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On an inverse scattering problem for a class of Dirac operators with spectral parameter in the boundary condition

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ABSTRACT

In this work, we consider the inverse scattering problem for a class of one dimensional Dirac operators on the semi-infinite interval with the boundary condition depending polynomially on a spectral parameter. The scattering data of the given problem is defined and its properties are examined. The main equation is derived, its solvability is proved and it is shown that the potential is uniquely recovered in terms of the scattering data. A generalization of the Marchenko method is given for a class of Dirac operator.

1. Introduction

We consider the canonical system of Dirac differential equations

$$By' + mTy + \Omega(x)y = \lambda y, \quad (0 < x < \infty) \tag{1}$$

with the boundary condition

$$\left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right) y_1(0) - \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right) y_2(0) = 0, \tag{2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \qquad y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

$m > 0$ is the mass, λ is the spectral parameter and $p(x)$, $q(x)$ are real valued functions satisfying the following inequalities

$$|p(x)| \le \frac{c}{(1+x)^{2+\epsilon}} \qquad |q(x)| \le \frac{c}{(1+x)^{1+\epsilon}} \tag{3}$$

where c and ϵ are positive numbers. Assume that for $\alpha_i, \beta_j \in \mathbb{R}$ $(i, j = 0, 1, 2)$ the relations

$$\alpha_0\beta_1 - \alpha_1\beta_0 < 0, \qquad \alpha_0\beta_2 - \alpha_2\beta_0 = 0, \qquad \alpha_1\beta_2 - \alpha_2\beta_1 < 0 \tag{4}$$

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hold. It is easily obtained that the function

$$f_0(x,\lambda)=\begin{pmatrix}\dfrac{\lambda+m}{\dfrac{k}{-i}}\end{pmatrix}e^{ikx}$$

is a solution of Eq. (1) when $\Omega(x)=0$, where $k=\lambda\sqrt{1-\frac{m^2}{\lambda^2}},\ \ |\lambda|>m$.

Assume that the components of matrix function $\Omega(x)$ satisfy the condition (3). It is known from [1,2] that Eq. (1) has an unique vector solution

$$f(x,\lambda)=\begin{pmatrix}f_1(x,\lambda)\\ f_2(x,\lambda)\end{pmatrix}$$

that tends to $f_0(x,\lambda)$ as $x\to\infty$, $\operatorname{Im}\lambda\geq 0$, and there exists a matrix function $A(x,t)$ such that

$$f(x,\lambda)=f_0(x,\lambda)+\int_x^\infty A(x,t)\,f_0(t,\lambda)\,dt. \tag{5}$$

The components of the matrix kernel $A(x,t)$ satisfy the inequalities

$$|A_{ii}|\leq\frac{c_2}{(1+t)^{1+\epsilon}},\quad i=1,2 \tag{6}$$

$$\left|A_{ij}\right|\leq\frac{c_1}{(1+x)(1+t)^{1+\epsilon}},\quad i\neq j \tag{7}$$

and

$$BA(x,x)-A(x,x)B=\Omega(x). \tag{8}$$

Denote by

$$\varphi(x,\lambda)=\begin{pmatrix}\varphi_1(x,\lambda)\\ \varphi_2(x,\lambda)\end{pmatrix}$$

a special solution of the equation system (1) satisfying the initial conditions

$$\varphi_1(0,\lambda)=\left(\beta_0+\beta_1\lambda+\beta_2\lambda^2\right),\qquad \varphi_2(0,\lambda)=\left(\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2\right).$$

Let

$$E(\lambda)=\left(\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2\right)f_1(0,\lambda)-\left(\beta_0+\beta_1\lambda+\beta_2\lambda^2\right)f_2(0,\lambda).$$

The function

$$S(\lambda)=\frac{\overline{E(\lambda)}}{E(\lambda)}$$

is the scattering function of the boundary value problem (1)–(3). It is proved that the function $E(\lambda)$ has only finitely many zeros $\lambda_1,\lambda_2,\ldots,\lambda_n$ in the interval $(-m,m)$. We show that the matrix kernel $A(x,y)$ of the solution (5) satisfies the integral equation

$$A(x,y)+F(x+y)+\int_x^\infty A(x,t)\,F(t+y)\,dt=0,\quad x<y<\infty, \tag{9}$$

where

$$F(x+y)=\Re\frac{1}{2\pi}\int_{|\lambda|>m}(S_0(\lambda)-S(\lambda))\begin{pmatrix}\dfrac{\lambda+m}{k} & -i\\ -i & -\dfrac{k}{\lambda+m}\end{pmatrix}e^{ik(x+y)}d\lambda-\sum_{j=1}^n m_j^2 f\left(x,\lambda_j\right)\widetilde{f}\left(y,\lambda_j\right),$$

$$S_0(\lambda)=\frac{\alpha_2-i\beta_2}{\alpha_2+i\beta_2}$$

and

$$m_j^{-2}\equiv-\frac{f_2^2(0,\lambda_j)}{\left(\alpha_0+\alpha_1\lambda_j+\alpha_2\lambda_j^2\right)^2}\left[\alpha_1\beta_0-\alpha_0\beta_1+(\alpha_2\beta_1-\alpha_1\beta_2)\lambda_j^2\right]+\int_0^\infty f^*\left(x,\lambda_j\right)f\left(x,\lambda_j\right)dx$$

are the norming numbers, where $\widetilde{f}$ denotes the transposed vector of f and f^* the transposed vector of $\overline{f}$.

Obviously, to write the integral equation (9) it is sufficient to know the function $F(x)$ and to find the function $F(x)$ it is sufficient to know only the set of values $\{S(\lambda)\ (-\infty < \lambda < +\infty);\ \lambda_k;\ m_k\ (k = 1, \dots, n)\}$. The set of values $\{S(\lambda), \lambda_k, m_k\}$ is called the scattering data for the boundary value problem (1)–(3). The inverse problem of the scattering theory for the boundary value problem (1)–(3) means to find the potential matrix $\Omega(x)$ by knowing scattering data of the problem (1)–(3). If Eq. (9), which is constructed only on the basis of the scattering data, has a unique solution $A(x, y)$, then the matrix function $\Omega(x)$ can be found from (8). The solvability of the integral equation (9) is examined and the algorithm given of recovering the potential $\Omega(x)$.

The one dimensional Dirac system has been extensively studied. The inverse problems for one dimensional Dirac operators on the semi-infinite interval $[0, \infty)$ by using the spectral function and the scattering data were examined in [2,3]. The inverse problem of scattering theory for a Dirac equation system of order $2n$ was completely solved in [1]. For the first order differential equation system, this problem has been studied by many authors in [4–11]. Notice that, when the polynomial coefficients of $y_1(0)$ and $y_2(0)$ in the condition (2) are changed, then the boundary value problem becomes completely different. The main aim of this article is to show the application of the Marchenko method [12] to the boundary value problem (1)–(3) when the condition (4) is satisfied. We follow the works of Gasymov and Levitan [3] (also see [1]).

This paper is organized as follows. In Section 2, the scattering data corresponding to the boundary value problem (1)–(3) is defined and its properties investigated. In Section 3, the Marchenko type main equation, which the kernel of the transformation operator satisfies, is derived. Finally, in Section 4, the solvability of the main equation and the unique recovery of the potential matrix from the given scattering data are shown.

The inverse problem on the semi-infinite interval $[0, \infty)$ for Sturm–Liouville operators with the spectral parameter in the boundary condition was investigated for spectral data in [13–18] and for different spectral characteristics in [19–21]. The inverse problems for Dirac operators and for Sturm–Liouville operators with spectral parameter in the boundary conditions on a finite interval were studied in [22–24,10].

The boundary value problem (1)–(3) can be reduced to an operator pencil equation in a Hilbert space. Inverse scattering problems formed for the first order differential equations system with a polynomial pencil are considered in applications of nonlinear Schrödinger equations [25–28]. In [26] Kaup and Newell showed that, with this pencil

$$L_\lambda \Psi(x, \lambda) \equiv \left\{ i\sigma_3 \frac{d}{dx} + \lambda \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} - \lambda^2 \right\} \Psi(x, \lambda),$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the modified nonlinear Schrödinger equation

$$iq_t + q_{xx} + i\epsilon \left(|q|^2 q \right)_x = 0, \qquad p = \epsilon q, \quad \epsilon = \pm 1$$

was solvable. Konopelchenko in [27] showed that with the help of a pencil

$$L_\lambda \Psi(x, \lambda) \equiv \left\{ i\sigma_3 \frac{d}{dx} + (\mu\lambda + \nu) \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} + \left(\mu\lambda^2 + 2\nu\lambda \right) \right\} \Psi(x, \lambda) = 0$$

the other version of the modified nonlinear Schrödinger equation

$$iq_t + q_{xx} + 2\nu^2 |q|^2 q - i\mu \left(|q|^2 q \right)_x = 0$$

was solvable. There are applications of these equations in plasma physics. Kuznetchov and Mikhailov [29] examined the Tirring model with a pencil-formed first order equation system.

Let $y(x, \lambda)$ and $z(x, \lambda)$ be vector solutions of the equation system (1). The expression

$$W[y(x, \lambda), z(x, \lambda)] = \widetilde{y}(x, \lambda) Bz(x, \lambda) = y_1 z_2 - y_2 z_1$$

is called the Wronskian of the vector functions $y(x, \lambda)$ and $z(x, \lambda)$.

It is clear that for all λ in the intervals $(-\infty, -m)$ and (m, ∞), $f(x, \lambda)$ and $\overline{f(x, \lambda)}$ constitute a fundamental solutions of equation system (1). The Wronskian of these functions doesn't depend on x and

$$W\left[f(x, \lambda), \overline{f(x, \lambda)}\right] = f_1(x, \lambda)\overline{f_2(x, \lambda)} - f_2(x, \lambda)\overline{f_1(x, \lambda)} = 2i\frac{\lambda + m}{k}.$$

2. The scattering data

Lemma 1. *For all λ in the intervals $(-\infty, -m)$ and (m, ∞), the following identity is valid:*

$$2i\frac{\lambda + m}{k}\frac{\varphi(x, \lambda)}{E(\lambda)} = \overline{f(x, \lambda)} - S(\lambda) f(x, \lambda) \tag{10}$$

where

$$S(\lambda) = \frac{\overline{E(\lambda)}}{E(\lambda)} \tag{11}$$

and

$$S^{-1}(\lambda) = \overline{S(\lambda)}, \qquad |S(\lambda)| = 1.$$

Proof. Since $f(x, \lambda)$ and $\overline{f(x, \lambda)}$ constitute a fundamental solution system for real λ, $|\lambda| > m$, we write

$$\varphi(x, \lambda) = c_1(\lambda) f(x, \lambda) + c_2(\lambda) \overline{f(x, \lambda)}, \tag{12}$$

where $c_1(\lambda)$ and $c_2(\lambda)$ are functions which we have to find. It is easily obtained that

$$W[f(x, \lambda), \varphi(x, \lambda)] = E(\lambda),$$
$$W\left[\overline{f(x, \lambda)}, \varphi(x, \lambda)\right] = \overline{E(\lambda)}.$$

Taking account of the following equalities

$$W[f(x, \lambda), \varphi(x, \lambda)] = c_2(\lambda)\, 2i \frac{\lambda + m}{k},$$
$$W\left[\overline{f(x, \lambda)}, \varphi(x, \lambda)\right] = -c_1(\lambda)\, 2i \frac{\lambda + m}{k},$$

$c_1(\lambda)$ and $c_2(\lambda)$ are found and substituted in (12). Thus, we get

$$\varphi(x, \lambda) = -\frac{k}{2i(\lambda + m)} \overline{E(\lambda)} f(x, \lambda) + \frac{k}{2i(\lambda + m)} E(\lambda) \overline{f(x, \lambda)}. \tag{13}$$

Now, it is necessary to show $E(\lambda) \neq 0$ for all real $|\lambda| > m$. Assume the contrary, then there exists λ_0, $|\lambda_0| > m$, such that

$$E(\lambda_0) = 0$$

or

$$\left(\alpha_0 + \alpha_1 \lambda_0 + \alpha_2 \lambda_0^2\right) f_1(0, \lambda_0) = \left(\beta_0 + \beta_1 \lambda_0 + \beta_2 \lambda_0^2\right) f_2(0, \lambda_0).$$

Also,

$$W\left[f(0, \lambda_0), \overline{f(0, \lambda_0)}\right] = 2i \frac{\lambda_0 + m}{k}$$

is satisfied. Using these relations, we obtain

$$0 = 2i \frac{\lambda_0 + m}{k}$$

and since $\lambda_0 \neq -m$ we reach a contradiction. From (13), we have (10), so $S(\lambda)$ is expressed by (11). From the expression of $S(\lambda)$ for real λ it follows that

$$S(\lambda) = \left[\overline{S(\lambda)}\right]^{-1},$$

or, which is the same,

$$|S(\lambda)| = 1.$$

Thus, the lemma is proved. □

The function $S(\lambda)$ defined by (11) is called the *scattering function* of the boundary value problem (1)–(3).

Let us examine properties of $E(\lambda)$ on the upper half-plane.

Lemma 2. *The function $E(\lambda)$ is analytic in the upper half-plane $\Im\lambda > 0$, continuous along the real axis except at $\lambda = m$ and has only a finite number of zeros in the interval $(-m, m)$. All the zeros in $(-m, m)$ are simple.*

Proof. Since the functions $f_1(0, \lambda)$ and $f_2(0, \lambda)$ are continuous for all real $\lambda \neq m$ and analytic in the upper plane ($\Im\lambda > 0$), then from the formula of $E(\lambda)$ it follows that the function $E(\lambda)$ has the same properties.

Let μ ($\Im\mu > 0$ or $\mu \in (-m, m)$) be one of the zeros of the function $E(\lambda)$. The function $f^*(x, \mu)$ denotes the transposed vector function $\overline{f(x, \mu)}$. When we go over to the conjugate transpose of the identity

$$Bf'(x, \mu) + mTf(x, \mu) + \Omega(x) f(x, \mu) = \mu f(x, \mu) \tag{14}$$

we obtain

$$-f^{*\prime}(x,\mu)B + mf^*(x,\mu)T + f^*(x,\mu)\Omega(x) = \overline{\mu}f^*(x,\mu). \tag{15}$$

Multiply (14) on the left by $f^*(x,\mu)$ and (15) on the right by $f(x,\mu)$ and subtract the second from the first and finally integrate this relation to x from 0 to ∞. As a result, we get

$$W\left[\overline{f(x,\mu)}, f(x,\mu)\right]_{x=0}^{\infty} + (\overline{\mu}-\mu)\int_0^\infty f^*(x,\mu)f(x,\mu)\,dx = 0.$$

On the other hand since μ is a zero of $E(\lambda)$ we have

$$f_1(0,\mu) = \frac{\beta_0+\beta_1\mu+\beta_2\mu^2}{\alpha_0+\alpha_1\mu+\alpha_2\mu^2}f_2(0,\mu).$$

Therefore,

$$W\left[\overline{f(0,\mu)}, f(0,\mu)\right] = \frac{\left[\alpha_1\beta_0-\alpha_0\beta_1+(\mu+\overline{\mu})(\alpha_2\beta_0-\alpha_0\beta_2)+(\alpha_2\beta_1-\alpha_1\beta_2)|\mu|^2\right]}{\left|\alpha_0+\alpha_1\mu+\alpha_2\mu^2\right|^2}(\mu-\overline{\mu})\,|f_2(0,\mu)|^2.$$

If these relations are substituted, we get

$$(\mu-\overline{\mu})\left\{\frac{\left[\alpha_1\beta_0-\alpha_0\beta_1+(\mu+\overline{\mu})(\alpha_2\beta_0-\alpha_0\beta_2)+(\alpha_2\beta_1-\alpha_1\beta_2)|\mu|\right]}{\left|\alpha_0+\alpha_1\mu+\alpha_2\mu^2\right|^2}|f_2(0,\mu)|^2 \right.$$
$$\left. + \int_0^\infty f^*(x,\mu)f(x,\mu)\,dx\right\} = 0.$$

Since $\alpha_0\beta_1-\alpha_1\beta_0<0$, $\alpha_0\beta_2-\alpha_2\beta_0=0$, $\alpha_1\beta_2-\alpha_2\beta_1<0$, the expression on the inside of the bracket is positive and it implies $\mu=\overline{\mu}$. This shows $\mu\in(-m,m)$.

Let us prove that there are only finitely many zeros. Let δ denote the infimum of the distances between two neighboring zeros of $E(\lambda)$, and show next that $\delta>0$. Let's assume the contrary and let $\{\lambda_k\}$ and $\left\{\widetilde{\lambda_k}\right\}$ be two sequences of zeros of the function $E(\lambda)$ such that

$$\lim_{k\to\infty}\left(\widetilde{\lambda_k}-\lambda_k\right)=0, \quad -m\le\lambda_k<\widetilde{\lambda_k}<0.$$

For A large enough, the inequalities

$$if_1(x,\lambda) > \frac{1}{2}\frac{\lambda+m}{\sqrt{m^2-\lambda^2}}e^{-\sqrt{m^2-\lambda^2}x}, \qquad if_2(x,\lambda) > \frac{1}{2}e^{-\sqrt{m^2-\lambda^2}x}$$

hold uniformly with respect to $x\in[A,\infty)$ and $\lambda\in(-m,m)$. Thus, we obtain

$$\int_A^\infty f^*(x,\lambda_k)f\left(x,\widetilde{\lambda_k}\right)dx > \frac{1}{4}\frac{1}{\sqrt{m^2-\lambda_k^2}+\sqrt{m^2-\widetilde{\lambda_k}^2}}e^{-\left(\sqrt{m^2-\lambda_k^2}+\sqrt{m^2-\widetilde{\lambda_k}^2}\right)A}.$$

Since $\lambda_k,\widetilde{\lambda_k}\to -m$ when $k\to\infty$ we find

$$\lim_{k\to\infty}\int_A^\infty f^*(x,\lambda_k)f\left(x,\widetilde{\lambda_k}\right)dx = +\infty. \tag{16}$$

On the other hand, it follows that

$$0 = \int_0^\infty f^*(x,\lambda_k)f\left(x,\widetilde{\lambda_k}\right)dx + \frac{\alpha_1\beta_0-\alpha_0\beta_1+(\alpha_2\beta_1-\alpha_1\beta_2)\lambda_k\widetilde{\lambda_k}}{\left(\alpha_0+\alpha_1\lambda_k+\alpha_2\lambda_k^2\right)\left(\alpha_0+\alpha_1\widetilde{\lambda_k}+\alpha_2\widetilde{\lambda_k}^2\right)}$$
$$= \int_0^A f^*(x,\lambda_k)\left[f\left(x,\widetilde{\lambda_k}\right)-f(x,\lambda_k)\right]dx + \int_0^A f^*(x,\lambda_k)f(x,\lambda_k)\,dx$$
$$+ \int_A^\infty f^*(x,\lambda_k)f\left(x,\widetilde{\lambda_k}\right)dx + \frac{\alpha_1\beta_0-\alpha_0\beta_1+(\alpha_2\beta_1-\alpha_1\beta_2)\lambda_k\widetilde{\lambda_k}}{\left(\alpha_0+\alpha_1\lambda_k+\alpha_2\lambda_k^2\right)\left(\alpha_0+\alpha_1\widetilde{\lambda_k}+\alpha_2\widetilde{\lambda_k}^2\right)}$$

and letting $k\to\infty$, we get

$$\lim_{k\to\infty}\int_A^\infty f^*(x,\lambda_k)f\left(x,\widetilde{\lambda_k}\right)dx \le A. \tag{17}$$

Comparing (16) and (17) we reach a contradiction. We conclude that $\delta > 0$ and so the function $E(\lambda)$ has only a finite number of zeros.

Finally, we show that all zeros of the function $E(\lambda)$ in $(-m, m)$ are simple. By the derivation of the identity

$$Bf'(x,\lambda) + mTf(x,\lambda) + \Omega(x)f(x,\lambda) = \lambda f(x,\lambda) \tag{18}$$

with respect to λ, assuming that λ lies in $(-m, m)$, and go over to the conjugate transpose, we have

$$-\left[\dot{f}^*(x,\lambda)\right]' B + m\dot{f}^*(x,\lambda)\,T + \dot{f}^*(x,\lambda)\,\Omega(x) = \lambda\dot{f}^*(x,\lambda) + f^*(x,\lambda). \tag{19}$$

(Here $\dot{f}$ denotes differentiation with respect to λ.) Multiplying (18) on the left by $\dot{f}^*(x,\lambda)$ and (19) on the right by $f(x,\lambda)$ and subtracting the second from the first, we obtain

$$\dot{f}^*(x,\lambda)\,Bf'(x,\lambda) + \left[\dot{f}^*(x,\lambda)\right]' Bf(x,\lambda) = -f^*(x,\lambda)\,f(x,\lambda).$$

Integrating this relation with respect to x over $(0,\infty)$ it follows that

$$\dot{f}^*(0,\lambda)\,Bf(0,\lambda) = \int_0^\infty f^*(x,\lambda)\,f(x,\lambda)\,dx.$$

Let λ_j be a zero of the function $E(\lambda)$. Using the expression of the function $E(\lambda)$, we obtain

$$\dot{f}^*\left(0,\lambda_j\right)Bf\left(0,\lambda_j\right) = \frac{-E(\dot{\lambda}_j)f_2\left(0,\lambda_j\right)}{\left(\alpha_0+\alpha_1\lambda_j+\alpha_2\lambda_j^2\right)} + \frac{f_2^2\left(0,\lambda_j\right)}{\left(\alpha_0+\alpha_1\lambda_j+\alpha_2\lambda_j^2\right)^2}\left[\alpha_1\beta_0-\alpha_0\beta_1++\left(\alpha_2\beta_1-\alpha_1\beta_2\right)\lambda_j^2\right]$$

and so

$$\frac{-E(\dot{\lambda}_j)f_2\left(0,\lambda_j\right)}{\left(\alpha_0+\alpha_1\lambda_j+\alpha_2\lambda_j^2\right)} = -\frac{f_2^2\left(0,\lambda_j\right)}{\left(\alpha_0+\alpha_1\lambda_j+\alpha_2\lambda_j^2\right)^2}\left[\alpha_1\beta_0-\alpha_0\beta_1+\left(\alpha_2\beta_1-\alpha_1\beta_2\right)\lambda_j^2\right] + \int_0^\infty f^*\left(x,\lambda_j\right)f\left(x,\lambda_j\right)dx. \tag{20}$$

Since $\alpha_1\beta_0-\alpha_0\beta_1 > 0$, $\alpha_2\beta_1-\alpha_1\beta_2 > 0$ and $f_2\left(0,\lambda_j\right)$ is pure imaginary, the right side is positive and so $E(\dot{\lambda}_j) \neq 0$. This shows that the zeros of $E(\lambda)$ are simple. The lemma is proved. □

The numbers

$$m_j^{-2} = -\frac{f_2^2\left(0,\lambda_j\right)}{\left(\alpha_0+\alpha_1\lambda_j+\alpha_2\lambda_j^2\right)^2}\left[\alpha_1\beta_0-\alpha_0\beta_1+\left(\alpha_2\beta_1-\alpha_1\beta_2\right)\lambda_j^2\right] + \int_0^\infty f^*\left(x,\lambda_j\right)f\left(x,\lambda_j\right)dx$$

is called the *norming numbers* of boundary value problem (1)–(3).

3. The main equation

The inverse scattering problem consists in recovering the coefficient $\Omega(x)$ from the scattering data of the boundary value problem (1)–(3). The main equation has an important role in the solving of the inverse problem. Let us derive the main equation by using the results in Lemmas 1 and 2. By considering (10) we have

$$\begin{aligned}
&2i\frac{\lambda+m}{k}\frac{\varphi(x,\lambda)}{E(\lambda)} - \begin{pmatrix}\frac{\lambda+m}{k}\\ i\end{pmatrix}e^{-ikx} + S_0(\lambda)\begin{pmatrix}\frac{\lambda+m}{k}\\ -i\end{pmatrix}e^{ikx}\\
&= \int_x^\infty A(x,t)\begin{pmatrix}\frac{\lambda+m}{k}\\ i\end{pmatrix}e^{-ikt}dt - S_0(\lambda)\int_x^\infty A(x,t)\begin{pmatrix}\frac{\lambda+m}{k}\\ -i\end{pmatrix}e^{ikt}dt\\
&\quad + \left(S_0(\lambda)-S(\lambda)\right)\begin{pmatrix}\frac{\lambda+m}{k}\\ -i\end{pmatrix}e^{ikx} + \left(S_0(\lambda)-S(\lambda)\right)\int_x^\infty A(x,t)\begin{pmatrix}\frac{\lambda+m}{k}\\ -i\end{pmatrix}e^{ikt}dt.
\end{aligned}$$

Multiplying this equality by $\frac{1}{2\pi}\frac{k}{\lambda+m}\left(\frac{\lambda+m}{k}, -i\right)e^{iky}$ and integrating with respect to λ over $(-\infty,-m)$ and (m,∞), we get

$$\begin{aligned}
&\Re\frac{1}{2\pi}\int_{|\lambda|>m} 2i\frac{\varphi(x,\lambda)}{E(\lambda)}\left(\frac{\lambda+m}{k}, -i\right)e^{iky}d\lambda - \Re\frac{1}{2\pi}\int_{|\lambda|>m}\begin{pmatrix}\frac{\lambda+m}{k}\\ i\end{pmatrix}e^{-ikx}\frac{k}{\lambda+m}\left(\frac{\lambda+m}{k}, -i\right)e^{iky}d\lambda\\
&\quad + \Re\frac{1}{2\pi}\int_{|\lambda|>m} S_0(\lambda)\begin{pmatrix}\frac{\lambda+m}{k}\\ -i\end{pmatrix}e^{ikx}\frac{k}{\lambda+m}\left(\frac{\lambda+m}{k}, -i\right)e^{iky}d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \Re \frac{1}{2\pi} \int_{|\lambda|>m} \int_x^\infty A(x,t) \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ i & \dfrac{k}{\lambda+m} \end{pmatrix} e^{-ik(t-y)} dt d\lambda \\
&\quad - \Re \frac{1}{2\pi} \int_{|\lambda|>m} S_0(\lambda) \int_x^\infty A(x,t) \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ -i & -\dfrac{k}{\lambda+m} \end{pmatrix} e^{ik(t+y)} dt d\lambda \\
&\quad + \Re \frac{1}{2\pi} \int_{|\lambda|>m} (S_0(\lambda) - S(\lambda)) \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ -i & -\dfrac{k}{\lambda+m} \end{pmatrix} e^{ik(x+y)} d\lambda \\
&\quad + \Re \frac{1}{2\pi} \int_{|\lambda|>m} (S_0(\lambda) - S(\lambda)) \int_x^\infty A(x,t) \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ -i & -\dfrac{k}{\lambda+m} \end{pmatrix} e^{ik(t+y)} dt d\lambda.
\end{aligned}
\tag{21}
$$

It is easily shown that

$$
\Re \frac{1}{2\pi} \int_{|\lambda|>m} \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ i & \dfrac{k}{\lambda+m} \end{pmatrix} e^{-ik(t-y)} d\lambda = \delta(t-y) I_2,
$$

where

$$
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

and $\delta(x)$ is the Dirac delta function. Thus,

$$
\begin{aligned}
&\Re \frac{1}{2\pi} \int_{|\lambda|>m} \int_x^\infty A(x,t) \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ i & \dfrac{k}{\lambda+m} \end{pmatrix} e^{-ik(t-y)} dt d\lambda \\
&= \int_x^\infty A(x,t) \Re \frac{1}{2\pi} \int_{|\lambda|>m} \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ i & \dfrac{k}{\lambda+m} \end{pmatrix} e^{-ik(t-y)} d\lambda dt \\
&= \int_x^\infty A(x,t) \delta(t-y) \, dt = A(x,y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Re \frac{1}{2\pi} \int_{|\lambda|>m} S_0(\lambda) \int_x^\infty A(x,t) \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ -i & -\dfrac{k}{\lambda+m} \end{pmatrix} e^{ik(t+y)} dt d\lambda \\
&= \int_x^\infty A(x,t) \Re \frac{1}{2\pi} \int_{|\lambda|>m} \frac{\alpha_2 - i\beta_2}{\alpha_2 + i\beta_2} \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ -i & -\dfrac{k}{\lambda+m} \end{pmatrix} e^{ik(t+y)} d\lambda dt \\
&= \frac{\alpha_2^2 - \beta_2^2}{\alpha_2^2 + \beta_2^2} \int_x^\infty A(x,t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta(-t-y) \, dt = 0.
\end{aligned}
$$

Hence, on the right of (21) we get

$$
A(x,y) + F_s(x+y) + \int_x^\infty A(x,t) F_s(t+y) \, dt, \quad y > x,
$$

where

$$F_s(x) = \Re \frac{1}{2\pi} \int_{|\lambda|>m} (S_0(\lambda) - S(\lambda)) \begin{pmatrix} \dfrac{\lambda+m}{k} & -i \\ -i & -\dfrac{k}{\lambda+m} \end{pmatrix} e^{ikx} d\lambda.$$

On the other hand, using the residue theorem and the formula (20) we get

$$\begin{aligned} \Re \frac{1}{2\pi} \int_{|\lambda|>m} 2i \frac{\varphi(x,\lambda)}{E(\lambda)} \left(\frac{\lambda+m}{k}, -i \right) e^{iky} d\lambda &= -\sum_{j=1}^{n} \operatorname{Res} \left(\frac{\varphi(x,\lambda)}{E(\lambda)} \left(\frac{\lambda+m}{k}, -i \right) e^{iky} \right) \Big|_{\lambda=\lambda_j} \\ &= -\sum_{j=1}^{n} \frac{\varphi(x,\lambda_j)}{\dot{E}(\lambda_j)} \left(\frac{\lambda_j+m}{k}, -i \right) e^{iky} \\ &= -\sum_{j=1}^{n} \frac{(\alpha_0+\alpha_1\lambda_j+\alpha_2\lambda_j^2) f(x,\lambda_j)}{\dot{E}(\lambda_j) f_2(0,\lambda_j)} \left(\frac{\lambda_j+m}{k}, -i \right) e^{iky} \\ &= \sum_{j=1}^{n} m_j^2 f(x,\lambda_j) \left(\frac{\lambda_j+m}{k}, -i \right) e^{iky} \\ &= \sum_{j=1}^{n} m_j^2 \left[\begin{pmatrix} \dfrac{\lambda_j+m}{k} \\ -i \end{pmatrix} e^{ikx} + \int_x^{\infty} A(x,t) \begin{pmatrix} \dfrac{\lambda_j+m}{k} \\ -i \end{pmatrix} e^{ikt} dt \right] \\ &\quad \times \left(\frac{\lambda_j+m}{k}, i \right) e^{-iky}. \end{aligned}$$

Substituting this value on the left side of (21), we have the equation

$$A(x,y) + F(x+y) + \int_x^{\infty} A(x,t) F(t+y)\, dt = 0, \quad y > x \tag{22}$$

where

$$F(x+y) = F_s(x+y) - \sum_{j=1}^{n} m_j^2 f_0(x,\lambda_j) \widetilde{f_0}(y,\lambda_j). \tag{23}$$

Thus, we have proved the following theorem.

Theorem 1. *For every $x \geq 0$ the kernel $A(x,t)$ of the solution (5) satisfies the integral equation (22).*

The integral equation (22) is called the *main equation* of the inverse scattering problem for the boundary value problem (1)–(3).

4. Solvability of the main equation

In the main equation (22) we can take kernel $A(x,t)$ as unknown and regard it as an equation of Fredholm type in the space $L_2(x,\infty)$ for every fixed x.

Theorem 2. *For every fixed x the main equation (22) has a unique matrix solution with elements in $L_2(x,\infty)$.*

Proof. Suppose that the collection $\{S(\lambda), (-\infty < \lambda < +\infty); \lambda_k; m_k\ (k = \overline{1,n})\}$ is the scattering data of the boundary value problem (1)–(3). The matrix function $F(x)$ is written by the formula (23). The transition function $F(x)$ possesses similar properties to the transition function of the problem without the spectral parameter in the boundary conditions. Thus, applying Theorem 2.3.1 in [1] it is obtained that Eq. (22) has a unique solution $A(x,y)$. This proves the theorem. □

Corollary 3. *The scattering data of the boundary value problem (1)–(3) uniquely determines potential $\Omega(x)$ in Eq. (1).*

Proof. The main equation (22) is constructed only on the basis of the scattering data, and by Theorem 2, it has a unique solution $A(x,y)$ for every $x \geq 0$. Solving the main equation, we find the matrix kernel $A(x,y)$ of transformation operator (5), and the potential matrix

$$\Omega(x) = BA(x,x) - A(x,x)B. \quad \square$$

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