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Equi-statistical σ-convergence of positive linear operators

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ABSTRACT

In this paper we introduce the notion of equi-statistical σ-convergence which is stronger than the uniform convergence (in the ordinary sense), statistical uniform convergence and statistical uniform σ-convergence. Then, we also give its use in the Korovkin-type approximation theory. We also compute the rate of equi-statistical σ-convergence of a sequence of positive linear operators.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence by Fast [1]. Let K be the subset of positive integers. Then the natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K_n|$$

if it exists, where $K_n = \{k \in K : k \leq n\}$ and the vertical bars denote the cardinality of the set K_n. A sequence $x = \{x_k\}$ of numbers is statistically convergent to L provided that, for every $\varepsilon > 0$, $\delta\{k : |x_k - L| \geq \varepsilon\} = 0$ holds. In this case we write $st - \lim x = L$.

Recently various kinds of statistical convergences which are stronger than the statistical convergence have been introduced by Mursaleen and Edely [2]. We first recall this convergence method.

Let σ be a one-to-one mapping from the set $\mathbb{N}$ into itself. A continuous linear functional φ defined on the space l_∞ of all bounded sequences is called an invariant mean (or σ-mean) [3] if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = \{x_k\}$ has $x_k \geq 0$ for all k,
(ii) $\varphi(e) = 1$, where $e = (1, 1, \ldots)$,
(iii) $\varphi(x) = \varphi((x_{\sigma(n)}))$ for all $x \in l_\infty$.

Thus, the σ-mean extends the limit functional on c of all convergent sequences in the sense that $\varphi(x) = \lim x$ for all $x \in c$ [4]. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal. It is known [5] that

$$V_\sigma = \left\{ x \in l_\infty : \lim_{p\to\infty} t_{pm}(x) = L \text{ uniformly in } m, L = \sigma - \lim x \right\}$$

where

$$t_{pm}(x) := \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^p(m)}}{p+1}.$$

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We say that a bounded sequence $x = \{x_k\}$ is σ-convergent if and only if $x \in V_\sigma$. Let

$$V_\sigma^s = \left\{ x \in l_\infty : st - \lim_{p\to\infty} t_{pm}(x) = L \text{ uniformly in } m, L = \sigma - \lim x \right\}.$$

A sequence $x = \{x_k\}$ is said to be statistically σ-convergent to L if and only if $x \in V_\sigma^s$ (see, for details, [2]). In this case we write $\delta(\sigma) - \lim x_k = L$. That is,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ p \le n : \left| t_{pm}(x) - L \right| \ge \varepsilon \right\} \right| = 0, \quad \text{uniformly in } m.$$

Note that if x is statistically convergent then x is statistically σ-convergent. However, its converse is not always true. Such an example was given in [2].

Let f and f_n belong to $C(X)$, which is the space of all continuous real valued functions on a compact subset X of the real numbers. The usual supremum norm on the spaces $C(X)$ is given by

$$\|f\|_{C(X)} := \sup_{x\in X} |f(x)| \quad \text{for } f \in C(X).$$

In [6] a kind of convergence for a sequence lying between pointwise and uniform statistical convergence is presented. We first recall these convergence methods.

Definition 1 ([6]). $\{f_n\}$ is said to be statistically pointwise convergent to f on X if $st - \lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$, i.e., for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{|\{k \le n : |f_k(x) - f(x)| \ge \varepsilon\}|}{n} = 0.$$

In this case, we denote this limit by $f_n \to f$ $(stat)$ on X.

Definition 2 ([6]). $\{f_n\}$ is said to be equi-statistically convergent to f on X if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \left(\sup_{x\in X} \frac{|\{k \le n : |f_k(x) - f(x)| \ge \varepsilon\}|}{n} \right) = 0.$$

In this case, we denote this limit by $f_n \to f$ (equi-stat) on X.

Definition 3 ([6]). $\{f_n\}$ is said to be statistically uniformly convergent to f on X if $st - \lim_{n\to\infty} \|f_n - f\|_{C(X)} = 0$, or

$$\lim_{n\to\infty} \frac{\left| \left\{ k \le n : \|f_k - f\|_{C(X)} \ge \varepsilon \right\} \right|}{n} = 0.$$

In this case, we denote this limit by $f_n \rightrightarrows f$ $(stat)$ on X.

Now we give the following definitions:

Definition 4. $\widetilde{f}(x) := \{f_n(x)\}$ is said to be statistically pointwise σ-convergent to f on X if

$$st - \lim_{p\to\infty} t_{pm}\left(\widetilde{f}(x)\right) = f(x) \quad \text{uniformly in } m$$

for each $x \in X$, i.e. for every $\varepsilon > 0$ and for each $x \in X$

$$\lim_{n\to\infty} \frac{\left| \left\{ p \le n : \left| t_{pm}\left(\widetilde{f}(x)\right) - f(x) \right| \ge \varepsilon \right\} \right|}{n} = 0 \quad \text{uniformly in } m$$

where

$$t_{pm}\left(\widetilde{f}(x)\right) := \frac{f_m(x) + f_{\sigma(m)}(x) + f_{\sigma^2(m)}(x) + \cdots + f_{\sigma^p(m)}(x)}{p+1}.$$

Then, this is denoted by $f_n \to f$ $(stat(\sigma))$ on X.

Definition 5. $\widetilde{f}(x) := \{f_n(x)\}$ is said to be equi-statistically σ-convergent to f on X if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \left(\sup_{x\in X} \frac{\left| \left\{ p \le n : \left| t_{pm}\left(\widetilde{f}(x)\right) - f(x) \right| \ge \varepsilon \right\} \right|}{n} \right) = 0 \quad \text{uniformly in } m.$$

This limit is denoted by $f_n \to f$ (equi-stat (σ)) on X.

Definition 6. $\widetilde{f}(x) := \{f_n(x)\}$ is said to be statistically uniformly σ-convergent to f on X if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\left|\left\{p \leq n : \left\| t_{pm}\left(\widetilde{f}(x)\right) - f(x) \right\|_{C(X)} \geq \varepsilon \right\}\right|}{n} = 0 \quad \text{uniformly in } m.$$

This limit is denoted by $f_n \rightrightarrows f$ $(stat(\sigma))$ on X.

Using the above definitions, the next result follows immediately.

Lemma 7. *$f_n \rightrightarrows f$ on X (in the ordinary sense) implies $f_n \rightrightarrows f$ $(stat)$ on X, which also implies $f_n \rightrightarrows f$ $(stat(\sigma))$ on X. Furthermore $f_n \rightrightarrows f$ $(stat(\sigma))$ on X implies $f_n \to f$ (equi-stat (σ)) on X and $f_n \to f$ (equi-stat (σ)) on X implies $f_n \to f$ $(stat(\sigma))$ on X. Also, $f_n \to f$ on X (in the ordinary sense) implies $f_n \to f$ $(stat)$ on X, which also implies $f_n \to f$ $(stat(\sigma))$ on X and $f_n \to f$ (equi-stat) on X implies $f_n \to f$ (equi-stat (σ)) on X.*

However one can construct an example which guarantees that the converses of Lemma 7 are not always true. Such an example is the following:

Example 8. Consider the case $\sigma(n) = n + 1$. Let $g(x) = 0$, $x \in [0, 1]$, and define continuous functions $g_k : [0, 1] \to \mathbb{R}$, $k \in \mathbb{N}$, by the formula

$$g_k(x) = \begin{cases} 2^{2(k+1)}\left(x - \dfrac{1}{2^k}\right)^2, & \text{if } x \in \left[\dfrac{1}{2^k}, \dfrac{1}{2^{k-1}} - \dfrac{1}{2^{k+1}}\right], \\ 2^{2(k+1)}\left(x - \dfrac{1}{2^{k-1}}\right)^2, & \text{if } x \in \left[\dfrac{1}{2^{k-1}} - \dfrac{1}{2^{k+1}}, \dfrac{1}{2^{k-1}}\right] \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Then observe that $g_k \to g = 0$ (equi-stat (σ)) on $[0, 1]$; however $\{g_k\}$ is not statistically uniformly σ-convergent to the function $g = 0$ on the interval $[0, 1]$. Also, $\{g_k\}$ is not uniformly convergent (in the ordinary sense) and statistically uniformly convergent to the function $g = 0$ on $[0, 1]$.

For a sequence $\{L_n\}$ of positive linear operators on $C(X)$, Korovkin [7] first introduced the sufficient conditions for the uniform convergence of $L_n(f)$ to a function f by using the test function e_i defined by $e_i(x) = x^i$ $(i = 0, 1, 2)$. Later many researchers investigated these conditions for various operators defined on different spaces. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results have been proved [8–14]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem.

In this paper, we obtain the Korovkin-type approximation theorem by means of the concept of equi-statistical σ-convergence in Definition 5.

2. A Korovkin-type approximation theorem

In this section, using equi-statistical σ-convergence, we prove the following Korovkin-type approximation theorem.

Theorem 9. *Let X be a compact subset of the real numbers and let $L := \{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into itself. Then for all $f \in C(X)$,*

$$L_n(f) \to f \text{ (equi-stat } (\sigma)) \quad \text{on } X \tag{2}$$

if and only if

$$L_n(e_i) \to e_i \text{ (equi-stat } (\sigma)) \quad \text{on } X \tag{3}$$

with $e_i(x) = x^i$, $i = 0, 1, 2$.

Proof. Since each $e_i \in C(X)$, $i = 0, 1, 2$, the implication (2) ⇒ (3) is obvious.

Now suppose that (3) holds. Let $f \in C(X)$ and $x \in X$ be fixed. By the continuity of the point x, we may write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in X$ satisfying $|y - x| < \delta$. Since

$$|f(y) - f(x)| = |f(y) - f(x)|\, \chi_{X_\delta}(y) + |f(y) - f(x)|\, \chi_{X \setminus X_\delta}(y)$$

where $X_\delta = [x - \delta, x + \delta] \cap X$ and χ_A denotes the characteristic function of the set A, then we have

$$|f(y) - f(x)| \leq \varepsilon + 2M \frac{(y - x)^2}{\delta^2} \tag{4}$$

for all $y \in X$, where $M := \|f\|_{C(X)}$. Using the positivity and monotonicity of L_n, we obtain from (4)

$$\begin{aligned}
\left|t_{pm}\left(L\left(f;x\right)\right)-f\left(x\right)\right| &= \left|\frac{L_m(f;x)+L_{\sigma(m)}(f;x)+\cdots+L_{\sigma^p(m)}(f;x)}{p+1}-f\left(x\right)\right| \\
&= \left|\frac{L_m(f;x)+L_{\sigma(m)}(f;x)+\cdots+L_{\sigma^p(m)}(f;x)}{p+1}-\frac{L_m(e_0;x)+L_{\sigma(m)}(e_0;x)+\cdots+L_{\sigma^p(m)}(e_0;x)}{p+1}f\left(x\right)\right. \\
&\quad \left.+\frac{L_m(e_0;x)+L_{\sigma(m)}(e_0;x)+\cdots+L_{\sigma^p(m)}(e_0;x)}{p+1}f\left(x\right)-f\left(x\right)\right| \\
&\le \frac{L_m\left(\left|f\left(y\right)-f\left(x\right)\right|;x\right)+L_{\sigma(m)}\left(\left|f\left(y\right)-f\left(x\right)\right|;x\right)+\cdots+L_{\sigma^p(m)}\left(\left|f\left(y\right)-f\left(x\right)\right|;x\right)}{p+1} \\
&\quad +\left|f\left(x\right)\right|\left|t_{pm}\left(L\left(e_0;x\right)\right)-e_0\left(x\right)\right| \\
&\le \varepsilon+\left(\varepsilon+M+\frac{2M}{\delta^2}\left\|e_2\right\|_{C(X)}\right)\left|t_{pm}\left(L\left(e_0;x\right)\right)-e_0\left(x\right)\right| \\
&\quad +\frac{4M}{\delta^2}\left\|e_1\right\|_{C(X)}\left|t_{pm}\left(L\left(e_1;x\right)\right)-e_1\left(x\right)\right|+\frac{2M}{\delta^2}\left|t_{pm}\left(L\left(e_2;x\right)\right)-e_2\left(x\right)\right|.
\end{aligned}$$

Then, we get

$$\left|t_{pm}\left(L\left(f;x\right)\right)-f\left(x\right)\right| \le \varepsilon+K\left\{\left|t_{pm}\left(L\left(e_0;x\right)\right)-e_0\left(x\right)\right|+\left|t_{pm}\left(L\left(e_1;x\right)\right)-e_1\left(x\right)\right|+\left|t_{pm}\left(L\left(e_2;x\right)\right)-e_2\left(x\right)\right|\right\} \tag{5}$$

where $K := \varepsilon+M+\frac{2M}{\delta^2}\left(\|e_2\|_{C(X)}+2\|e_1\|_{C(X)}+1\right)$. Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Then, it follows from (5) that

$$\begin{aligned}
\left|\left\{p \le n : \left|t_{pm}\left(L\left(f;x\right)\right)-f\left(x\right)\right| \ge r\right\}\right| &\le \left|\left\{p \le n : \left|t_{pm}\left(L\left(e_0;x\right)\right)-e_0\left(x\right)\right| \ge \frac{r-\varepsilon}{3K}\right\}\right| \\
&\quad +\left|\left\{p \le n : \left|t_{pm}\left(L\left(e_1;x\right)\right)-e_1\left(x\right)\right| \ge \frac{r-\varepsilon}{3K}\right\}\right| \\
&\quad +\left|\left\{p \le n : \left|t_{pm}\left(L\left(e_2;x\right)\right)-e_2\left(x\right)\right| \ge \frac{r-\varepsilon}{3K}\right\}\right|
\end{aligned}$$

which gives

$$\begin{aligned}
\sup_{x\in X}\frac{\left|\left\{p \le n : \left|t_{pm}\left(L\left(f;x\right)\right)-f\left(x\right)\right| \ge r\right\}\right|}{n} &\le \sup_{x\in X}\frac{\left|\left\{p \le n : \left|t_{pm}\left(L\left(e_0;x\right)\right)-e_0\left(x\right)\right| \ge \frac{r-\varepsilon}{3K}\right\}\right|}{n} \\
&\quad +\sup_{x\in X}\frac{\left|\left\{p \le n : \left|t_{pm}\left(L\left(e_1;x\right)\right)-e_1\left(x\right)\right| \ge \frac{r-\varepsilon}{3K}\right\}\right|}{n} \\
&\quad +\sup_{x\in X}\frac{\left|\left\{p \le n : \left|t_{pm}\left(L\left(e_2;x\right)\right)-e_2\left(x\right)\right| \ge \frac{r-\varepsilon}{3K}\right\}\right|}{n}.
\end{aligned} \tag{6}$$

Then using the hypothesis (3) and considering Definition 5 in (6),

$$\lim_{n\to\infty}\left(\sup_{x\in X}\frac{\left|\left\{p \le n : \left|t_{pm}\left(L\left(f;x\right)\right)-f\left(x\right)\right| \ge r\right\}\right|}{n}\right)=0 \quad \text{uniformly in } m$$

for every $r > 0$. The proof is complete. □

Remark 10. Now we show that our result Theorem 9 is stronger than the classical Korovkin theorem and the statistical Korovkin theorem.

Let $X = [0, 1]$ and consider the following Bernstein–Kantorovich operators:

$$U_n\left(f;x\right)=(n+1)\sum_{k=0}^{n}\binom{n}{k}x^k\left(1-x\right)^{n-k}\int_{k/(n+1)}^{(k+1)/(n+1)}f\left(t\right)\mathrm{d}t$$

where $x \in [0, 1]$, $f \in C[0, 1]$ and $n \in \mathbb{N}$. Using these polynomials, we introduce the following positive linear operators on $C[0, 1]$:

$$T_n\left(f;x\right)=\left(1+g_n\left(x\right)\right)U_n\left(f;x\right), \quad x\in[0,1], f\in C[0,1] \tag{7}$$

where $g_n(x)$ is given by (1). Then observe that

$$T_n(e_0;x) = (1+g_n(x))\, e_0(x)$$
$$T_n(e_1;x) = (1+g_n(x))\left(e_1(x)\frac{n}{n+1}+\frac{1}{2(n+1)}\right)$$
$$T_n(e_2;x) = (1+g_n(x))\left[e_2(x)\frac{n(n-1)}{(n+1)^2}+\frac{2nx}{(n+1)^2}+\frac{1}{3(n+1)^2}\right]$$

where $e_i(x) = x^i$, $i = 0, 1, 2$. Consider the case $\sigma(n) = n+1$. Since $g_n \to g = 0$ (equi-stat (σ)) on $[0, 1]$, we conclude that

$$T_n(e_i) \to e_i \text{ (equi-stat } (\sigma)) \text{ on } [0,1] \quad \text{for each } i = 0, 1, 2.$$

So the $\{T_n\}$ satisfy all hypotheses of Theorem 9 and we immediately see that

$$T_n(f) \to f \text{ (equi-stat } (\sigma)) \text{ on } [0,1] \quad \text{for all } f \in C[0,1].$$

However, since $\{g_n\}$ is not uniformly convergent (in the ordinary sense) to the function $g = 0$ on $[0, 1]$, we can say that $T_n(e_i)$ is not uniformly convergent to e_i for $i = 0, 1, 2$. Also since $\{g_n\}$ is not statistically uniformly convergent to the function $g = 0$ on $[0, 1]$, we can say that $T_n(e_i)$ is not statistically uniformly convergent to e_i on $[0, 1]$ for $i = 0, 1, 2$. So the classical Korovkin theorem and statistical Korovin theorem do not work for our operators defined by (7). Therefore, this application clearly shows that our Theorem 9 is a non-trivial generalization of the classical and the statistical cases of the Korovkin-type results introduced in [7,13], respectively.

3. Rate of equi-statistical σ-convergence

In this section, we study the rates of equi-statistical σ-convergence of positive linear operators defined on $C(X)$ with the help of the modulus of continuity.

Definition 11. A sequence $\widetilde{f}(x) := \{f_n(x)\}$ is equi-statistically σ-convergent to a function f with the rate of $\beta \in (0, 1)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\left(\sup_{x\in X}\frac{\left|\left\{p \le n : \left|t_{pm}\left(\widetilde{f}(x)\right) - f(x)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}}\right) = 0 \quad \text{uniformly in } m$$

where

$$t_{pm}\left(\widetilde{f}(x)\right) := \frac{f_m(x)+f_{\sigma(m)}(x)+f_{\sigma^2(m)}(x)+\cdots+f_{\sigma^p(m)}(x)}{p+1}.$$

In this case, it is denoted by

$$f_n - f = o(n^{-\beta}) \text{ (equi-stat } (\sigma)) \quad \text{on } X.$$

Using this definition, we obtain the following auxiliary result.

Lemma 12. *Let $\widetilde{f}(x) := \{f_n(x)\}$ and $\widetilde{g}(x) := \{g_n(x)\}$ be function sequences belonging to $C(X)$. Assume that $f_n - f = o(n^{-\beta_1})$ (equi-stat (σ)) on X and $g_n - g = o(n^{-\beta_2})$ (equi-stat (σ)) on X. Let $\beta := \min\{\beta_1, \beta_2\}$. Then we have*

(i) $(f_n - f) \mp (g_n - g) = o(n^{-\beta})$ (equi-stat (σ)) on X,
(ii) $\lambda(f_n - f) = o(n^{-\beta_1})$ (equi-stat (σ)) on X, for any real number λ.

Proof. (i) Assume that $f_n - f = o(n^{-\beta_1})$ (equi-stat (σ)) on X and $g_n - g = o(n^{-\beta_2})$ (equi-stat (σ)) on X. Then, since $\beta := \min\{\beta_1, \beta_2\}$ for $\varepsilon > 0$, observe that

$$\begin{aligned}
&\frac{\left|\left\{p \le n : \left|\left(t_{pm}\left(\widetilde{f}(x)\right) - f(x)\right) \mp \left(t_{pm}\left(\widetilde{g}(x)\right) - g(x)\right)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} \\
&\quad \le \frac{\left|\left\{p \le n : \left|t_{pm}\left(\widetilde{f}(x)\right) - f(x)\right| \ge \frac{\varepsilon}{2}\right\}\right| + \left|\left\{p \le n : \left|t_{pm}\left(\widetilde{g}(x)\right) - g(x)\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}} \\
&\quad \le \frac{\left|\left\{p \le n : \left|t_{pm}\left(\widetilde{f}(x)\right) - f(x)\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_1}} + \frac{\left|\left\{p \le n : \left|t_{pm}\left(\widetilde{g}(x)\right) - g(x)\right| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_2}}.
\end{aligned} \tag{8}$$

Now, by taking the limit as $n \to \infty$ in (8) and using the hypotheses, we conclude that

$$\lim_{n\to\infty}\left(\sup_{x\in X}\frac{\left|\left\{p \le n : \left|\left(t_{pm}\left(\widetilde{f}(x)\right) - f(x)\right) \mp \left(t_{pm}\left(\widetilde{g}(x)\right) - g(x)\right)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}}\right) = 0, \quad \text{uniformly in } m,$$

which completes the proof of (i). Since the proof of (ii) is similar, we omit it. □

Now we recall that the modulus of continuity of a function $f \in C(X)$ is defined by

$$\omega(f;\delta) = \sup_{|y-x|\le\delta, x,y\in X} |f(y)-f(x)| \quad (\delta>0).$$

It is also well know that, for any $\lambda>0$ and for all $f \in C(X)$,

$$\omega(f;\lambda\delta) \le (1+[\lambda])\,\omega(f;\delta)$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.

Then we have the following result.

Theorem 13. *Let $L := \{L_n\}$ be a sequence of positive linear operators acting from $C(X)$ into itself. Assume that the following conditions hold:*

$$L_n(e_0) - e_0 = o(n^{-\beta_1})\ (\textit{equi-stat}\ (\sigma)) \quad \textit{on}\ X \tag{9}$$

and

$$\omega\left(f;\gamma_{pm}\right) = o(n^{-\beta_2})\ (\textit{equi-stat}\ (\sigma)) \quad \textit{on}\ X \tag{10}$$

where $\gamma_{pm} := \sqrt{t_{pm}(L(\varphi;x))}$ with $\varphi(y) = (y-x)^2$. Then we have for all $f \in C(X)$

$$L_n(f) - f = o(n^{-\beta})\ (\textit{equi-stat}\ (\sigma)) \quad \textit{on}\ X$$

where $\beta := \min\{\beta_1, \beta_2\}$.

Proof. Let $f \in C(X)$ and $x \in X$ be fixed. Using the properties of ω and the positivity and monotonicity of L_n, we get, for any $\delta>0$, that

$$\begin{aligned}
\left|t_{pm}(L(f;x)) - f(x)\right| &= \left|\frac{L_m(f;x) + L_{\sigma(m)}(f;x) + \cdots + L_{\sigma^p(m)}(f;x)}{p+1} - f(x)\right| \\
&\le \frac{L_m(|f(y)-f(x)|;x) + L_{\sigma(m)}(|f(y)-f(x)|;x) + \cdots + L_{\sigma^p(m)}(|f(y)-f(x)|;x)}{p+1} \\
&\quad + |f(x)|\left|t_{pm}(L(e_0;x)) - e_0(x)\right| \\
&\le \frac{L_m\left(\omega\left(f;\delta\frac{|y-x|}{\delta}\right);x\right) + L_{\sigma(m)}\left(\omega\left(f;\delta\frac{|y-x|}{\delta}\right);x\right) + \cdots + L_{\sigma^p(m)}\left(\omega\left(f;\delta\frac{|y-x|}{\delta}\right);x\right)}{p+1} \\
&\quad + |f(x)|\left|t_{pm}(L(e_0;x)) - e_0(x)\right| \\
&\le \frac{L_m\left(1+\left[\frac{|y-x|}{\delta}\right];x\right) + L_{\sigma(m)}\left(1+\left[\frac{|y-x|}{\delta}\right];x\right) + \cdots + L_{\sigma^p(m)}\left(1+\left[\frac{|y-x|}{\delta}\right];x\right)}{p+1}\omega(f;\delta) \\
&\quad + |f(x)|\left|t_{pm}(L(e_0;x)) - e_0(x)\right| \\
&\le \frac{L_m\left(1+\frac{(y-x)^2}{\delta^2};x\right) + L_{\sigma(m)}\left(1+\frac{(y-x)^2}{\delta^2};x\right) + \cdots + L_{\sigma^p(m)}\left(1+\frac{(y-x)^2}{\delta^2};x\right)}{p+1}\omega(f;\delta) \\
&\quad + |f(x)|\left|t_{pm}(L(e_0;x)) - e_0(x)\right| \\
&\le \omega(f;\delta) + \left|t_{pm}(L(e_0;x)) - e_0(x)\right|\omega(f;\delta) \\
&\quad + \frac{L_m(\varphi;x) + L_{\sigma(m)}(\varphi;x) + \cdots + L_{\sigma^p(m)}(\varphi;x)}{p+1}\frac{\omega(f;\delta)}{\delta^2} + |f(x)|\left|t_{pm}(L(e_0;x)) - e_0(x)\right| \\
&\le \omega(f;\delta) + \left|t_{pm}(L(e_0;x)) - e_0(x)\right|\omega(f;\delta) + \frac{\omega(f;\delta)}{\delta^2}t_{pm}(L(\varphi;x)) + |f(x)|\left|t_{pm}(L(e_0;x)) - e_0(x)\right|.
\end{aligned}$$

Then we have

$$\left|t_{pm}(L(f;x)) - f(x)\right| \le 2\omega(f;\delta) + \left|t_{pm}(L(e_0;x)) - e_0(x)\right|\omega(f;\delta) + M\left|t_{pm}(L(e_0;x)) - e_0(x)\right| \tag{11}$$

where $\delta := \gamma_{pm} := \sqrt{t_{pm}(L(\varphi;x))}$ and $M := \|f\|_{C(X)}$. Hence, given $\varepsilon>0$, it follows from (11) and Lemma 12 that

$$\begin{aligned}
&\frac{\left|\left\{p\le n : \left|t_{pm}(L(f;x)) - f(x)\right| \ge \varepsilon\right\}\right|}{n^{1-\beta}} \\
&\quad \le \frac{\left|\left\{p\le n : \omega(f;\delta) \ge \frac{\varepsilon}{6}\right\}\right|}{n^{1-\beta_2}} + \frac{\left|\left\{p\le n : \left|t_{pm}(L(e_0;x)) - e_0(x)\right| \ge \sqrt{\frac{\varepsilon}{3}}\right\}\right|}{n^{1-\beta_1}}
\end{aligned}$$

$$+\frac{\left|\left\{p\le n:\omega\left(f;\delta\right)\ge\sqrt{\frac{\varepsilon}{3}}\right\}\right|}{n^{1-\beta_2}}+\frac{\left|\left\{p\le n:\left|t_{pm}\left(L\left(e_0;x\right)\right)-e_0\left(x\right)\right|\ge\frac{\varepsilon}{3M}\right\}\right|}{n^{1-\beta_1}} \tag{12}$$

where $\beta := \min\{\beta_1, \beta_2\}$. Letting $n \to \infty$ in (12), we conclude from (9) and (10) that

$$\lim_{n\to\infty}\left(\sup_{x\in X}\frac{\left|\left\{p\le n:\left|t_{pm}\left(L\left(f;x\right)\right)-f\left(x\right)\right|\ge\varepsilon\right\}\right|}{n^{1-\beta}}\right)=0,\quad \text{uniformly in } m,$$

which means

$$L_n\left(f\right)-f=o(n^{-\beta})\ (\text{equi-stat}\left(\sigma\right))\quad \text{on } X.\quad \square$$

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