Results Math 72 (2017), 1073–1085

1422-6383/17/031073-13
published online August 18, 2016
DOI 10.1007/s00025-016-0591-2

Results in Mathematics

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Weighted Equi-Statistical Convergence of the Korovkin Type Approximation Theorems

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Abstract. In this paper, the concepts of weighted statistical pointwise convergence, weighted statistical uniform convergence and weighted equi-statistical convergence are introduced. Then, using weighted equi-statistical convergence, a general Korovkin type theorem is obtained. Also, an example such that our new approximation result works but its classical case does not work is constructed.

Mathematics Subject Classification. 40A05, 40G15, 41A36.

Keywords. Statistical convergence, equi-statistical convergence, weighted equi-statistical convergence, Korovkin type approximation theorem, Voronovskaya type theorem.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence by Steinhaus [13] and Fast [5]. If K is a subset of the natural numbers $\mathbb{N}$, K_n will denote the set $\{k \leq n : k \in K\}$, and $|K_n|$ will denote the cardinality of K_n. The natural density of K [12] is given by $\delta(K) := \lim_n \frac{1}{n}|K_n|$, if it exists. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero [5]. Hence $x = (x_k)$ is statistically convergent to L if and only if $(C_1\chi_{K(\varepsilon)})_n \to 0$, (as $n \to \infty$, for every $\varepsilon > 0$), where C_1 is the Cesáro matrix of order one and $\chi_{K(\varepsilon)}$ is the characteristic function of the set $K(\varepsilon)$.

The concept of weighted statistical convergence was defined by Karakaya and Chishti [7]. Recently, Mursaleen et al. [11] modify the definition of weighted statistical convergence. In [6] Ghosal showed that both definitions of weighted

Birkhäuser

statistical convergence are not well defined in general. So Ghosal modify the definition of weighted statistical convergence as follows:

Definition 1.1. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $p_1 > 0$, $\liminf_{n\to\infty} p_n > 0$ and $P_n = \sum_{k=1}^{n} p_k$ where $n \in \mathbb{N}$, $P_n \to \infty$ as $n \to \infty$. The sequence $x = (x_k)$ is said to be weighted statistically convergent (or $S_{\overline{N}}$-convergent) to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{P_n} \left|\{k \le P_n : p_k \left|x_k - L\right| \ge \varepsilon\}\right| = 0.$$

In this case, we write $st_{\overline{N}} - \lim x = L$ and we denote the set of all weighted statistically convergent sequences by $S_{\overline{N}}$.

Remark 1.1. If $p_k = 1$ for all k, then weighted statistical convergence is reduced to statistical convergence.

Example 1.1. Let $x = (x_k)$ is a sequence defined by

$$x = (x_k) = \begin{cases} \sqrt{k}, & \text{if } k = n^2, \qquad n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Let $p_k = k$ $(k = 1, 2, \ldots)$. Then

$$P_n = \sum_{k=1}^{n} p_k = \frac{n(n+1)}{2}.$$

Since

$$\lim_{n\to\infty} \frac{1}{P_n} \left|\{k \le P_n : p_k \left|x_k - 0\right| \ge \varepsilon\}\right| \le \lim_{n\to\infty} \frac{1}{P_n} \sqrt{P_n} = 0.$$

So (x_k) is weighted statistical convergent to 0 but not ordinary convergent.

In [6] Ghosal showed that both convergence which are weighted statistical convergence and statistical convergence do not imply each other in general.

2. Weighted Equi-Statistical Convergence

Let $C(D)$ be the space of all functions f continuous on a compact subset D of the real numbers. Also, $C(D)$ is a Banach space with norm

$$\|f\| := \sup_{x\in D} |f(x)|, \qquad f \in C(D).$$

Let f and f_n belong to $C(D)$.

Balcerzak, Dems and Komisarski [1] introduced the notion of equi-statisitcal convergence which is stronger than the statistical uniform convergence. First we recall this concept.

Definition 2.1 [1]. A sequence (f_n) is said to be equi-statistically convergent to f on D if for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{\|\Phi_n(.,\varepsilon)\|}{n} = 0$$

where

$$\Phi_n(x,\varepsilon) := |\{k \le n : |f_k(x) - f(x)| \ge \varepsilon\}|.$$

In this case we write $f_n \to f \quad (equi - st)$.

Throughout the paper, let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $p_1 > 0$, $\liminf_{n\to\infty} p_n > 0$ and $P_n = \sum_{k=1}^{n} p_k$ where $n \in \mathbb{N}$, $P_n \to \infty$ as $n \to \infty$.

Now, we give the following definitions:

Definition 2.2. A sequence (f_n) is said to be weighted equi-statistically convergent to f on D if for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{\Omega_n(x,\varepsilon)}{P_n} = 0$$

uniformly with respect to $x \in D$, which means that

$$\lim_{n\to\infty} \frac{\|\Omega_n(.,\varepsilon)\|}{P_n} = 0$$

where

$$\Omega_n(x,\varepsilon) := |\{k \le P_n : p_k |f_k(x) - f(x)| \ge \varepsilon\}|.$$

In this case we write $f_n \to f \quad (w - equi - st)$.

Definition 2.3. A sequence (f_n) is said to be weighted statistically pointwise convergent to f on D if for every $\varepsilon > 0$ and for each $x \in D$, we have

$$\lim_{n\to\infty} \frac{1}{P_n} |\{k \le P_n : p_k |f_k(x) - f(x)| \ge \varepsilon\}| = 0.$$

In this case, we write $f_n \to f \quad (w - st)$.

Definition 2.4. A sequence (f_n) is said to be weighted statistically uniform convergent to f on D if for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{P_n} |\{k \le P_n : p_k \|f_k - f\| \ge \varepsilon\}| = 0.$$

In this case, we write $f_n \rightrightarrows f \quad (w - st)$.

Remark 2.1. If $p_k = 1$ for all k, then weighted equi-statistically convergence,weighted statistically pointwise convergence and weighted statistically uniform convergence are reduced to equi-statistically convergence, statistically pointwise convergence and statistically uniform convergence, respectively.

The following implications of the above definitions and concepts are trivial.

Lemma 2.1. *Each of the following implications holds true*

$$f_n \rightrightarrows f \ \ (w - st) \Longrightarrow f_n \to f \ \ (w - equi - st) \Longrightarrow f_n \to f \ \ (w - st)\,.$$

Furthermore, in general the reverse implications do not hold true. Such an example is given as follows:

Example 2.1 [1]. For each $n \in \mathbb{N}$ define $h_n \in C\,[0,1]$ by

$$h_n\,(x) = \begin{cases} 2^{n+1}\left(x - \frac{1}{2^n}\right), & \text{if } x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}\right] \\ -2^{n+1}\left(x - \frac{1}{2^{n-1}}\right), & \text{if } x \in \left[\frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}\right] \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $p_k = \sqrt{k} \ \ (k = 1, 2, \ldots)$. Then observe that

$$\frac{1}{P_n}\left|\{k \le P_n : p_k\,|h_k\,(x)| \ge \varepsilon\}\right| \le \frac{1}{P_n} \to 0 \quad (n \to \infty)\,.$$

So $h_n \to h = 0 \ \ (w - equi - st)$ on $[0,1]$. On the other hand (h_n) is not weighted statistically (or ordinary) uniform convergent to the function $h = 0$ on the interval $[0,1]$.

Let (T_n) be a sequence of positive linear operators from $C\,(D)$ into $C\,(D)$. Korovkin [9] first noticed that , $\lim_{n\to\infty} \|T_n\,(f;x) - f\,(x)\| = 0$ for all $f \in C\,(D)$ if and only if $\lim_{n\to\infty} \|T_n\,(f_i;x) - f_i\,(x)\| = 0$ for $i = 0,1,2,$ where $f_0\,(x) = 1$, $f_1\,(x) = x$, $f_2\,(x) = x^2$. Korovkin type approximation theorem play a very active role in recent research on summability theory. So many mathematicians extended the Korovkin-type approximation theorems by using various test functions in several setups, including Banach spaces, abstract Banach lattices, function spaces, Banach algebras and so on.

Karakuş, Demirci and Duman [8] proved a Korovkin-type approximation theorem by means of equi-statistical convergence.

Recently, weighted statistical convergence has been used in the Korovkin-type approximation theory [2–4, 10, 11]. Then it was demonstrated that those results are more powerful than the classical Korovkin theorem.

In this paper, we establish a Korovkin type result for a sequence of positive linear operators acting on $C\,(D)$ by using weighted equi-statistical convergence. Also, by considering Lemma 2.1 and the above example, we will construct a sequence of positive linear operators such that while our new result work, its classical case do not work.

Theorem 2.2. *Let (T_n) be a sequence of positive linear operators from $C\,(D)$ into $C\,(D)$. Then, for all $f \in C\,(D)$,*

$$T_n\ \ (f) \to f \quad (w - equi - st) \ \ on\ D \tag{2.2}$$

if and only if

$$T_n\,(f_i) \to f_i \quad (w - equi - st) \ \ on\ D \quad with \ \ f_i\,(x) = x^i \quad (i = 0,1,2)\,. \tag{2.3}$$

Proof. Let us suppose that the first assertion (2.2) is true. Since each function $f_i \in C(D)$, $i = 0, 1, 2$, the implication (2.2)$\Longrightarrow$(2.3) follow immediately from the first assertion (2.2). Now we will prove the converse. Assume now that (2.3) holds. Let $f \in C(D)$ and $x \in D$ be fixed. Let $\varepsilon > 0$ be given. By the continuity of f at the point x, there is a $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \quad \forall \; |t - x| < \delta.$$

Since

$$|f(t) - f(x)| = |f(t) - f(x)| \chi_{D_\delta}(t) + |f(t) - f(x)| \chi_{D \setminus D_\delta}(t)$$

where $D_\delta = [x - \delta, x + \delta] \cap D$ and χ_A denotes the characteristic function of the set A. Then we have

$$|f(t) - f(x)| \leq \varepsilon + \frac{2C}{\delta^2} \Lambda$$

for all $t \in D$, where $C := \|f\|$ and $\Lambda = (t - x)^2$. By positivity and linearity of the operator T_n, we conclude that

$$\begin{aligned}
&|T_n(f; x) - f(x)| \\
&\quad = |T_n(f(t); x) - T_n(f(x); x) + T_n(f(x); x) - f(x)| \\
&\quad \leq T_n(|f(t) - f(x)|; x) + |f(x)| \, |T_n(1; x) - 1| \\
&\quad \leq T_n\left(\varepsilon + \frac{2C}{\delta^2}\Lambda; x\right) + \|f\| \, |T_n(f_0; x) - f_0(x)| \\
&\quad \leq \varepsilon + (\varepsilon + C) |T_n(f_0; x) - f_0(x)| + \frac{2C}{\delta^2} T_n(\Lambda; x).
\end{aligned} \tag{2.4}$$

We can write the term "$T_n(\Lambda; x)$" in (2.4) as follows:

$$\begin{aligned}
T_n(\Lambda; x) &= T_n\left((t - x)^2; x\right) \\
&= T_n\left(t^2; x\right) - 2x T_n(t; x) + x^2 T_n(1; x) \\
&\leq \left|T_n\left(t^2; x\right) - x^2\right| + 2|x| \, |T_n(t; x) - x| \\
&\qquad + x^2 |T_n(1; x) - 1|.
\end{aligned} \tag{2.5}$$

So we get

$$\begin{aligned}
|T_n(f; x) - f(x)| \leq \varepsilon + \left(\varepsilon + C + \frac{2C}{\delta^2}\left\|x^2\right\|\right) |T_n(f_0; x) - f_0(x)| \\
+ \frac{4C}{\delta^2} \|x\| \, |T_n(f_1; x) - f_1(x)| + \frac{2C}{\delta^2} |T_n(f_2; x) - f_2(x)|.
\end{aligned}$$

Because of ε is arbitrary, we obtain

$$\begin{aligned}
|T_n(f; x) - f(x)| \leq K \{ &|T_n(f_0; x) - f_0(x)| + |T_n(f_1; x) - f_1(x)| \\
&+ |T_n(f_2; x) - f_2(x)| \}
\end{aligned}$$

where $K := \varepsilon + C + \frac{2C}{\delta^2}\left[\left\|x^2\right\| + 2\left\|x\right\| + 1\right]$. Hence;

$$p_k \left|T_n(f;x) - f(x)\right| \leq K \left\{p_k \left|T_n(f_0;x) - f_0(x)\right| + p_k \left|T_n(f_1;x) - f_1(x)\right| \right.$$
$$\left. + p_k \left|T_n(f_2;x) - f_2(x)\right|\right\}.$$

For a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$ and we will define the following sets:

$$\begin{aligned}
\Omega_n(x,r) &:= \left|\left\{k \leq P_n : p_k \left|T_n(f;x) - f(x)\right| \geq r\right\}\right|, \\
\Omega_{1,n}(x,r) &:= \left|\left\{k \leq P_n : p_k \left|T_n(f_0;x) - f_0(x)\right| \geq \frac{r}{3K}\right\}\right|, \\
\Omega_{2,n}(x,r) &:= \left|\left\{k \leq P_n : p_k \left|T_n(f_1;x) - f_1(x)\right| \geq \frac{r}{3K}\right\}\right|, \\
\Omega_{3,n}(x,r) &:= \left|\left\{k \leq P_n : p_k \left|T_n(f_2;x) - f_2(x)\right| \geq \frac{r}{3K}\right\}\right|.
\end{aligned}$$

Then

$$\frac{\Omega_n(x,r)}{P_n} \leq \frac{\Omega_{1,n}(x,r)}{P_n} + \frac{\Omega_{2,n}(x,r)}{P_n} + \frac{\Omega_{3,n}(x,r)}{P_n}.$$

So we have

$$\frac{\left\|\Omega_n(.,r)\right\|}{P_n} \leq \frac{\left\|\Omega_{1,n}(.,r)\right\|}{P_n} + \frac{\left\|\Omega_{2,n}(.,r)\right\|}{P_n} + \frac{\left\|\Omega_{3,n}(.,r)\right\|}{P_n}.$$

Thus, by condition (2.3) and considering Definition 2.2 we obtain

$$\lim_{n\to\infty} \frac{\left\|\Omega_n(.,r)\right\|}{P_n} = 0 \quad \text{for every } r > 0$$

which completes the proof. □

We remark that our Theorem 2.2 is stronger than that of classical Korovkin approximation theorem. For this claim, we consider the following example.

Example 2.2. Let $D = [0,1]$ and consider the classical Bernstein operators

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

Let $p_k = \sqrt{k}$. We define the sequence

$$L_n(f;x) = (1 + h_n(x))\, B_n(f;x), \qquad x \in [0,1] \quad \text{and} \quad f \in C[0,1], \tag{2.6}$$

where $h_n(x)$ is given by (2.1). Note that (h_n) is weighted equi-statistical convergent to $h = 0$ but it is not uniformly convergent to the function $h = 0$ on the interval $[0,1]$. Then, observe that

$$\begin{aligned}
L_n(1;x) &= 1 + h_n(x) \\
L_n(t;x) &= (1 + h_n(x))\, x \\
L_n\left(t^2;x\right) &= (1 + h_n(x))\left(x^2 + \frac{x(1-x)}{n}\right).
\end{aligned}$$

We have that the sequence (L_n) satisfies the conditions (2.3). Hence by Theorem 2.2 we have

$$L_n(f;x) \to f \qquad (w - equi - st) \text{ for all } f \in C[0,1].$$

Since (h_n) is not uniformly convergent to the function $h = 0$ on $[0,1]$, the classical Korovkin theorem does not work for our operators defined by (2.6). Consequently, Theorem 2.2 is a non-trivial extension of the classical Korovkin Theorem.

3. A Voronovskaya Type Theorem

In this section, we will show that the positive linear operators L_n defined by (2.6) satisfy a Voronovskaya type property in the weighted equi-statistial sense. We first prove the following lemma:

Lemma 3.1. *Let $x \in [0,1]$, $p_k = \sqrt{k}\,(k \in \mathbb{N})$ and $\theta(y) := y - x$. Then we have*

$$n^2 L_n\left(\theta^4\right) \to 3x^2\left(x^2 - 2x + 1\right) \quad (w - equi - st) \;\; \textit{on} \quad [0,1].$$

Proof. Using (2.6), a simple calculation shows that

$$n^2 L_n\left(\theta^4; x\right) = (1 + h_n(x))\left\{3x^4 - 6x^3 + 3x^2 + \frac{1}{n}\left(-6x^4 + 12x^3 - 7x^2 + x\right)\right\}.$$

Then we get

$$\left|n^2 L_n\left(\theta^4; x\right) - 3x^2\left(x^2 - 2x + 1\right)\right| \le 12 h_n(x) + \frac{26\,(1 + h_n(x))}{n}, \qquad (x \in [0,1]).$$

From Example 2.2, we have

$$12 h_n + \frac{26}{n} + \frac{26 h_n}{n} \to 0 \quad (w - equi - st) \text{ on } \quad [0,1].$$

So we get

$$n^2 L_n\left(\theta^4\right) \to 3x^2\left(x^2 - 2x + 1\right) \quad (w - equi - st) \text{ on } \quad [0,1].$$

This completes the proof. □

We establish the following Voronovskaya type result for the operators L_n given by (2.6).

Theorem 3.2. *For every $f \in C[0,1]$ such that $f', f'' \in C[0,1]$, we have*

$$n\{L_n f - f\} \to \frac{x(1-x)}{2} f'' \quad (w - equi - st) \;\; \textit{on} \quad [0,1].$$

Proof. Suppose that $f, f', f'' \in C[0,1]$ and $x \in [0,1]$. Define

$$\xi_x(y) = \begin{cases} \frac{f(y) - f(x) - (y-x) f'(x) - \frac{1}{2}(y-x)^2 f''}{(y-x)^2}, & \text{if } y \ne x \\ 0, & \text{if } y = x. \end{cases}$$

Then

$$\xi_x(x) = 0 \text{ and } \xi_x \in C[0,1].$$

Hence by Taylor's theorem we get

$$f(y) = f(x) + (y-x) f'(x) + \frac{(y-x)^2}{2} f''(x) + (y-x)^2 \ \xi_x(y).$$

Now operating L_n given by (2.6) upon both sides of the above equation, we conclude that

$$\begin{aligned} L_n(f;x) - f(x) &= f(x) h_n(x) + f'(x) L_n(\theta;x) \\ &\quad + \frac{f''(x)}{2} L_n\left(\theta^2;x\right) + L_n\left(\theta^2 \xi_x;x\right) \\ &= f(x) h_n(x) + \left(\frac{x(1-x)}{2n}\right)(1+h_n(x)) f''(x) \\ &\quad + L_n\left(\theta^2 \xi_x;x\right) \end{aligned}$$

which yields

$$\left| n(L_n(f;x) - f(x)) - \frac{x(1-x)}{2} f''(x) \right| \le K(n+1) h_n(x) + n\left|L_n\left(\theta^2\xi_x;x\right)\right| \tag{3.1}$$

where $\theta(y) = y - x$ and $K = \|f\|_{C[0,1]} + \|f''\|_{C[0,1]}$.

By applying the familiar Cauchy-Schwarz inequality for the second term on the right-hand side of (3.1), we obtain

$$n\left|L_n\left(\theta^2\xi_x;x\right)\right| \le \left(n^2 L_n\left(\theta^4;x\right)\right)^{\frac{1}{2}} \left(L_n\left(\xi_x^2;x\right)\right)^{\frac{1}{2}}. \tag{3.2}$$

Let $\eta_x(y) := \xi_x^2(y)$. Then we observe that

$$\eta_x(x) = 0 \text{ and } \eta_x(.) \in C[0,1].$$

Then it follows from Theorem 2.2 that

$$L_n(\eta_x) \to 0 \quad (w - equi - st) \text{ on } [0,1] \tag{3.3}$$

Now, from (3.2) and (3.3) and by Lemma 3.1, we have

$$nL_n\left(\theta^2\xi_x;x\right) \to 0 \quad (w - equi - st) \text{ on } [0,1]. \tag{3.4}$$

For a given $\varepsilon > 0$, we define

$$\begin{aligned} \Omega_n(x,\varepsilon) &: = \left|\left\{k \le P_n : p_k \left| k(L_k(f;x) - f(x)) - \frac{x(1-x)}{2} f''(x) \right| \ge \varepsilon \right\}\right|, \\ \Omega_{1,n}(x,\varepsilon) &: = \left|\left\{k \le P_n : p_k \left|(k+1) h_k(x)\right| \ge \frac{\varepsilon}{2K}\right\}\right|, \\ \Omega_{2,n}(x,\varepsilon) &: = \left|\left\{k \le P_n : p_k \left|kL_k\left(\theta^2\xi_x;x\right)\right| \ge \frac{\varepsilon}{2}\right\}\right|. \end{aligned}$$

Then it follows from (3.1) that

$$\frac{\Omega_n(x,\varepsilon)}{P_n} \le \frac{\Omega_{1,n}(x,\varepsilon)}{P_n} + \frac{\Omega_{2,n}(x,\varepsilon)}{P_n}$$

which yields;

$$\frac{\|\Omega_n(.,\varepsilon)\|_{C[0,1]}}{P_n} \leq \frac{\|\Omega_{1,n}(.,\varepsilon)\|_{C[0,1]}}{P_n} + \frac{\|\Omega_{2,n}(.,\varepsilon)\|_{C[0,1]}}{P_n}. \tag{3.5}$$

Also, by the definition h_n we observe that

$$(n+1)\,h_n \to h = 0 \quad (w - equi - st) \text{ on } \ [0,1]. \tag{3.6}$$

Now, by (3.4) and (3.6), the right-hand side of (3.5) tends to zero as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \frac{\|\Omega_n(.,\varepsilon)\|_{C[0,1]}}{P_n} = 0.$$

This completes the proof of Theorem 3.2. □

Remark 3.1. Since the function sequence $\{h_n\}$ given by (2.1) is not uniformly convergent to the function $h = 0$ on $[0,1]$, we notice that our operators L_n defined by (2.6) do not satisfy the Voronovskaya-type property in the usual sense.

4. Rates of Weighted Equi-Statistical Convergence

In this section we study the rates of weighted equi-statistical convergence of a sequence of positive linear operators defined on $C(D)$ with the help of modulus of continuity

First, we recall that the modulus of continuity of a function $f \in C(D)$ is defined by

$$w(f;\delta) := \sup_{|y-x|\leq\delta,\, x,y\in D} |f(y) - f(x)| \qquad (\delta > 0).$$

Now we begin by presenting the following definition.

Definition 4.1. A sequence (f_n) is weighted equi-statistically convergent to a function f with the rate of $\beta \in (0,1)$, if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\Omega_n(x,\varepsilon)}{n^{1-\beta}P_n} = 0$$

uniformly with respect to $x \in D$ or equivalently if (for every $\varepsilon > 0$)

$$\lim_{n\to\infty} \frac{\|\Omega_n(.,\varepsilon)\|}{n^{1-\beta}P_n} = 0$$

where

$$\Omega_n(x,\varepsilon) := |\{k \leq P_n : p_k\,|f_k(x) - f(x)| \geq \varepsilon\}|.$$

In this case, it is denoted as follows:

$$f_n - f = o\left(n^{-\beta}\right) \quad (w - equi - st) \text{ on } \ D.$$

Now we prove the following basic lemma which we need.

Lemma 4.1. *Let (f_n) and (g_n) be function sequences belonging to $C(D)$. Suppose that*

$$f_n - f = o\left(n^{-\beta_0}\right) \quad (w - equi - st) \ \ on \ \ D$$

and

$$g_n - g = o\left(n^{-\beta_1}\right) \quad (w - equi - st) \ \ on \ \ D.$$

Let $\beta = \min\{\beta_0, \beta_1\}$. Then each of the following statement holds true.

(i) $(f_n + g_n) - (f + g) = o\left(n^{-\beta}\right)$ $(w - equi - st)$ on D,
(ii) $(f_n - f)(g_n - g) = o\left(n^{-\beta}\right)$ $(w - equi - st)$ on D,
(iii) $\mu(f_n - f) = o\left(n^{-\beta_0}\right)$ $(w - equi - st)$ on D for any real number μ,
(iv) $\sqrt{|f_n - f|} = o\left(n^{-\beta_0}\right)$ $(w - equi - st)$ on D.

Proof. i) Suppose that $f_n - f = o\left(n^{-\beta_0}\right)$ $(w - equi - st)$ on D and $g_n - g = o\left(n^{-\beta_1}\right)$ $(w - equi - st)$ on D. In order to prove the statement (i), for every $\varepsilon > 0$ and $x \in D$, we define the following sets:

$$\begin{aligned}
\Omega_n(x,\varepsilon) &: = \left|\left\{k \le P_n : p_k \left|(f_k + g_k)(x) - (f+g)(x)\right| \ge \varepsilon\right\}\right| \\
\Omega_{0,n}(x,\varepsilon) &: = \left|\left\{k \le P_n : p_k \left|f_k(x) - f(x)\right| \ge \frac{\varepsilon}{2}\right\}\right| \\
\Omega_{1,n}(x,\varepsilon) &: = \left|\left\{k \le P_n : p_k \left|g_k(x) - g(x)\right| \ge \frac{\varepsilon}{2}\right\}\right|.
\end{aligned}$$

Then we have that

$$\frac{\Omega_n(x,\varepsilon)}{n^{1-\beta}P_n} \le \frac{\Omega_{0,n}(x,\varepsilon)}{n^{1-\beta}P_n} + \frac{\Omega_{1,n}(x,\varepsilon)}{n^{1-\beta}P_n}. \tag{4.1}$$

Since $\beta = \min\{\beta_0, \beta_1\}$, by (4.1) we get

$$\frac{\Omega_n(x,\varepsilon)}{n^{1-\beta}P_n} \le \frac{\Omega_{0,n}(x,\varepsilon)}{n^{1-\beta_0}P_n} + \frac{\Omega_{1,n}(x,\varepsilon)}{n^{1-\beta_1}P_n}.$$

So we can write

$$\frac{\|\Omega_n(.,\varepsilon)\|}{n^{1-\beta}P_n} \le \frac{\|\Omega_{0,n}(.,\varepsilon)\|}{n^{1-\beta_0}P_n} + \frac{\|\Omega_{1,n}(.,\varepsilon)\|}{n^{1-\beta_1}P_n}. \tag{4.2}$$

Now by taking limit as $n \to \infty$ in (4.2) and using the hypothesis we conclude that

$$\lim_{n\to\infty} \frac{\|\Omega_n(.,\varepsilon)\|}{n^{1-\beta}P_n} = 0$$

which evidently completes the proof of the statement (i).

The proofs of (ii), (iii) and (iv) of Lemma 4.1 can also be given along the same or similar lines.

Now we get the following result.

Theorem 4.2. *Let T_n be a sequence of positive linear operators acting from $C(D)$ into $C(D)$. Assume that the following conditions hold true:*

(a) $T_n(f_0) - f_0 = o\left(n^{-\beta_0}\right)$ $(w - equi - st)$ *on* D.
(b) $w(f;\delta_n) = o\left(n^{-\beta_1}\right)$ $(w - equi - st)$ *on* D, *where*

$$\delta_n(x) = \sqrt{T_n\left(\theta^2; x\right)} \quad \text{with} \quad \theta(y) = (y - x).$$

Then, for all $f \in C(D)$

$$T_n(f) - f = o\left(n^{-\beta}\right) \quad (w - equi - st) \text{ on } D$$

where $\beta = \min\{\beta_0, \beta_1\}$.

Proof. Let $f \in C(D)$ and $x \in X$. We know that

$$|f(t) - f(x)| \leq \left(1 + \frac{|t - x|}{\delta}\right) w(f;\delta) \qquad (\text{for } \ f \in C(D)). \tag{4.3}$$

If we use (4.3), then it is easy to see that

$$\begin{aligned} |T_n(f;x) - f(x)| &= |T_n(f(t);x) - T_n(f(x);x) + T_n(f(x);x) - f(x)| \\ &\leq T_n(|f(t) - f(x)|;x) + \|f\| \, |T_n(f_0;x) - f_0(x)| \\ &\leq T_n\left(w\left(f; \frac{\delta|t - x|}{\delta}\right);x\right) + C\,|T_n(f_0;x) - f_0(x)| \\ &\leq T_n\left(\left(1 + \frac{|t - x|}{\delta}\right) w(f;\delta);x\right) + C\,|T_n(f_0;x) - f_0(x)| \\ &= w(f;\delta)\, T_n(f_0(t);x) \\ &\quad + \frac{w(f;\delta)}{\delta} T_n(|t - x|;x) + C\,|T_n(f_0;x) - f_0(x)| \end{aligned}$$

where $C = \|f\|$. From Cauchy Schwartz inequality

$$\begin{aligned} T_n(|t - x|;x) &\leq \left(T_n\left((t - x)^2;x\right)\right)^{\frac{1}{2}} (T_n(f_0;x))^{\frac{1}{2}} \\ &= \sqrt{T_n\left(\theta^2;x\right)}\sqrt{T_n(f_0;x)}. \end{aligned}$$

So we get

$$\begin{aligned} &|T_n(f;x) - f(x)| \\ &\quad \leq C\,|T_n(f_0;x) - f_0(x)| + w(f;\delta)\left[T_n(f_0;x) + \frac{1}{\delta}\sqrt{T_n\left(\theta^2;x\right)}\sqrt{T_n(f_0;x)}\right]. \end{aligned}$$

If we take $\delta = \delta_n$ then;

$$|T_n(f;x) - f(x)| \leq C\,|T_n(f_0;x) - f_0(x)| + w(f;\delta_n)\left[T_n(f_0;x) + \frac{1}{\delta}\sqrt{T_n(f_0;x)}\right].$$

So we get

$$\begin{aligned}\left|T_n(f;x)-f(x)\right| &\le C\left|T_n(f_0;x)-f_0(x)\right| + w(f;\delta_n)\left[\left|T_n(f_0;x)-f_0(x)\right|\right.\\ &\quad \left. +2+\sqrt{\left|T_n(f_0;x)-f_0(x)\right|}\right]\\ &\le C\left|T_n(f_0;x)-f_0(x)\right| + 2w(f;\delta_n)\\ &\quad +\left|T_n(f_0;x)-f_0(x)\right| w(f;\delta_n)\\ &\quad +w(f;\delta_n)\sqrt{\left|T_n(f_0;x)-f_0(x)\right|}.\end{aligned}$$

Now, by using the hypothesis (a) and (b) and Lemma 4.1, in the above inequality the proof is completed. □

5. Concluding remark

In this concluding section of our investigation, we present a remark and observations.

Remark 5.1. Suppose that we replace the conditions (a) and (b) in Theorem 4.2 by the following condition:

$$T_n(f_i)-f_i = o\left(n^{-\beta_i}\right) \quad (w-equi-st) \ \ on \ \ D \text{ with } f_i(x)=x^i \quad (i=0,1,2). \tag{5.1}$$

Then, since

$$\begin{aligned}T_n\left(\theta^2;x\right) &\le C\left\{\left|T_n(f_0;x)-f_0(x)\right| + \left|T_n(f_1;x)-f_1(x)\right|\right.\\ &\quad \left. +\left|T_n(f_2;x)-f_2(x)\right|\right\},\end{aligned} \tag{5.2}$$

where $C = 1+2\|f_1\|+\|f_2\|$. Now it follows from (5.1), (5.2) and Lemma 4.1 that

$$\delta_n = \sqrt{T_n\left(\theta^2\right)} = o\left(n^{-\sigma}\right) \quad (w-equi-st) \ \text{ on } D$$

where

$$\sigma = \min\left\{\beta_0,\beta_1,\beta_2\right\}.$$

Hence

$$w(f;\delta_n) = o\left(n^{-\sigma}\right) \quad (w-equi-st) \ \text{ on } D. \tag{5.3}$$

Using (5.3) in Theorem 4.2 we can get, for all $f \in C(D)$,

$$T_n(f)-f = o\left(n^{-\sigma}\right) \quad (w-equi-st) \ \text{ on } D.$$

Therefore, if we use the condition (5.1) in Theorem 4.2 instead of the conditions (a) and (b), then we obtain the rates of weighted equi-statistical convergence of the sequence of positive linear operators in Theorem 2.2.

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Received: December 28, 2015.
Accepted: August 11, 2016.