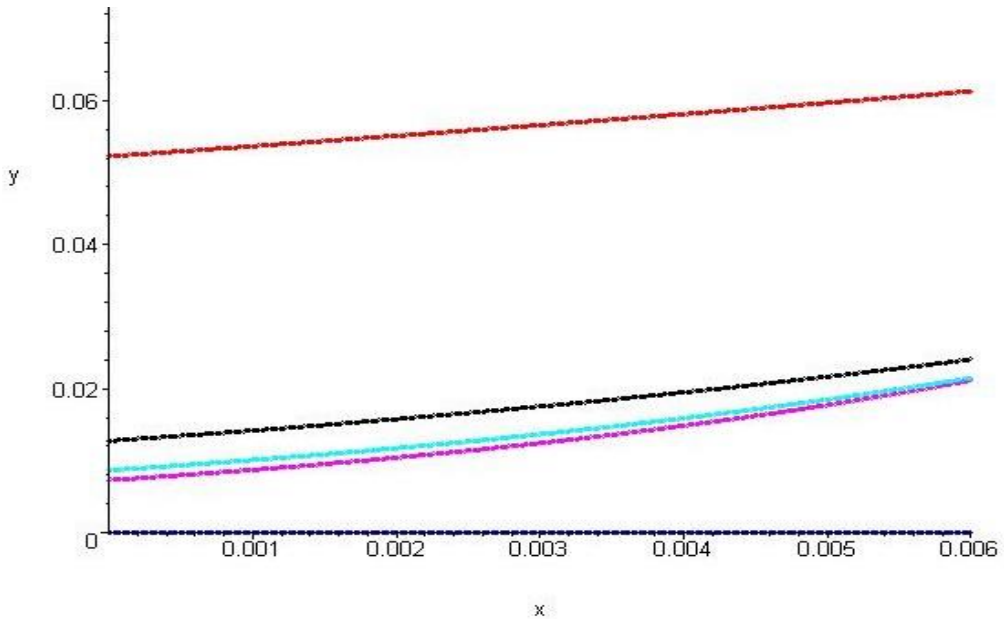


Fig.1. Graph of approximation to function f

Example 3.2 Below is the graph of the approximation of the function $f(x) = x^2$ (blue) with the operator $\tilde{L}_n(f, x)$ in Fig.2. Here $(\phi_n) = n$, $(g_n) = n^2$, $\mu = 2$, $\psi = 10$, $\tilde{L}_5(f, x)$ with red, $\tilde{L}_{10}(f, x)$ with black, $\tilde{L}_{12}(f, x)$ with cyan, $\tilde{L}_{13}(f, x)$ with magenta drawing is made.

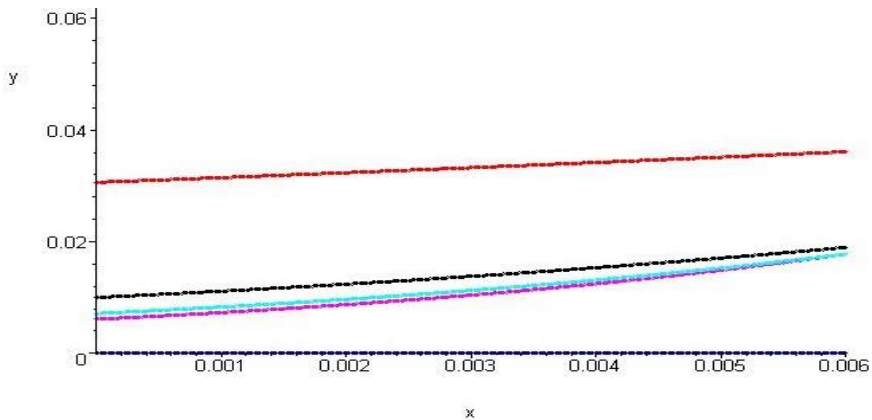


Fig.2. Graph of approximation to function f

Example 3.3 Below is the graph of the approximation of the function $f(x) = 5x^2$ (blue) with the operator $\tilde{L}_n(f, x)$ in Fig.3. Here $(\phi_n) = 1$, $(g_n) = n$, $\mu = 1$, $\psi = 1.4$, $\tilde{L}_1(f, x)$ with red, $\tilde{L}_5(f, x)$ with black, $\tilde{L}_{10}(f, x)$ with cyan, $\tilde{L}_{15}(f, x)$ with magenta drawing is made.

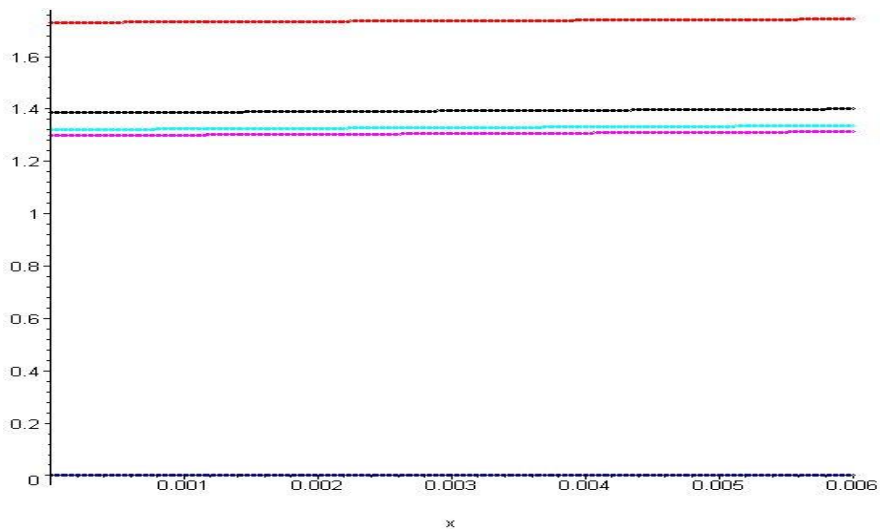


Fig.3 Graph of approximation to function f

Example 3.4 Below is the graph of the approximation of the function $f(x) = 5x^2$ (blue) with the operator $\tilde{L}_n(f, x)$ in Fig.4. Here $(\phi_n) = 1$, $(g_n) = n$, $\mu = 2$, $\psi = 3$, $\tilde{L}_1(f, x)$ with red, $\tilde{L}_5(f, x)$ with black, $\tilde{L}_{10}(f, x)$ with cyan, $\tilde{L}_{15}(f, x)$ with magenta drawing is made.

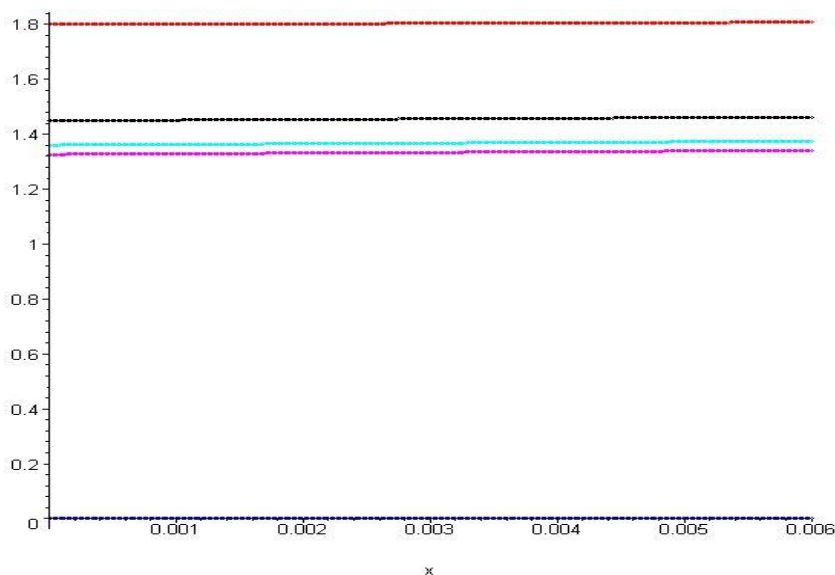


Fig.4 graph of approximation to function f

Example 3.5

Below is the table for the rate of convergence to the function $f(x) = 5e^{(x^2+1)}$ with the operator $\tilde{L}_n(f, x)$.

Table 1. The error bound of function $f(x) = 5e^{(x^2+1)}$ for $(\phi_n) = 1, (g_n) = n$.

n	Error calculation of approximation to the function with $\tilde{G}_n(f, x)$
	$0.2025819086 \cdot 10^8$
10^2	0.144099965310^8
10^3	0.138300075710^8
10^4	0.137721496510^8
10^5	0.137663654310^8
10^6	0.137657870010^8
10^7	0.137657292910^8
10^8	0.137657234010^8
10^9	0.137657229410^8

Example 3.6

Below is the table for the rate of convergence to the function $f(x) = \frac{x^2+1}{3e^{x+1}}$ with the operator $\tilde{L}_n(f, x)$.

Table 2. The error bound of function $f(x) = \frac{x^2+1}{3e^{x+1}}$ for $(\phi_n) = 1$,
 $(g_n) = n$.

n	Error calculation of approximation to the function with $\tilde{G}_n(f, x)$
	0.9267182048
10^2	0.9051598220
10^3	0.9025763898
10^4	0.9023129756
10^5	0.9022865822
10^6	0.9022839432
10^7	0.9022836790
10^8	0.9022836524
10^9	0.9022836504

CONCLUSIONS

In our study, where important approximation features were obtained, the selection of the moving range increased the usability of our operator for researches where it was insufficient to deal with a fixed interval. This study, which includes the generalization of some previously defined operators, includes the range limits associated with the establishment of the operator and the changing and generalizing approach features accordingly. For example (Bozma and Bars, 2022) and (Bilgin and Bars, 2022). Since it does not contain derivative expressions as in the classical Gadjeiv-Ibragimov operator, important properties related to the operator can be obtained by using only continuous functions and the properties of the sum formula. The described operator is a preferable tool for researchers looking for a suitable operator for daily life problems.

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CHAPTER XIII

Rough Statistical Convergence For Triple Difference Sequences In Neutrosophic Normed Spaces

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1.INTRODUCTION

We have tried to cope with uncertainties at many stages of our lives. In addition to situations of truth and falsehood, we frequently encounter situations involving uncertainty, doubt and indecision. It is not easy to say for sure whether some objects are inside or outside a category. In such cases, we can overcome the situation by using partial or gradual membership. All these concepts revealed neutrosophic philosophy, which is considered a newly location of philosophy. On the other hand, the Neutrosophic concept, which was transferred to these fields due to the classical concept of probability being insufficient for mathematics and statistics, took the studies in the field of mathematics one step further. In fact, the concepts of fuzzy and intuitionistic are naturally included in the definition of neutrosophic. After Cantor's set system, was generalized first with fuzzy sets by Zadeh and then with intuitionistic sets by Atanassov, a

generalization was made as neutrosophic sets by Smarandache. You can refer to the sources (Kausar et. al, 2023), (Bilgin and Bozma, 2020) for studies on fuzzy and (for studies on intuitionistic fuzzy sets Bilgin, and Bozma, 2021), (Malik and Akram, 2018). Statistical convergence on neutrosophic normed space was first introduced by (Kirisci and Şimşek, 2020). Very soon after (Granados and Dhital, 2021) Neutrosophic statistical convergence is defined using double-indexed sequences in normed space. Neutrosophic triple normed space is presented by (Şahin and Kargın, 2017). Then, many kinds of convergence were transferred to Neutrosophic normed space. Since the subject of convergence has an important place in both mathematics and daily life problems, a lot of studies have been carried out since 2017 on the types of convergence on neutrosophic normed spaces. One is rough convergence.

This type of convergence, which is thought to be helpful in checking the accuracy of numerical analysis and computer programming solutions and thus has the possibility of being applied in daily life, has taken its place in the literature with the studies carried out by (Phu, 2001) for normed spaces (finite dimensional). Later, this type of convergence was combined with the statistical type of convergence by Aytar, who has studies containing important evaluations about rough convergence from different perspectives. e.g. (Aytar, 2008), (Olmez and Aytar, 2021).

Rough statistical convergence is given for difference sequences in (Demir and Gumus, 2022). On the other hand, Rough statistical convergence of triplet sequences is given in (Debnath and Subramanian, 2017). In (Kisi and Gurdal, 2021), statistical convergence is defined of triplet difference sequences on neutrosophic normed space. Rough statistical convergence is studied using tripled sequences on neutrosophic normed spaces in (Bilgin, 2022).

After combining the notion of rough convergence of triplet sequences with statistical convergence theory, now, the definition of rough statistical convergence of triplet difference sequences on

neutrosophic normed space was established to complete the corresponding blank in the published.

2. PRIOR INFORMATION

(t_n) is named to be statistically convergent to t if for all $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |t_k - t| \geq \varepsilon\}) = 0.$$

Here, the three-dimensional version of natural density is given (see e.g Sahiner et all, 2007) as:

$\mathfrak{D} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is named having a natural density $\delta_3(\mathfrak{D})$, here

$$\delta_3(\mathfrak{D}) = \lim_{p,q,r \rightarrow \infty} \frac{|\mathfrak{D}(p, q, r)|}{pqr}$$

exists. Here, $|\mathfrak{D}(p, q, r)|$ demonstrate the numbers of (n_1, n_2, n_3) in $\mathfrak{D}$ where, $p \geq n_1, q \geq n_2$ and $r \geq n_3$.

(Sahiner et all, 2007) gave statistical convergence of triple sequence. $(\acute{z}_{n_1 n_2 n_3})$ is named to be statistical convergent to $\acute{z}$ if for every $\varepsilon > 0$,

$$\delta_3(\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |\acute{z}_{n_1 n_2 n_3} - \acute{z}| \geq \varepsilon\}) = 0.$$

Then, it is shown by $st - \lim_{n_1, n_2, n_3 \rightarrow \infty} \acute{z}_{n_1 n_2 n_3} = \acute{z}$.

Let us recall the definition given by (Debnath and Subramanian, 2017).

$(\acute{z}_{n_1 n_2 n_3})$ is called to be rough convergent to $\acute{z}$ is shown by $\acute{z}_{n_1 n_2 n_3} \xrightarrow{r} \acute{z}$ such that for every $\varepsilon > 0$, $\exists n_0 \in \mathbb{N} : n_1, n_2, n_3 \geq n_0$ then $|\acute{z}_{n_1 n_2 n_3} - \acute{z}| < r + \varepsilon$.

The rough limit set of $(\acute{z}_{n_1 n_2 n_3})$ is demonstrated with $Lim^r_{n_1 n_2 n_3} := \left\{ \acute{z} : \acute{z}_{n_1 n_2 n_3} \xrightarrow{r} \acute{z} \right\}$. It can be easily seen that rough limit set is not unique.

Let us recall some necessary information about the neutrosophic normed space. Let $\mathbb{X} \neq \emptyset$, $\sigma_t(t), \kappa_u(t)$ and $\beth_f(t)$ are the degrees of truth, falsity and uncertainty. We defined a neutrosophic set $\aleph$ as: For all t in $\mathbb{X}$; $\sigma_t(t), \kappa_u(t)$ and $\beth_f(t) \in [0,1]$,

$$\aleph = \left\{ \left(t, \sigma_t(t), \kappa_u(t), \beth_f(t) \right) : t \in \mathbb{X} \right\}, 0 \leq \sigma_t(t) + \kappa_u(t) + \beth_f(t) \leq 2.$$

Here, it must be pointed out that, $\kappa_u(t)$ is an independent component, $\sigma_t(t)$ and $\beth_f(t)$ are dependent components. (Smarandache, 1998).

Now, we remember definition of Neutrosophic Normed spaces given in (Kirisici and Simsek, 2020).

Let $\mathbb{X}$ be a linear spaces, $\star$ and $\diamond$ demonstrate the continuous t -norm and continuous t -conorm on $\mathbb{R}$. The notation of Neutrosophic Normed is $\left\{ \left((t, v), \sigma_t(t, v), \kappa_u(t, v), \beth_f(t, v) \right) : (t, v) \in \mathbb{X} \times (0, \infty) \right\}$. Here σ_t , κ_u and $\beth_f$ shown the degree of correctness, uncertainty and falsity of (t, v) on $\mathbb{X} \times (0, \infty)$ fulfills the below criteria: For each $t_1, t_2 \in \mathbb{X}$,

- i. For every $v \in \mathbb{R}^+$ $\sigma_t(t, v) + \kappa_u(t, v) + \beth_f(t, v) \leq 2$,
- ii. For every $v_1, v_2 \in \mathbb{R}^+$,
 $\sigma_t(t_1, v_1) \star \sigma_t(t_2, v_2) \leq \sigma_t(t_1 + t_2, v_1 + v_2)$,
 $\kappa_u(t_1, v_1) \diamond \kappa_u(t_2, v_2) \geq \kappa_u(t_1 + t_2, v_1 + v_2)$,
 $\beth_f(t_1, v_1) \diamond \beth_f(t_2, v_2) \geq \beth_f(t_1 + t_2, v_1 + v_2)$.
- iii. For every $p \in \mathbb{R}^+$, $\sigma_t(t, v) = 1 \Leftrightarrow t = 0$, $\kappa_u(t, v) = 0 \Leftrightarrow t = 0$,
 $\beth_f(t, v) = 0 \Leftrightarrow t = 0$,
- iv. For each $u \neq 0$, $\sigma_t(uv, p) = \sigma_t\left(v, \frac{p}{|u|}\right)$, $\kappa_u(uv, p) = \kappa_u\left(v, \frac{p}{|u|}\right)$
and $\beth_f(uv, p) = \beth_f\left(v, \frac{p}{|u|}\right)$.
- v. $\kappa_u(v, \cdot)$ and $\beth_f(v, \cdot)$ are continuous non-increasing function and $\sigma_t(v, \cdot)$ is continuous non-decreasing function,
- vi. $\lim_{s \rightarrow \infty} \sigma_t(v, p) = 1$, $\lim_{s \rightarrow \infty} \kappa_u(v, p) = 0$ and $\lim_{s \rightarrow \infty} \beth_f(v, p) = 0$.

vii. If $s \leq 0$, $\sigma_t(\mathfrak{o}, \mathfrak{p}) = 0$, $\kappa_u(\mathfrak{o}, \mathfrak{p}) = 1$ and $\beth_f(\mathfrak{o}, \mathfrak{p}) = 1$.

So, $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ is named Neutrosophic Normed Spaces. Here σ_t and κ_u are interdependent and $\beth_f$ is an independent components.

Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ be a Neutrosophic Normed Spaces, $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences. $(\dot{z}_{n_1 n_2 n_3})$ is named to be rough convergent to $\dot{z}$ for some $\mathfrak{r} \in \mathbb{R}^+$ such that each $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ and $\gamma \in (0, 1)$:for every $n_1, n_2, n_3 \geq n_0$ if $\sigma_t(\dot{z}_{n_1 n_2 n_3} - \dot{z}, \mathfrak{r} + \varepsilon) > 1 - \gamma$, $\kappa_u(\dot{z}_{n_1 n_2 n_3} - \dot{z}, \mathfrak{r} + \varepsilon) < \gamma$ and $\beth_f(\dot{z}_{n_1 n_2 n_3} - \dot{z}, \mathfrak{r} + \varepsilon) < \gamma$. Then, it is denote with $\mathfrak{r} - \lim_{n_1, n_2, n_3 \rightarrow \infty} \dot{z}_{n_1 n_2 n_3} = \dot{z}$. (Bilgin, 2022).

Let $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$. $(\dot{z}_{n_1 n_2 n_3})$ is called to be rough statistically convergent to $\dot{z}$ for some $\mathfrak{r} \in [0, \infty)$ such that all $\varepsilon > 0$ and $\gamma \in (0, 1)$ where

$$\delta_3(\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t(\dot{z}_{n_1 n_2 n_3} - \dot{z}, \mathfrak{r} + \varepsilon) \leq 1 - \gamma \text{ or } \kappa_u(\dot{z}_{n_1 n_2 n_3} - \dot{z}, \mathfrak{r} + \varepsilon) \geq \gamma \text{ and } \beth_f(\dot{z}_{n_1 n_2 n_3} - \dot{z}, \mathfrak{r} + \varepsilon) \geq \gamma\}) = 0.$$

Afterwards, it is indicated by $st - \mathfrak{r} - \lim_{n_1, n_2, n_3 \rightarrow \infty} \dot{z}_{n_1 n_2 n_3} = \dot{z}$. (Bilgin, 2022).

For $\mathfrak{r} = 0$, rough statistical convergence consistent matching the statistical convergence on $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$. Let rough statistical limit set of $(\dot{z}_{n_1 n_2 n_3})$ is denoted with;

$$\mathcal{S}t - \mathfrak{r} - LIM(\dot{z}_{n_1 n_2 n_3}) = \{\dot{z} : st - \mathfrak{r} - \lim_{n_1, n_2, n_3 \rightarrow \infty} \dot{z}_{n_1 n_2 n_3} = \dot{z}\}.$$

3. MAIN RESULT

Definition 3.1 Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ be a Neutrosophic Normed Spaces, $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences. $(\dot{z}_{n_1 n_2 n_3})$ is named to be

rough Δ -convergent to $\acute{z}$ for some $r \in \mathbb{R}^+$ such that each $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ and $\gamma \in (0,1)$: for every $n_1, n_2, n_3 \geq n_0$ if

$$\sigma_t(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, r + \varepsilon) > 1 - \gamma \quad , \quad \kappa_u(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, r + \varepsilon) < \gamma, \\ \beth_f(\Delta_{\acute{z}_{n_1 n_2 n_3}} - \acute{z}, r + \varepsilon) < \gamma.$$

for all $n_1, n_2, n_3 \geq n_0$, where $n_1, n_2, n_3 \in \mathbb{N}$,

$$\Delta_{\acute{z}_{n_1 n_2 n_3}} = \acute{z}_{n_1 n_2 n_3} - \acute{z}_{n_1(n_2+1)n_3} - \acute{z}_{n_1 n_2(n_3+1)} + \\ \acute{z}_{n_1(n_2+1)(n_3+1)} - \acute{z}_{(n_1+1)n_2 n_3} \\ + \acute{z}_{(n_1+1)(n_2+1)n_3} + \acute{z}_{(n_1+1)n_2(n_3+1)} - \acute{z}_{(n_1+1)(n_2+1)(n_3+1)}.$$

In this case, it is denoted with $r - \lim_{n_1, n_2, n_3 \rightarrow \infty} \Delta_{\acute{z}_{n_1 n_2 n_3}} = \acute{z}$.

Definition 3.2 Let $(\acute{z}_{n_1 n_2 n_3})$ be a triple sequence in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$. $(\acute{z}_{n_1 n_2 n_3})$ is called to be rough statistically convergent to $\acute{z}$ for some $r \in [0, \infty)$ such that each $\varepsilon > 0$ and $\gamma \in (0,1)$ if

$$\delta_3(\{(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t(\acute{z}_{n_1 n_2 n_3} - \acute{z}, r + \varepsilon) \\ \leq 1 - \gamma \text{ or } \kappa_u(\acute{z}_{n_1 n_2 n_3} - \acute{z}, r + \varepsilon) \\ \geq \gamma \text{ and } \beth_f(\acute{z}_{n_1 n_2 n_3} - \acute{z}, r + \varepsilon) \geq \gamma\}) = 0$$

for all $n_1, n_2, n_3 \geq n_0$, where $n_1, n_2, n_3 \in \mathbb{N}$,

$$\Delta_{\acute{z}_{n_1 n_2 n_3}} = \acute{z}_{n_1 n_2 n_3} - \acute{z}_{n_1(n_2+1)n_3} - \acute{z}_{n_1 n_2(n_3+1)} + \\ \acute{z}_{n_1(n_2+1)(n_3+1)} - \acute{z}_{(n_1+1)n_2 n_3} \\ + \acute{z}_{(1+n_1)(1+n_2)n_3} + \acute{z}_{(1+n_1)n_2(n_3+1)} - \acute{z}_{(1+n_1)(1+n_2)(n_3+1)}.$$

Then, it is denoted with $\mathcal{R}t - r - \lim_{n_1, n_2, n_3 \rightarrow \infty} \Delta_{\acute{z}_{n_1 n_2 n_3}} = \acute{z}$.

For $\mathbb{r} = 0$, rough statistical convergence in accordance to statistical convergence on $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$. Let rough statistical limit set of $(\Delta_{\dot{z}_{n_1 n_2 n_3}})$ is denoted with;

$$\mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right) = \left\{ \dot{z} : \mathcal{St} - \mathbb{r} - \lim_{n_1, n_2, n_3 \rightarrow \infty} \Delta_{\dot{z}_{n_1 n_2 n_3}} = \dot{z} \right\}.$$

Definition 3.3 Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ be a Neutrosophic Normed Spaces, $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences. $(\dot{z}_{n_1 n_2 n_3})$ is named to be rough statistically bounded for some $\mathbb{r} \in \mathbb{R}^+$ such that each $\varepsilon > 0$ and $\gamma \in (0, 1)$ if there exists a $\mathcal{T} > 0$ such that

$$\begin{aligned} \delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \right. \right. \\ \left. \left. \leq 1 - \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \geq \gamma \text{ and } \right. \right. \\ \left. \left. \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \geq \gamma \right\} \right) = 0 \end{aligned}$$

$$\begin{aligned} \text{where} \quad \Delta_{\dot{z}_{n_1 n_2 n_3}} &= \dot{z}_{n_1 n_2 n_3} - \dot{z}_{n_1 (n_2+1) n_3} - \dot{z}_{n_1 n_2 (n_3+1)} + \\ &\dot{z}_{n_1 (n_2+1) (n_3+1)} - \dot{z}_{(n_1+1) n_2 n_3} \\ &+ \dot{z}_{(n_1+1) (n_2+1) n_3} + \dot{z}_{(n_1+1) n_2 (n_3+1)} - \dot{z}_{(n_1+1) (n_2+1) (n_3+1)}. \end{aligned}$$

Now, using these descriptions, the next significant theorems for triple sequences in Neutrosophic normed spaces can be proved.

Lemma 3.1 Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ be a Neutrosophic Normed Spaces, For a triple sequences $(\dot{z}_{mno})$ and some $\mathbb{r} \geq 0$, if $\mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right) \neq \emptyset$ then $(\dot{z}_{n_1 n_2 n_3})$ is rough bounded sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$.

Proof Let $(\dot{z}_{mno})$ be a triple sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ and some > 0 , $\mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right) \neq \emptyset$. Then there exists $\dot{z}$ and $\dot{z} \in \mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$. For all $\varepsilon > 0$ and $0 < \gamma < 1$, it is written

$\delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \right. \right.$
 $\left. \leq 1 - \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \geq \gamma \right.$
 $\left. \text{and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \geq \gamma \right\} \right) = 0.$
 So, $(\dot{z}_{n_1 n_2 n_3})$ is statistically bounded in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$.

Lemma 3.2 Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ be a Neutrosophic Normed Spaces, $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences. If $(\dot{z}_{n_1 n_2 n_3})$ is rough bounded sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ then, for some $\mathbb{r} \geq 0$, if $\mathcal{S}t - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right) \neq \emptyset$.

Proof $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ and $(\dot{z}_{n_1 n_2 n_3})$ is bounded sequences. For each $\varepsilon > 0$, $\gamma \in (0, 1)$ and some $\mathbb{r} \geq 0$ there exists a $\mathcal{T} > 0$ such that

$$\begin{aligned}
 \delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \right. \right. \\
 \left. \leq 1 - \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \geq \gamma \right.
 \end{aligned}$$

$$\left. \text{and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \geq \gamma \right\} \right) = 0.$$

Let a set of the form

$$\begin{aligned}
 \mathfrak{M} = \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \leq 1 - \right. \\
 \left. \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \geq \gamma \right. \\
 \left. \text{and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) \geq \gamma \right\}
 \end{aligned}$$

is defined. For $(n_1, n_2, n_3) \in \mathfrak{M}^c$, it is written $\sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) > 1 - \gamma$ and $\kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) < \gamma$, $\beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathcal{T} \right) < \gamma$. Furthermore,

$$\begin{aligned}
 \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathbb{r} + \mathcal{T} \right) > 1 - \gamma \quad \text{and} \quad \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathbb{r} + \mathcal{T} \right) < \\
 \gamma, \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}}, \mathbb{r} + \mathcal{T} \right) < \gamma.
 \end{aligned}$$

So $0 \in \mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}})$. Hence $\mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}}) \neq \emptyset$.

In the next part, some topological characteristics of the set of $\mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}})$ will be examined.

Theorem 3.1 Let $(\Delta_{\dot{z}_{n_1 n_2 n_3}})$ is a triple sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \star, \diamond)$ then $\mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}})$ is closed sets.

Proof It's simple to demonstrate that $\mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}}) = \emptyset$, so let $\mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}}) \neq \emptyset$. Then, choosing a triple sequences $(\dot{z}_{n_1 n_2 n_3})$ and $0 < \gamma'$, where $\gamma \diamond \gamma > \gamma', 1 - \gamma' < (1 - \gamma) \star (1 - \gamma)$ and $st - \mathbb{R} - \lim_{n_1, n_2, n_3 \rightarrow \infty} \Delta_{\dot{z}_{n_1 n_2 n_3}} = \dot{z}$. It will be shown that $\dot{z} \in \mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}})$. Let $\varepsilon > 0$ and we use definition; there exists a $n_0 \in \mathbb{N}$ such that for $n_1, n_2, n_3 \geq n_0$,

$$\sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \frac{\varepsilon}{3} \right) > 1 - \gamma \quad , \quad \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \frac{\varepsilon}{3} \right) < \gamma \quad \text{and} \\ \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \frac{\varepsilon}{3} \right) < \gamma.$$

If choosing $\dot{z}_{m' n' o'} \in \mathcal{St} - \mathbb{R} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}})$ where $\ddot{n}_1, \ddot{n}_2, \ddot{n}_3 > n_0$ such that

$$\delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{R} + \frac{\varepsilon}{3} \right) \leq 1 - \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{R} + \frac{\varepsilon}{3} \right) \geq \gamma \right\} \right) = 0.$$

$$\text{and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{R} + \frac{\varepsilon}{3} \right) \geq \gamma \} = 0.$$

For $(a, b, c) \in \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) > 1 - \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \right\}.$

Furthermore,

$$1 - \gamma' < (1 - \gamma) \star (1 - \gamma) < \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \text{ and } \kappa_u \left(\Delta_{\dot{z}_{abc}} - \dot{z}, \mathbb{r} + \varepsilon \right) \leq \gamma \diamond \gamma < \gamma',$$

$$\beth_f \left(\dot{z}_{n_1 n_2 n_3} - \dot{z}, \mathbb{r} + \varepsilon \right) \leq \gamma \diamond \gamma < \gamma'. \text{ So,}$$

$$(a, b, c) \in \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) > 1 - \gamma \text{ or}$$

$$\kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) < \gamma \right\}. \text{ Then,}$$

$$\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) > 1 - \gamma, \right.$$

$$\left. \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\ddot{n}_1 \ddot{n}_2 \ddot{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \right\}$$

$$\subseteq \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \frac{\varepsilon}{3} \right) > 1 - \gamma, \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - y, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \right.$$

$$\left. \text{and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \right\}.$$

Then,

$$\delta_3 \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \leq 1 - \gamma \text{ or} \right.$$

$$\left. \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \geq \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \geq \gamma \right\}$$

$$\begin{aligned}
&\leq \delta_3 \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\vec{n}_1 \vec{n}_2 \vec{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) \right. \\
&\quad \left. \leq 1 - \gamma \text{ or } \right. \\
&\quad \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\vec{n}_1 \vec{n}_2 \vec{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) \\
&\quad \left. \geq \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \Delta_{\dot{z}_{\vec{n}_1 \vec{n}_2 \vec{n}_3}}, \mathbb{r} + \frac{\varepsilon}{3} \right) \geq \gamma \right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\delta_3 \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \leq 1 - \right. \\
&\quad \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \geq \gamma \text{ and } \\
&\quad \left. \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) \geq \gamma \right\} = 0.
\end{aligned}$$

Consequently, it is shown that $\dot{z} \in \mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$.

Definition 3.4 Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, *, \diamond)$ be a Neutrosophic Normed Spaces. For some $\mathbb{r} \in [0, \infty)$, every $\varepsilon > 0$ and $\gamma \in (0, 1)$,

$$\begin{aligned}
&\delta_3 \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) > 1 - \right. \\
&\quad \gamma \text{ and } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \varepsilon \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \right. \\
&\quad \left. \varepsilon \right) < \gamma \left. \right\} \neq 0,
\end{aligned}$$

then, $\dot{z}$ is termed rough statistical cluster point of $(\dot{z}_{n_1 n_2 n_3})$. It is denote with $\mathcal{St} - \mathbb{r} - \Delta cls$ point of $(\dot{z}_{n_1 n_2 n_3})$. Let $\mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{r}}$ is demonstrated the set of each $\mathcal{St} - \mathbb{r} - \Delta cls$ point of $(\dot{z}_{n_1 n_2 n_3})$ in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, *, \diamond)$.

Theorem 3.2 Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \min, \max)$ be a Neutrosophic Normed Spaces, $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences. Then, for some $\mathbb{r} \in [0, \infty)$, every $\varepsilon > 0$ and $\gamma \in (0, 1)$ the set $\mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{r}}$ is closed.

Proof Let $\mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{R}} \neq \emptyset$ be taken as the proof for $\mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{R}} = \emptyset$ is clear. Now, choosing $(u_{n_1 n_2 n_3}) \subseteq \mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{R}}$ and $\mathbb{R} - \lim_{n_1, n_2, n_3 \rightarrow \infty} u_{n_1 n_2 n_3} = u$. If $u \in \mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{R}}$ is proven. If we use definition of rough convergence of sequences, for every $\varepsilon > 0$ and $\gamma \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that for $m, n, o > n_0$, $\sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - u, \frac{\varepsilon}{3} \right) > 1 - \gamma$, $\kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - u, \frac{\varepsilon}{3} \right) < \gamma$ and $\beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - u, \frac{\varepsilon}{3} \right) < \gamma$. Choosing $\tilde{n} \in \mathbb{N}$ where $\tilde{n} \geq n_0$. So, $\sigma_t \left(\dot{z}_{\tilde{n}} - u, \frac{\varepsilon}{3} \right) > 1 - \gamma$, $\kappa_u \left(\dot{z}_{\tilde{n}} - u, \frac{\varepsilon}{3} \right) < \gamma$ and $\beth_f \left(\dot{z}_{\tilde{n}} - u, \frac{\varepsilon}{3} \right) < \gamma$. Using $(u_{n_1 n_2 n_3}) \subseteq \mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{R}}$, it is written $\dot{z}_{\tilde{n}} \in \mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{R}}$. So

$$\delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) > 1 - \gamma, \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) < \gamma \right\} \right) \neq 0.$$

Taking

$$\begin{aligned} (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3) &\in \{(n_1, n_2, n_3) \\ &\in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) > 1 - \gamma, \\ \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) < \gamma\}. \end{aligned}$$

Hence, it is written that

$$\begin{aligned} \sigma_t \left(\Delta_{\dot{z}_{\tilde{n}_1 \tilde{n}_2 \tilde{n}_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) &> 1 - \gamma, \kappa_u \left(\Delta_{\dot{z}_{\tilde{n}_1 \tilde{n}_2 \tilde{n}_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) < \gamma \text{ and} \\ \beth_f \left(\Delta_{\dot{z}_{\tilde{n}_1 \tilde{n}_2 \tilde{n}_3}} - \dot{z}_{\tilde{n}}, \mathbb{R} + \frac{\varepsilon}{3} \right) &< \gamma. \text{ Then, similarly} \\ \sigma_t \left(\Delta_{\dot{z}_{\tilde{n}_1 \tilde{n}_2 \tilde{n}_3}} - u, \mathbb{R} + \varepsilon \right) &> 1 - \gamma, \kappa_u \left(\Delta_{\dot{z}_{\tilde{n}_1 \tilde{n}_2 \tilde{n}_3}} - u, \mathbb{R} + \varepsilon \right) < \gamma \text{ and} \\ \beth_f \left(\Delta_{\dot{z}_{\tilde{n}_1 \tilde{n}_2 \tilde{n}_3}} - u, \mathbb{R} + \varepsilon \right) &< \gamma. \end{aligned}$$

Then,

$$\begin{aligned}
(\widetilde{n}_1, \widetilde{n}_2, \widetilde{n}_3) &\in \left\{ (n_1, n_2, n_3) \right. \\
&\quad \left. \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \varepsilon \right) > 1 - \gamma, \right. \\
&\quad \left. \kappa_u \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \varepsilon \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \varepsilon \right) < \gamma \right\}.
\end{aligned}$$

So,

$$\begin{aligned}
&\delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - \dot{z}_{\widetilde{n}}, \mathbb{r} + \frac{\varepsilon}{3} \right) > 1 - \right. \right. \\
&\gamma, \kappa_u \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - \dot{z}_{\widetilde{n}}, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - \dot{z}_{\widetilde{n}}, \mathbb{r} + \frac{\varepsilon}{3} \right) < \\
&\gamma \left. \right\} \leq \delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \frac{\varepsilon}{3} \right) > \right. \right. \\
&1 - \gamma, \kappa_u \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \left. \right\} \Big).
\end{aligned}$$

Using definition of natural density

$$\begin{aligned}
&\delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \frac{\varepsilon}{3} \right) > 1 - \right. \right. \\
&\gamma, \kappa_u \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \frac{\varepsilon}{3} \right) < \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{\widetilde{n}_1 \widetilde{n}_2 \widetilde{n}_3}} - u, \mathbb{r} + \frac{\varepsilon}{3} \right) < \\
&\gamma \left. \right\} \Big) \neq 0.
\end{aligned}$$

So, $u \in \mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^{\mathbb{r}}$.

Theorem 3.3 Let $(\dot{z}_{n_1 n_2 n_3})$ is a triple sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \min, \max)$ then the set $\mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$ is convex.

Proof

For $\dot{z}_1, \dot{z}_2 \in \mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$, $\varepsilon > 0$ and some $0 < \alpha < 1$, it will be shown that $((1 - \alpha)y_1 + \alpha y_2) \in \mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$. Let $R, \widetilde{R}$ be defined in next form:

$$\begin{aligned}
R = & \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1, \frac{\mathbb{I} + \varepsilon}{3(1 - \alpha)} \right) \right. \\
& \leq 1 - \gamma \text{ or} \\
& \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1, \frac{\mathbb{I} + \varepsilon}{3(1 - \alpha)} \right) \\
& \geq \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1, \frac{\mathbb{I} + \varepsilon}{3(1 - \alpha)} \right) \geq \gamma \Big\},
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{R} = & \left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2, \frac{\mathbb{I} + \varepsilon}{3\alpha} \right) \right. \\
& \leq 1 - \gamma \text{ or} \\
& \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2, \frac{\mathbb{I} + \varepsilon}{3\alpha} \right) \geq \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2, \frac{\mathbb{I} + \varepsilon}{3\alpha} \right) \geq \gamma \Big\}.
\end{aligned}$$

Using $\dot{z}_1, \dot{z}_2 \in \mathcal{St} - \mathbb{I} - LIM(\Delta_{\dot{z}_{n_1 n_2 n_3}})$, it is written $\delta_3(R) =$

$\delta_3(\tilde{R}) = 0$. Let $(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3) \in R^c \cap \tilde{R}^c$, then

$$\begin{aligned}
& \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - ((1 - \alpha)\dot{z}_1 + \alpha\dot{z}_2), \mathbb{I} + \varepsilon \right) \\
& = \sigma_t \left((1 - \alpha) \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1 \right) + \alpha \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2 \right), \mathbb{I} \right. \\
& \quad \left. + \varepsilon \right) \\
& \geq \min \left\{ \sigma_t \left((1 - \alpha) \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1 \right), \frac{\mathbb{I} + \varepsilon}{2} \right), \sigma_t \left(\alpha \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right. \right. \right. \\
& \quad \left. \left. - \dot{z}_2 \right), \frac{\mathbb{I} + \varepsilon}{2} \right) \Big\} \\
& = \min \left\{ \sigma_t \left(\left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1 \right), \frac{\mathbb{I} + \varepsilon}{2(1 - \alpha)} \right), \sigma_t \left(\left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2 \right), \frac{\mathbb{I} + \varepsilon}{2\alpha} \right) \right\} \\
& \quad > 1 - \gamma,
\end{aligned}$$

$$\begin{aligned}
& \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - ((1 - \alpha)\dot{z}_1 + \alpha\dot{z}_2), \mathbb{I} + \varepsilon \right) \\
&= \kappa_u \left((1 - \alpha) \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1 \right) + \alpha \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2 \right), \mathbb{I} + \varepsilon \right) \\
&\leq \max \left\{ \kappa_u \left((1 - \alpha) \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1 \right), \frac{\mathbb{I} + \varepsilon}{2} \right), \kappa_u \left(\alpha \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right. \right. \right. \\
&\quad \left. \left. \left. - \dot{z}_2 \right), \frac{\mathbb{I} + \varepsilon}{2} \right) \right\} < \gamma
\end{aligned}$$

and

$$\begin{aligned}
& \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - ((1 - \alpha)\dot{z}_1 + \alpha\dot{z}_2), \mathbb{I} + \varepsilon \right) \\
&= \beth_f \left((1 - \alpha) \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1 \right) + \alpha \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2 \right), \mathbb{I} + \varepsilon \right) \\
&\leq \max \left\{ \beth_f \left(\left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_1 \right), \frac{\mathbb{I} + \varepsilon}{2(1 - \alpha)} \right), \beth_f \left(\alpha \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}_2 \right), \frac{\mathbb{I} + \varepsilon}{2} \right) \right\} \\
&\quad < \gamma.
\end{aligned}$$

Then, it is written that

$$\begin{aligned}
& \delta_3 \left(\left\{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - ((1 - \alpha)\dot{z}_1 + \right. \right. \right. \\
&\quad \left. \left. \left. \alpha\dot{z}_2), \mathbb{I} + \varepsilon \right) \leq 1 - \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - ((1 - \right. \right. \right. \\
&\quad \left. \left. \left. \alpha)\dot{z}_1 + \alpha\dot{z}_2), \mathbb{I} + \varepsilon \right) \geq 1 - \gamma \text{ and } \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - ((1 - \alpha)\dot{z}_1 + \right. \right. \right. \\
&\quad \left. \left. \left. \alpha\dot{z}_2), \mathbb{I} + \varepsilon \right) \geq 1 - \gamma \right\} \right) = 0.
\end{aligned}$$

Consequently, $((1 - \alpha)\dot{z}_1 + \alpha\dot{z}_2) \in \mathcal{S}\mathcal{T} - \mathbb{I} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$. So, it is shown that the set $\mathcal{S}\mathcal{T} - \mathbb{I} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$ is convex.

Theorem 3.4 Let $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \min, \max)$ be a Neutrosophic Normed Spaces, $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences. For some $r > 0$, $\gamma \in (0,1)$ and a fixed $w \in \mathcal{X}$,

$$\begin{aligned} \sigma(w, \gamma, r) &= \{\dot{z}_{n_1 n_2 n_3} : \sigma_t(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, r) \\ &> 1 - \gamma, \kappa_u(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, r) \leq \gamma, \end{aligned}$$

$$\beth_f(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, r) \leq \gamma\},$$

also,

$$\begin{aligned} \overline{\sigma(w, \gamma, r)} &= \{\dot{z}_{n_1 n_2 n_3} : \sigma_t(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, r) \\ &\geq 1 - \gamma, \kappa_u(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, r) \\ &< \gamma, \beth_f(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, r) < \gamma\}. \end{aligned}$$

Then,

$$\mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}^r = \bigcup_{w \in \mathcal{C}_{\Delta_{\dot{z}_{n_1 n_2 n_3}}}} \overline{\sigma(w, \gamma, r)}$$

Proof

Let $\dot{z} \in \bigcup_{w \in \mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}} \overline{\sigma(w, \gamma, r)}$. For some $r > 0$ and given $\gamma \in (0,1)$ there exists $w \in \mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}$ such that $\sigma_t(w - \dot{z}, r) > 1 - \gamma$, $\kappa_u(w - \dot{z}, r) < \gamma$ and $\beth_f(w - \dot{z}, r) < \gamma$.

Then, for $\bar{r} > 0$ if we use $w \in \mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}$, then there exists a set

$$\begin{aligned} P &= (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \bar{r}) \\ &> 1 - \gamma, \kappa_u(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \bar{r}) < \gamma, \end{aligned}$$

$$\beth_f(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \bar{r}) < \gamma$$

and $\delta_3(P) \neq 0$. So, for $(n_1, n_2, n_3) \in P$,

$$\sigma_t(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, r + \bar{r}) \geq \min \left\{ \sigma_t(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \bar{r}), \sigma_t(w - \dot{z}, r) \right\} > 1 - \gamma,$$

$$\begin{aligned} \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \mathbb{r} + \bar{\mathbb{r}} \right) \\ \leq \max \left\{ \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \bar{\mathbb{r}} \right), \kappa_u(w - \dot{z}, \mathbb{r}) \right\} < \gamma \end{aligned}$$

and

$$\begin{aligned} \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \mathbb{r} + \bar{\mathbb{r}} \right) \\ \leq \max \left\{ \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - w, \bar{\mathbb{r}} \right), \beth_f(w - \dot{z}, \mathbb{r}) \right\} < \gamma. \end{aligned}$$

From hence,

$$\begin{aligned} \delta_3 \left\{ \left((n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \bar{\mathbb{r}} \right) > 1 - \right. \right. \\ \left. \left. \gamma, \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \bar{\mathbb{r}} \right) < \gamma \text{ and } \right. \right. \\ \left. \left. \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \mathbb{r} + \bar{\mathbb{r}} \right) < \gamma \right) \right\} \neq 0. \end{aligned}$$

Thus, $\dot{z} \in \mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}^{\mathbb{r}}$ and then $\bigcup_{w \in \mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}} \overline{\sigma(w, \gamma, \mathbb{r})} \subseteq \mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}^{\mathbb{r}}$.

It is easily demonstrated from definition that $\mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}^{\mathbb{r}} \subseteq \bigcup_{w \in \mathcal{C}_{\dot{z}_{n_1 n_2 n_3}}} \overline{\sigma(w, \gamma, \mathbb{r})}$

Theorem 3.5

Let $(\dot{z}_{n_1 n_2 n_3})$ be a triple sequences in $(\mathcal{X}, \sigma_t, \kappa_u, \beth_f, \min, \max)$ and $(\dot{z}_{n_1 n_2 n_3})$ is statistically convergent to $\dot{z}$. For some $\mathbb{r} > 0$, there exists $\gamma \in (0, 1)$:

$$\mathcal{S}t - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right) = \overline{\sigma(\gamma, \gamma, \mathbb{r})}.$$

Proof

For $\check{\gamma} > 0$, if we use statistical convergence of $(\dot{z}_{n_1 n_2 n_3})$, there exists

$$\begin{aligned} T = \{ (n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \check{\mathbb{r}} \right) \leq 1 - \\ \gamma \text{ or } \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \check{\mathbb{r}} \right) \geq \gamma \end{aligned}$$

and $\beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - \dot{z}, \check{\mathbb{r}} \right) \geq \gamma \}$ and $\delta_3(T) = 0$.

Let $s \in \overline{\sigma(\dot{z}, \gamma, \mathbb{r})}$ and for $(n_1, n_2, n_3) \in T^c$, then

$$\sigma_t \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - s, \mathbb{r} + \ddot{\mathbb{r}} \right) > 1 - \gamma, \kappa_u \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - s, \mathbb{r} + \ddot{\mathbb{r}} \right) < \gamma \quad \text{and} \\ \beth_f \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} - s, \mathbb{r} + \ddot{\mathbb{r}} \right) < \gamma.$$

That is $s \in \mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$. So, $\overline{\sigma(\dot{z}, \gamma, \mathbb{r})} \subseteq \mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right)$. Furthermore,
 $\mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right) \subseteq \overline{\sigma(\dot{z}, \gamma, \mathbb{r})}$. Consequently, $\mathcal{St} - \mathbb{r} - LIM \left(\Delta_{\dot{z}_{n_1 n_2 n_3}} \right) = \overline{\sigma(\dot{z}, \gamma, \mathbb{r})}$.

4. CONCLUSION

Rough convergence is an important concept for the fields of fuzzy, intuitionistic fuzzy and neutrosophic theory. We carry forward the notion of coarse statistical convergence defined on Neutrosophistic normed spaces using triple difference sequences. Thus, we extend some important well-known results. Important topological properties of the set of coarse statistical limit points are given.

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