Mathematical and Computer Modelling 53 (2011) 189–195

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Statistical A-summability of positive linear operators

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ARTICLE INFO

Article history:
Received 21 April 2010
Received in revised form 4 August 2010
Accepted 4 August 2010

Keywords:
Statistical convergence
Statistical A-summability
Positive linear operator
Korovkin-type approximation theorem

ABSTRACT

In this paper, using the concept of statistical A-summability which is stronger than the A-statistical convergence we provide a Korovkin-type approximation theorem. We also compute the rates of statistical A-summability of a sequence of positive linear operators.
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1. Introduction

The idea of statistical convergence was introduced by Fast [1], which is closely related to the concept of natural density or asymptotic density of subsets of the set of natural numbers $\mathbb{N}$. Let K be a subset of $\mathbb{N}$. The natural density of K is the nonnegative real number given by $\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|$ provided the limit exists, where $|B|$ denotes the cardinality of the set B (see [2] for details). Then, a sequence $x = \{x_k\}$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}) = 0.$$

This is denoted by $st - \lim_{k\to\infty} x_k = L$ (see [1,3]). It is easy to see that every convergent sequence is statistically convergent but not conversely.

If $x = \{x_k\}$ is a number sequence and $A = \{a_{jk}\}$ is an infinite matrix, then Ax is the sequence whose jth term is given by

$$A_j(x) := \sum_{k=1}^{\infty} a_{jk} x_k$$

provided the series converges for each $j \in \mathbb{N}$. Thus we say that x is A-summable to L if

$$\lim_{j\to\infty} A_j(x) = L.$$

We say that A is regular if $\lim_{j\to\infty} A_j(x) = L$ whenever $\lim_{k\to\infty} x_k = L$. The well-known necessary and sufficient conditions [4] (Silverman–Toeplitz) for A to be regular are

(R1) $\|A\| = \sup_j \sum_{k=1}^{\infty} |a_{jk}| < \infty$,
(R2) $\lim_{j\to\infty} a_{jk} = 0$ for each $k \in \mathbb{N}$,
(R3) $\lim_{j\to\infty} \sum_{k=1}^{\infty} a_{jk} = 1$.

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doi:10.1016/j.mcm.2010.08.003

Freedman and Sember [5] introduced the following extension of statistical convergence. Let $A = \{a_{jk}\}$ be a nonnegative regular matrix. The A-density of K is defined by

$$\delta_A(K) := \lim_{j\to\infty} \sum_{k=1}^{\infty} a_{jk}\chi_K(k)$$

provided the limit exists, where χ_K is the characteristic function of K. Then the sequence $x = \{x_k\}$ is said to be A-statistically convergent to the number L if, for every $\varepsilon > 0$,

$$\delta_A(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

or equivalently

$$\lim_{j\to\infty} \sum_{k:|x_k - L|\geq\varepsilon} a_{jk} = 0.$$

We denote this limit by $st_A - \lim_{k\to\infty} x_k = L$ (see [5–7]). The case in which $A = C_1$, the Cesàro matrix of order one, reduces to statistical convergence, and also if $A = I$, the identity matrix, then it coincides with the ordinary convergence.

Recently, the idea of statistical $(C, 1)$-summability was introduced in [8], statistical $(H, 1)$-summability in [9] by Moricz, and statistical $(\overline{N}, p)$-summability by Moricz and Orhan [10]. Then these statistical summability methods were generalized by defining the statistical A-summability in [11].

Now we recall this convergence method.

Definition 1.1. Let $A = \{a_{jk}\}$ be a nonnegative regular matrix and $x = \{x_k\}$ be a sequence. We say that x is statistically A-summable to L if for every $\varepsilon > 0$,

$$\delta\left(\left\{j \in \mathbb{N} : \left|A_j(x) - L\right| \geq \varepsilon\right\}\right) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{j \leq n : \left|A_j(x) - L\right| \geq \varepsilon\right\}\right| = 0.$$

Thus $x = \{x_k\}$ is statistically A-summable to L if and only if Ax is statistically convergent to L. In this case we write $(A)_{st} - \lim_{k\to\infty} x_k = L$ or, $st - \lim_{j\to\infty} A_j(x) = L$.

Using Definition 1.1, we see that, if a sequence is bounded and A-statistically convergent to L then it is A-summable to L and hence statistically A-summable to L. However, its converse is not always true. Such examples were given in [11].

We note that the use of statistical convergence in approximation theory has been shown to be effective than the classical methods (see, for instance [12–16]). On the other hand, most of the classical approximation operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. In fact, there are some sharp exceptions such as the interpolation operator of Hermite–Fejér (see [17]). These operators do not converge at points of simple discontinuity. For such a misbehavior, the matrix summability methods of Cesàro type are strong enough to correct the lack of convergence (see [18]).

In the present paper, we mainly use the statistical A-summability method in the approximation theory settings. We also compute the rates of statistical A-summability of a sequence of positive linear operators.

2. A Korovkin-type approximation theorem

We denote by $C_M[a, b]$ the space of all continuous functions on $[a, b]$ and bounded on the entire line, that is

$$|f(x)| \leq K_f, \quad -\infty < x < \infty,$$

where K_f is a constant depending on f. Also by $B[a, b]$ we denote the space of all bounded functions on $[a, b]$. This space is equipped with the supremum norm

$$\|f\|_{B[a,b]} = \sup_{x\in[a,b]} |f(x)|, \quad (f \in B[a, b]).$$

Let L be a linear operator from $C_M[a, b]$ into $B[a, b]$. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in [a, b]$ by $L(f(u); x)$ or, briefly, $L(f; x)$.

Throughout the paper, we also use the following test functions

$$e_0(x) = 1, \qquad e_1(x) = x, \qquad e_2(x) = x^2.$$

First we recall the classical and statistical cases of the Korovkin-type results introduced in [19,20], respectively.

Theorem 2.1. *Let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a, b]$ into $B[a, b]$. Then, for all $f \in C_M[a, b]$,*

$$\lim_{k\to\infty} \|L_k(f) - f\|_{B[a,b]} = 0$$

if and only if

$$\lim_{k\to\infty} \|L_k(e_i) - e_i\|_{B[a,b]} = 0, \quad (i = 0, 1, 2).$$

Theorem 2.2. *Let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a,b]$ into $B[a,b]$. Then, for all $f \in C_M[a,b]$,*

$$st - \lim_{k\to\infty} \|L_k(f) - f\|_{B[a,b]} = 0$$

if and only if

$$st - \lim_{k\to\infty} \|L_k(e_i) - e_i\|_{B[a,b]} = 0, \quad (i = 0, 1, 2).$$

Now we have the following main result.

Theorem 2.3. *Let $A = \{a_{jk}\}$ be a nonnegative regular matrix and let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a,b]$ into $B[a,b]$. Then, for all $f \in C_M[a,b]$,*

$$st - \lim_{j\to\infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{B[a,b]} = 0 \tag{2.1}$$

if and only if

$$st - \lim_{j\to\infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(e_i) - (e_i) \right\|_{B[a,b]} = 0, \quad (i = 0, 1, 2). \tag{2.2}$$

Proof. Since each $e_i(x) \in C_M[a,b]$, $(i = 0, 1, 2)$, the implication (2.1) $\Longrightarrow$ (2.2) is obvious. Now, to prove the implication (2.2) $\Longrightarrow$ (2.1) assume that (2.2) holds. Let $f \in C_M[a,b]$ and $x \in [a,b]$ be fixed. Since f is bounded on the entire line, we can write

$$|f(x)| \le H, \quad -\infty < x < \infty. \tag{2.3}$$

Also, since f is continuous on $[a,b]$, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ holds for all $y \in [a,b]$ satisfying $|y - x| < \delta$. Hence, we get

$$|f(y) - f(x)| \le \varepsilon + \frac{2H}{\delta^2}(y-x)^2 \tag{2.4}$$

for all y, x. Using the linearity and the positivity of the operators $\{L_k\}$ and (2.4), we get, for any $j \in \mathbb{N}$ that,

$$\begin{aligned}
\left| \sum_{k=1}^{\infty} a_{jk} L_k(f;x) - f(x) \right| &\le \sum_{k=1}^{\infty} a_{jk} L_k(|f(y) - f(x)|;x) + |f(x)| \left| \sum_{k=1}^{\infty} a_{jk} L_k(e_0;x) - e_0(x) \right| \\
&\le \sum_{k=1}^{\infty} a_{jk} L_k\left(\varepsilon + \frac{2H}{\delta^2}(y-x)^2;x\right) + |f(x)| \left| \sum_{k=1}^{\infty} a_{jk} L_k(e_0;x) - e_0(x) \right| \\
&\le \varepsilon + (\varepsilon + H) \left| \sum_{k=1}^{\infty} a_{jk} L_k(e_0;x) - e_0(x) \right| + \frac{2H}{\delta^2} \sum_{k=1}^{\infty} a_{jk} L_k\left((y-x)^2;x\right) \\
&\le \varepsilon + \left(\varepsilon + H + \frac{2Hc^2}{\delta^2}\right) \left| \sum_{k=1}^{\infty} a_{jk} L_k(e_0;x) - e_0(x) \right| \\
&\quad + \frac{4Hc}{\delta^2} \left| \sum_{k=1}^{\infty} a_{jk} L_k(e_1;x) - e_1(x) \right| + \frac{2H}{\delta^2} \left| \sum_{k=1}^{\infty} a_{jk} L_k(e_2;x) - e_2(x) \right|
\end{aligned}$$

where $c := \max_{x\in[a,b]} |x|$. Then taking the supremum over $x \in [a,b]$, we have

$$\begin{aligned}
\left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{B[a,b]} &\le \varepsilon + B \left\{ \left\| \sum_{k=1}^{\infty} a_{jk} L_k(e_0) - e_0 \right\|_{B[a,b]} \right. \\
&\quad \left. + \left\| \sum_{k=1}^{\infty} a_{jk} L_k(e_1) - e_1 \right\|_{B[a,b]} + \left\| \sum_{k=1}^{\infty} a_{jk} L_k(e_2) - e_2 \right\|_{B[a,b]} \right\}
\end{aligned} \tag{2.5}$$

where $B := \max\left\{\varepsilon + H + \frac{2Hc^2}{\delta^2}, \frac{4Hc}{\delta^2}, \frac{2H}{\delta^2}\right\}$. Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Then, it follows from (2.5) that

$$\frac{\left|\left\{j \le n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{B[a,b]} \ge r\right\}\right|}{n} \le \frac{\left|\left\{j \le n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(e_0) - e_0\right\|_{B[a,b]} \ge \frac{r-\varepsilon}{3B}\right\}\right|}{n}$$
$$+ \frac{\left|\left\{j \le n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(e_1) - e_1\right\|_{B[a,b]} \ge \frac{r-\varepsilon}{3B}\right\}\right|}{n} + \frac{\left|\left\{j \le n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(e_2) - e_2\right\|_{B[a,b]} \ge \frac{r-\varepsilon}{3B}\right\}\right|}{n}.$$

Then using the hypothesis (2.2), we get

$$\lim_{n\to\infty} \frac{\left|\left\{j \le n : \left\|\sum_{k=1}^{\infty} a_{jk} L_k(f) - f\right\|_{B[a,b]} \ge r\right\}\right|}{n} = 0$$

for every $r > 0$. The proof is complete. $\square$

Remark 2.1. Let A be the Cesàro matrix, i.e.,

$$a_{jk} = \begin{cases} \frac{1}{j}, & 1 \le k \le j, \\ 0, & \text{otherwise}, \end{cases}$$

and let

$$\alpha_k = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ -1, & \text{if } k \text{ is even}. \end{cases} \tag{2.6}$$

Observe now that, $A = \{a_{jk}\}$ is a nonnegative regular matrix and for the sequence $\alpha := \{\alpha_k\}$

$$st - \lim_{j\to\infty} A_j(\alpha) = 0.$$

However, the sequence $\{\alpha_k\}$ does not converge in the usual sense. Then, consider the following Bernstein–Kantorovich operators:

$$U_k(f; x) = (k+1)\sum_{n=0}^{k} \binom{k}{n} x^n (1-x)^{k-n} \int_{(n+1)/(k+1)}^{n/(k+1)} f(t)\,dt$$

where $x \in [0, 1]$, $f \in C_M[0, 1]$ and $k \in \mathbb{N}$. Using these polynomials, we introduce the following positive linear operators on $C_M[0, 1]$:

$$T_k(f; x) = (1 + \alpha_k) U_k(f; x), \quad x \in [0, 1], f \in C_M[0, 1] \tag{2.7}$$

where $\alpha = \{\alpha_k\}$ is the same as (2.6). Then observe that

$$T_k(e_0; x) = (1 + \alpha_k) e_0(x)$$
$$T_k(e_1; x) = (1 + \alpha_k)\left(e_1(x)\frac{k}{k+1} + \frac{1}{2(k+1)}\right)$$
$$T_k(e_2; x) = (1 + \alpha_k)\left[e_2(x)\frac{k(k-1)}{(k+1)^2} + \frac{2kx}{(k+1)^2} + \frac{1}{3(k+1)^2}\right].$$

Since $st - \lim_{j\to\infty} A_j(\alpha) = 0$, we conclude that

$$st - \lim_{j\to\infty} \left\|\sum_{k=1}^{\infty} a_{jk} T_k(e_i) - (e_i)\right\|_{B[0,1]} = 0, \quad \text{on } [0, 1] \text{ for each } i = 0, 1, 2.$$

So $\{T_k\}$ satisfy all hypotheses of Theorem 2.3 and we immediately see that

$$st - \lim_{j\to\infty} \left\|\sum_{k=1}^{\infty} a_{jk} T_k(f) - (f)\right\|_{B[0,1]} = 0, \quad \text{on } [0, 1] \text{ for all } f \in C_M[0, 1].$$

However, since $\{\alpha_k\}$ is not convergent and A-statistical convergent to 0, we can say that $\|T_k(e_i) - (e_i)\|_{B[0,1]}$ is not convergent and A-statistical convergent to zero for each $i = 0, 1, 2$. So the classical Korovkin theorem (Theorem 2.1) and the statistical Korovkin theorem (Theorem 2.2) do not work for our operators defined by (2.7).

3. Rate of convergence

In this section, using statistical A-summability we study the rate of convergence of positive linear operators with the help of the modulus of continuity.

Definition 3.1. Let $A = \{a_{jk}\}$ be a nonnegative regular matrix. A sequence $x = \{x_k\}$ is statistical A-summable to a number L at a rate of $\beta \in (0, 1)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\left|\left\{j \le n : \left|A_j(x) - L\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} = 0.$$

In this case, it is denoted by

$$x_k - L = o\left(n^{-\beta}\right) \quad ((A)_{st}).$$

Using this definition, we obtain the following auxiliary result.

Lemma 3.1. *Let $A = \{a_{jk}\}$ be a nonnegative regular matrix. Let $x = \{x_k\}$ and $y = \{y_k\}$ be bounded sequences. Assume that $x_k - L_1 = o\left(n^{-\beta_1}\right)$ $((A)_{st})$ and $y_k - L_2 = o\left(n^{-\beta_2}\right)$ $((A)_{st})$. Let $\beta := \min\{\beta_1, \beta_2\}$. Then we have*

(i) $(x_k - L_1) \mp (y_k - L_2) = o\left(n^{-\beta}\right)$ $((A)_{st})$
(ii) $\lambda (x_k - L_1) = o\left(n^{-\beta_1}\right)$ $((A)_{st})$*, for any real number* λ.

Proof. (i) Assume that $x_k - L_1 = o\left(n^{-\beta_1}\right)$ $((A)_{st})$ and $y_k - L_2 = o\left(n^{-\beta_2}\right)$ $((A)_{st})$. Then, since $\beta := \min\{\beta_1, \beta_2\}$, for $\varepsilon > 0$ observe that

$$\begin{aligned} \frac{\left|\left\{j \le n : \left|\left(A_j(x) - L_1\right) \mp \left(A_j(y) - L_2\right)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} &\le \frac{\left|\left\{j \le n : \left|A_j(x) - L_1\right| \ge \frac{\varepsilon}{2}\right\}\right| + \left|\left\{j \le n : \left|A_j(y) - L_2\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}} \\ &\le \frac{\left|\left\{j \le n : \left|A_j(x) - L_1\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_1}} + \frac{\left|\left\{j \le n : \left|A_j(y) - L_2\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_2}}. \end{aligned} \tag{3.1}$$

Now by taking the limit as $n \to \infty$ in (3.1) and using the hypothesis, we conclude that

$$\lim_{n\to\infty} \frac{\left|\left\{j \le n : \left|\left(A_j(x) - L_1\right) \mp \left(A_j(y) - L_2\right)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} = 0$$

which completes the proof of (i). Since the proof of (ii) is similar, we omit it. $\square$

Now we recall the concept of modulus of continuity. For $f \in C_M[a, b]$, the modulus of continuity of f, denoted by $\omega(f; \delta)$, is defined to be

$$\omega(f; \delta) := \sup\{|f(y) - f(x)| : y, x \in [a, b], |y - x| \le \delta\} \quad (\delta > 0).$$

It is also well known that, for any $\lambda > 0$ and for all $f \in C_M[a, b]$

$$\omega(f; \lambda\delta) \le (1 + [\lambda])\, \omega(f; \delta)$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.

Then we have the following result.

Theorem 3.2. *Let $A = \{a_{jk}\}$ be a nonnegative regular matrix and let $\{L_k\}$ be a sequence of positive linear operators acting from $C_M[a, b]$ into $B[a, b]$. Assume that the following conditions hold:*

(i) $\|L_k(e_0) - e_0\|_{B[a,b]} = o\left(n^{-\beta_1}\right)$ $((A)_{st})$ *on* $[a, b]$,
(ii) $\omega\left(f; \gamma_j\right) = o\left(n^{-\beta_2}\right)$ $((A)_{st})$ *on* $[a, b]$ *where* $\gamma_j := \sqrt{\left\|\sum_{k=1}^{\infty} a_{jk} L_k(\varphi)\right\|_{B[a,b]}}$ *with* $\varphi(y) = (y - x)^2$.

Then we have for all $f \in C_M[a, b]$

$$\|L_k(f) - f\|_{B[a,b]} = o\left(n^{-\beta}\right) \quad ((A)_{st}) \text{ on } [a, b]$$

where $\beta := \min\{\beta_1, \beta_2\}$.

Proof. Let $f \in C_M[a,b]$ and $x \in [a,b]$ be fixed. Using (2.3), the properties of ω, and the positivity and monotonicity of L_k, we get, for any $\delta > 0$ and $j \in \mathbb{N}$, that

$$\begin{aligned}\left|\sum_{k=1}^{\infty} a_{jk}L_k(f;x) - f(x)\right| &\leq \sum_{k=1}^{\infty} a_{jk}L_k(|f(y)-f(x)|;x) + |f(x)|\left|\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) - e_0(x)\right| \\
&\leq \sum_{k=1}^{\infty} a_{jk}L_k\left(\omega\left(f;\delta\frac{|y-x|}{\delta}\right);x\right) + |f(x)|\left|\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) - e_0(x)\right| \\
&\leq \sum_{k=1}^{\infty} a_{jk}L_k\left(\left(1+\left[\frac{|y-x|}{\delta}\right]\right);x\right)\omega(f;\delta) + H\left|\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) - e_0(x)\right| \\
&\leq \sum_{k=1}^{\infty} a_{jk}L_k\left(\left(1+\frac{(y-x)^2}{\delta^2}\right);x\right)\omega(f;\delta) + H\left|\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) - e_0(x)\right| \\
&\leq \left[\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) + \frac{1}{\delta^2}\sum_{k=1}^{\infty} a_{jk}L_k(\varphi;x)\right]\omega(f;\delta) + H\left|\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) - e_0(x)\right| \\
&\leq \left|\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) - e_0(x)\right|\omega(f;\delta) + \omega(f;\delta) + H\left|\sum_{k=1}^{\infty} a_{jk}L_k(e_0;x) - e_0(x)\right| + \frac{\omega(f;\delta)}{\delta^2}\sum_{k=1}^{\infty} a_{jk}L_k(\varphi;x).\end{aligned}$$

Then, taking the supremum over $x \in [a,b]$, we get

$$\left\|\sum_{k=1}^{\infty} a_{jk}L_k(f) - f\right\|_{B[a,b]} \leq \left\|\sum_{k=1}^{\infty} a_{jk}L_k(e_0) - e_0\right\|_{B[a,b]} \omega(f;\delta) + 2\omega(f;\delta) + H\left\|\sum_{k=1}^{\infty} a_{jk}L_k(e_0) - e_0\right\|_{B[a,b]} \tag{3.2}$$

where $\delta := \gamma_j := \sqrt{\left\|\sum_{k=1}^{\infty} a_{jk}L_k(\varphi)\right\|_{B[a,b]}}$. Hence, given $\varepsilon > 0$, it follows from (3.2) and Lemma 3.1 that

$$\begin{aligned}\frac{\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk}L_k(f) - f\right\|_{B[a,b]} \geq \varepsilon\right\}\right|}{n^{1-\beta}} &\leq \frac{\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk}L_k(e_0) - e_0\right\|_{B[a,b]} \geq \sqrt{\frac{\varepsilon}{3}}\right\}\right|}{n^{1-\beta_1}} \\
&\quad + \frac{\left|\left\{j \leq n : \omega(f;\gamma_j) \geq \sqrt{\frac{\varepsilon}{3}}\right\}\right|}{n^{1-\beta_2}} + \frac{\left|\left\{j \leq n : \omega(f;\gamma_j) \geq \frac{\varepsilon}{6}\right\}\right|}{n^{1-\beta_2}} + \frac{\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk}L_k(e_0) - e_0\right\|_{B[a,b]} \geq \frac{\varepsilon}{3H}\right\}\right|}{n^{1-\beta_1}}\end{aligned} \tag{3.3}$$

where $\beta := \min\{\beta_1, \beta_2\}$. Letting $n \to \infty$ in (3.3), we conclude from (i) and (ii) that

$$\sum_{n\to\infty} \lim \frac{\left|\left\{j \leq n : \left\|\sum_{k=1}^{\infty} a_{jk}L_k(f) - f\right\|_{B[a,b]} \geq \varepsilon\right\}\right|}{n^{1-\beta}} = 0$$

which means

$$\|L_k(f) - f\|_{B[a,b]} = o\left(n^{-\beta}\right) \quad ((A)_{st}) \text{ on } [a,b].$$

The proof is completed. □

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