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ORIGINAL RESEARCH

$P-$statistical summation process of sequences of convolution operators

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Abstract In the present paper, we make a study of $P-$statistical Korovkin theorem via $\mathcal{A}-$summation process for a sequence of positive linear convolution operators and we show that our results obtained via an interesting application are meaningful. We also analyze rate of convergence of these operators via modulus of continuity.

Keywords Summation process · Convolution operators · Statistical convergence · Power series methods · Korovkin theorem

Mathematics Subject Classification 40A35 · 41A25 · 41A36

1 Introduction and preliminaries

The Classical Korovkin theorem establishes the uniform convergence for a given sequence $\left(L_j\right)$ of positive linear operators in the space of all continuous real valued functions on compact interval assuming the classical convergence by the test functions [18]. In [16], Gadjiev and Orhan developed this theorem by taking the concept of statistical convergence ([15], [24]) instead of classical convergence. After, Demirci and Dirik [13] have been carried this convergence for double sequences of positive linear operators. With these developments, many new types of statistical convergence have been added to the literature ([4], [5], [9], [10], [11], [12]).

Recently, in [25], Ünver and Orhan have been presented the notion of $P-$statistical convergence. Moreover, since this convergence cannot be compared with statistical convergence, they obtained meaningful results. Also, Korovkin theorem have been studied.

The summability theory is significant in approximation theory. The advantage of this method is that it corrects the lack of convergence using the matrix summability methods (see details [2], [3], [6], [19], [21], [22]).

Convolution operators are a wide area in approximation theory and Fourier analysis ([21], [26]). As a consequence, important results have been obtained with researches in many fields about covolution operators. Duman [14] studied a Korovkin theorem for positive convolution operators via the concept of $\mathcal{B}-$statistical convergence in the year 2008.

The main motivation for this article is to study $P-$statistical Korovkin theorems via $\mathcal{A}-$summation process for a sequence of positive linear convolution operators. We compare this new convergence with statistical convergence, hence we will get a different perspective.

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Now, we pause to collect some basic concepts and notations.

Let $\mathcal{B}=(b_{k,j})$ be an infinite summability matrix and $x:=(x_j)$ be a sequence. The $\mathcal{B}-$transform of x is given by $(\mathcal{B}x)_k=\sum_{j=1}^{\infty} b_{k,j}x_j$ provided the series converges for each $k\in\mathbb{N}$ and denoted by $\mathcal{B}x:=((\mathcal{B}x)_k)$. Also, we say that $\mathcal{B}$ is regular if $\lim \mathcal{B}x=L$ whenever $\lim x=L$ [17]. Suppose that $\mathcal{B}$ is a non-negative regular summability matrix. Then it is said that the sequence x is $\mathcal{B}-$statistically convergent to L provided that, for every $\varepsilon>0$,

$$\lim_k \sum_{j:\,|x_j-L|\geq\varepsilon} b_{k,j}=0.$$

We shall write $st_{\mathcal{B}}-\lim x=L$. Observe that if we take $\mathcal{B}=C_1:=(c_{k,j})$ the Cesaro matrix defined by

$$c_{kj}:=\begin{cases}\frac{1}{k}, & \text{if } 1\leq j\leq k,\\ 0, & \text{otherwise,}\end{cases}$$

this convergence reduces to the concept of statistical convergence. In this manner can write $st-\lim x=L$.

Similarly, classical convergence is obtained by taking $\mathcal{B}=I$ where I is identity matrix. We shall write $\lim x=L$.

Now let's examine the definitions and properties of the power series method.

Let (p_j) be a non-negative real sequence with $p_0>0$. We assume that the power series $p(s):=\sum_{j=0}^{\infty} p_j s^j$ has radius of convergence $R\in(0,\infty.]$.

Let

$$C_p:=\left\{f:(-R,R)\to\mathbb{R}:\lim_{0<s\to R^-}\frac{1}{p(s)}f(s)\ \text{exists}\right\}$$

and

$$C_{p_p}:=\left\{x=(x_j):p_x(s):=\sum_{j=0}^{\infty}p_j s^j x_j \text{ has radius of convergence }\geq R\text{ and }p_x\in C_p\right\}.$$

The functional $P-\lim:C_{p_p}\to\mathbb{R}$ defined by

$$P-\lim x=\lim_{0<s\to R^-}\frac{1}{p(s)}\sum_{j=0}^{\infty}p_j s^j x_j$$

is a power series method also x is called $P-$convergent [8]. It is worth noting that the power series method is regular if and only if

$$\lim_{0<s\to R^-}\frac{p_j s^j}{p(s)}=0,\ \text{for each } j\in\mathbb{N}_0.$$

Here and in the sequel, P will be a regular power series method.

Let us now recall $P-$statistical convergence given by Ünver and Orhan ([25]), which cannot be compared with statistical convergence:

$\delta_P(E)$ is called the $P-$density of E, if there exists the following limit

$$\delta_P(E):=\lim_{0<s\to R^-}\frac{1}{p(s)}\sum_{j\in E}p_j s^j$$

where $E\subset\mathbb{N}_0$.

Let $x=(x_j)$ be a real sequence. Then x said to be $P-$statistically convergent to L if for any $\varepsilon>0$

$$\lim_{0<s\to R^-}\frac{1}{p(s)}\sum_{j:|x_j-L|\geq\varepsilon}p_j s^j=0.$$

We can write $st_P-\lim x=L$.

With the example below it can be seen that statistical convergence and $P-$ statistical convergence cannot be compared.

Example 1 Let (p_j) be defined as follows

$$p_j = \begin{cases} 1, & j = k^2, \\ 0, & \text{otherwise}, \end{cases} \quad k \in \mathbb{N}_0, \tag{1}$$

and take the sequence $\alpha = (\alpha_j)$ defined by

$$\alpha_j = \begin{cases} 0, & j = k^2, \\ 1, & \text{otherwise}, \end{cases} \quad k \in \mathbb{N}_0. \tag{2}$$

We calculate that, since for any $\varepsilon > 0$, $\lim\limits_{0<s\to R^-} \frac{1}{p(s)} \sum\limits_{j:|x_j - L| \geq \varepsilon} p_j s^j = 0$, α is $P-$ statistically convergent to 0. It is a simple matter to see that α is not statistically convergent to 0. On the other hand let $\beta = (\beta_j)$ be a sequence defined by

$$\beta_j = \begin{cases} 1, & j = k^2 \\ 0, & \text{otherwise} \end{cases} \quad k \in \mathbb{N}_0.$$

It is not difficult to observe that β statistically convergent to 0 however it is not $P-$statistically convergent.

Let $\mathcal{A} := (A^{(n)}) = \left(a_{k,j}^{(n)}\right)$ be a sequence of infinite non-negative real matrices. For a sequence of real numbers $x = (x_j)$,

$$\mathcal{A}x := \left\{ (Ax)_k^n : \ k, n \in \mathbb{N} \right\}$$

defined by $(Ax)_k^n := \sum\limits_{j=1}^{\infty} a_{k,j}^{(n)} x_j$ is called the $\mathcal{A}-$transform of x whenever the series converges for all k and n. A sequence x is $\mathcal{A}-$summable to L if

$$\lim_k \sum_{j=1}^{\infty} a_{k,j}^{(n)} x_j = L,$$

uniformly in n ([7], [23]).

If $A^{(n)} = \mathcal{B}$ for some matrix $\mathcal{B}$, then $\mathcal{A}-$summability is the ordinary matrix summability by $\mathcal{B}$. If $a_{k,j}^{(n)} = \frac{1}{k}$, for $n \leq j \leq k + n - 1$ $(n = 1, 2, ...)$, and $a_{k,j}^{(n)} = 0$ otherwise, then $\mathcal{A}-$summability reduces to almost convergence [20].

Let $C[a, b]$ be the space of all continuous real valued functions and denote by $B[a, b]$ the space all bounded real valued functions on $[a, b]$. Then, as it is well known, $C[a, b]$ is Banach space with norm

$$\|f\| := \sup_{x \in [a,b]} |f(x)|, \ \ f \in C[a, b].$$

Now, let L be a linear operator from $C[a, b]$ into $C[a, b]$. An operator L is positive linear provided that $f \geq 0$ implies $L(f; x) \geq 0$. Also, we express the value of $L(f)$ at a point $x \in [a, b]$ by $L(f; x)$. A sequence (L_j) of positive linear operators of $C[a, b]$ into $B[a, b]$ is said to be an $\mathcal{A}-$ summation process on $C[a, b]$ if $(L_j(f))$ is $\mathcal{A}-$summable to f for every $f \in C[a, b]$, namely,

$$\lim_k \left\| \sum_{j=1}^{\infty} a_{k,j}^{(n)} L_j(f) - f \right\| = 0,$$

uniformly in n, where the series converges for each k, n and f.

Now, let us state that the convolution operators defined on $C[a, b]$ are as follows:

$$L_j(f; x) = \int_a^b f(y) K_j(y - x) \, dy, \ \ j \in \mathbb{N}, \ x \in [a, b] \text{ and } f \in C[a, b], \tag{3}$$

where a and b are two real numbers such that $a < b$.

In what follows, K_j is a continuous function on $[a-b, b-a]$ and also that $K_j(u) \geq 0$ for all $j \in \mathbb{N}$ and for every $u \in [a-b, b-a]$. Note at this point that if $x, y \in [a, b]$ then $u := y - x \in [a-b, b-a]$. Also, the afore-mentioned L_j from (3) is a positive linear operator, for all $j \in \mathbb{N}$.

Let (L_j) be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$ such that for each $k, n \in \mathbb{N}$

$$\sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\| L_j(1) \right\| < \infty. \tag{4}$$

Furthermore, for each $k, n \in \mathbb{N}$ and $f \in C[a, b]$, let

$$B_k^{(n)}(f; x) = \sum_{j=1}^{\infty} a_{k,j}^{(n)} L_j(f; x)$$

which is well defined by (4), and $B_k^{(n)}(f) \in B[a, b]$.

In the remaining two sections we deal with Korovkin type approximation theorems, examples which show that our results are more sense, and the rate of convergence.

2 Approximation via $P-$statistical $\mathcal{A}-$summation process

In this section, we express our main theorem and striking examples. Before our main theorem, let's give some of the lemmas and theorem that are required.

We have initially considered the following theorem:

Theorem 1 *Let $\mathcal{A} = (A^{(n)})$ be a sequence of infinite matrices with non-negative real entries and let (L_j) be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$ for which (4) holds. If*

$$st_P - \lim \left\| B_k^{(n)}(f_0) - f_0 \right\| = 0 \textit{ with } f_0(y) = 1, \textit{ uniformly in } n$$

and

$$st_P - \lim \left\| B_k^{(n)}(\phi) \right\| = 0 \textit{ with } \phi(y) := (y - x)^2, \textit{ uniformly in } n$$

then for all $f \in C[a, b]$, we have

$$st_P - \lim \left\| B_k^{(n)}(f) - f \right\| = 0, \textit{ uniformly in } n.$$

Proof Let $f \in C[a, b]$ and $x \in [a, b]$. Since f is continuous on $[a, b]$, for every $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for y with $|y - x| \leq \delta$. Letting $I_\delta := [x - \delta, x + \delta] \cap [a, b]$, we can express it as

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f(x)| \chi_{I_\delta}(y) + |f(y) - f(x)| \chi_{[a,b] \setminus I_\delta}(y) \\ &\leq \varepsilon + 2M\delta^{-2}(y - x)^2 \end{aligned} \tag{5}$$

where $M := \|f\|$. In view of (5) and from positivity and linearity of the operator L_j, we have

$$\begin{aligned} \left| \sum_{j=1}^{\infty} a_{k,j}^{(n)} L_j(f; x) - f(x) \right| = \left| B_k^{(n)}(f; x) - f(x) \right| &\leq B_k^{(n)}(|f(y) - f(x)|; x) \\ &\quad + |f(x)| \left| B_k^{(n)}(f_0; x) - f_0(x) \right| \\ &\leq \varepsilon B_k^{(n)}(f_0; x) + \frac{2M}{\delta^2} B_k^{(n)}(\phi; x) \\ &\quad + M \left| B_k^{(n)}(f_0; x) - f_0(x) \right| \\ &\leq \varepsilon + \alpha \left\{ \left| B_k^{(n)}(f_0; x) - f_0(x) \right| + B_k^{(n)}(\phi; x) \right\} \end{aligned}$$

where $\alpha := \max\left\{\varepsilon + M, \frac{2M}{\delta^2}\right\}$. By this inequality, it is straightforward to see that

$$\left\| B_k^{(n)}(f) - f \right\| \le \varepsilon + \alpha \left\{ \left\| B_k^{(n)}(f_0) - f_0 \right\| + \left\| B_k^{(n)}(\phi) \right\| \right\}. \tag{6}$$

Given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Define

$$\begin{aligned}
D &:= \left\{ k : \left\| B_k^{(n)}(f) - f \right\| \ge r \right\}, \\
D_1 &:= \left\{ k : \left\| B_k^{(n)}(f_0) - f_0 \right\| \ge \frac{r-\varepsilon}{2\alpha} \right\}, \\
D_2 &:= \left\{ k : \left\| B_k^{(n)}(\phi) \right\| \ge \frac{r-\varepsilon}{2\alpha} \right\}.
\end{aligned}$$

Then in view of (6), it can easily be verified that $D \subseteq D_1 \cup D_2$. So, we have

$$\frac{1}{p(s)} \sum_{k\in D} p_k s^k \le \frac{1}{p(s)} \sum_{k\in D_1} p_k s^k + \frac{1}{p(s)} \sum_{k\in D_2} p_k s^k. \tag{7}$$

Letting $0 < s \to R^-$ in (7) and using the hypotheses we get

$$\frac{1}{p(s)} \sum_{k\in D} p_k s^k = 0,$$

which yields the proof. □

Let δ be a positive real number so that $\delta < \frac{b-a}{2}$, and let

$$\|f\|_\delta := \sup_{a+\delta \le x \le b-\delta} |f(x)|, \quad f \in C[a,b].$$

Now, as the main result of the paper, we prove the following lemmas:

Lemma 1 *Let $\mathcal{A} = \left(A^{(n)}\right)$ be the same as in Theorem 1 and δ be a fixed positive number such that $\delta < \frac{b-a}{2}$. Assume that $\left(L_j\right)$ is a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ for which (4) holds. If*

$$st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \int_{-\delta}^{\delta} K_j(y)\,dy \right\} = 1, \text{ uniformly in } n \tag{8}$$

and

$$st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \sup_{|y|\ge\delta} K_j(y) \right\} = 0, \text{ uniformly in } n \tag{9}$$

then we have

$$st_P - \lim \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta = 0, \text{ uniformly in } n, \text{ with } f_0(y) = 1.$$

Proof Fix $0 < \delta < \frac{b-a}{2}$ and let $x \in [a+\delta, b-\delta]$. In that case it is simple to see

$$-(b-a) \le a - x \le -\delta \tag{10}$$

and

$$\delta \le b - x \le b - a. \tag{11}$$

It can be writen, for all $j \in \mathbb{N}$,

$$B_k^{(n)}(f_0; x) = \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_a^b K_j(y-x)\,dy = \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{a-x}^{b-x} K_j(y)\,dy$$

Hence we get

$$\sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-\delta}^{\delta} K_j(y)\,dy \le B_k^{(n)}(f_0; x) \le \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-(b-a)}^{(b-a)} K_j(y)\,dy.$$

These inequalities imply that

$$\left\| B_k^{(n)}(f_0) - f_0 \right\|_{\delta} \le u_k^{(n)} \tag{12}$$

where

$$u_k^{(n)} := \max\left\{ \left| \left\{ \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-\delta}^{\delta} K_j(y)\,dy \right\} - 1 \right|, \left| \left\{ \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-(b-a)}^{(b-a)} K_j(y)\,dy \right\} - 1 \right| \right\}.$$

As an immediate consequence of the hypothesis, we have

$$st_P - \lim u_k^{(n)} = 0, \text{ uniformly in } n. \tag{13}$$

Now, for a given $\varepsilon > 0$, we get from (12) that

$$R := \left\{ k : \left\| B_k^{(n)}(f_0) - f_0 \right\|_{\delta} \ge \varepsilon \right\} \subseteq \left\{ k : u_k^{(n)} \ge \varepsilon \right\} =: R'.$$

Then, we have

$$\frac{1}{p(s)} \sum_{k \in R} p_k s^k \le \frac{1}{p(s)} \sum_{k \in R'} p_k s^k. \tag{14}$$

Taking the limit as $0 < s \to R^-$ in (14) and using (13) we conclude this result as follows. □

Lemma 2 *Let $\mathcal{A} = \left(A^{(n)}\right)$ be the same as in Theorem 1 and δ be a fixed positive number such that $\delta < \frac{b-a}{2}$. Assume that $\left(L_j\right)$ is a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ for which (4) holds. If*

$$st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \int_{-\delta}^{\delta} K_j(y)\,dy \right\} = 1, \text{ uniformly in } n$$

and

$$st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \sup_{|y| \ge \delta} K_j(y) \right\} = 0, \text{ uniformly in } n$$

then we have

$$st_P - \lim \left\| B_k^{(n)}(\phi) \right\|_{\delta} = 0, \text{ uniformly in } n,$$

where $\phi(y) := (y-x)^2$.

Proof For a fixed $0 < \delta < \frac{b-a}{2}$, let $x \in [a+\delta, b-\delta]$. It should also be noted that, for $x \in [a+\delta, b-\delta]$, since $\phi(y) = y^2 - 2xy + x^2$, it is obvious that $\phi \in C[a,b]$. So we can compute $L_j(\phi; x)$. Actually, $L_j(\phi; x) = L_j(f_2; x) - 2xL_j(f_1; x) + x^2 L_j(f_0; x)$ with $f_i(y) = y^i$ $(i = 0, 1, 2)$.

$$
\begin{aligned}
B_k^{(n)}(\phi; x) &= \sum_{j=1}^{\infty} a_{k,j}^{(n)} L_j(\phi; x) \\
&= \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_a^b (y-x)^2 K_j(y-x)\,dy \\
&= \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{a-x}^{b-x} y^2 K_j(y)\,dy \\
&\le \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-(b-a)}^{(b-a)} y^2 K_j(y)\,dy.
\end{aligned} \tag{15}
$$

Since the function f_2 is continuous at $y = 0$, given $\varepsilon > 0$, $\left(0 < \sqrt{\varepsilon} < \delta\right)$, $\phi(y) < \varepsilon$ whenever $|y| \le \sqrt{\varepsilon}$. Then we may write that

$$
\begin{aligned}
B_k^{(n)}(\phi; x) &\le \varepsilon \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{|y|\le\sqrt{\varepsilon}} K_j(y)\,dy + \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{|y|\ge\sqrt{\varepsilon}} y^2 K_j(y)\,dy \\
&\le \varepsilon \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-\delta}^{\delta} K_j(y)\,dy \\
&\quad + \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \sup_{|y|\ge\sqrt{\varepsilon}} K_j(y) \right\} \int_{-(b-a)}^{(b-a)} y^2 dy \\
&= \varepsilon \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-\delta}^{\delta} K_j(y)\,dy + M \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \sup_{|y|\ge\sqrt{\varepsilon}} K_j(y) \right\},
\end{aligned}
$$

where $M = \int_{-(b-a)}^{(b-a)} y^2 dy$. Now taking the $P-$statistical limit and by the hypotheses, the proof is completed. □

It should also be noted that the following main result follows from Theorem 1, Lemma 1 and Lemma 2 at once.

Theorem 2 *Let P be a regular power series method. Let $\mathcal{A} = \left(A^{(n)}\right)$ be a sequence of infinite matrices with non-negative real entries and δ be a fixed positive number such that $\delta < \frac{b-a}{2}$. Assume that $\left(L_j\right)$ is a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ for which (4) holds. If*

$$
st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \int_{-\delta}^{\delta} K_j(y)\,dy \right\} = 1, \ \textit{uniformly in } n, \tag{16}
$$

and

$$
st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \sup_{|y|\ge\delta} K_j(y) \right\} = 0, \ \textit{uniformly in } n \tag{17}
$$

then we have

$$st_P - \lim \left\| B_k^{(n)}(f) - f \right\|_\delta = 0, \text{ uniformly in } n. \tag{18}$$

Using the similar techniques in the proof of Theorem 1, Lemma 1 and Lemma 2 as Theorem 2, then we can get the following results:

Corollary 1 *Let δ be a fixed positive number such that $\delta < \frac{b-a}{2}$. If the conditions*

$$st_P - \lim \int_{-\delta}^{\delta} K_j(y-x)\,dy = 1$$

and

$$st_P - \lim \left(\sup_{|y| \geq \delta} K_j(y) \right) = 0$$

hold, then for all $f \in C[a,b]$, we have

$$st_P - \lim \left\| L_j(f) - f \right\|_\delta = 0.$$

Corollary 2 *Let $\mathcal{A} = \left(A^{(n)}\right)$ be the same as in Theorem 1 and δ be a fixed positive number such that $\delta < \frac{b-a}{2}$. Assume that $\left(L_j\right)$ is a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ for which (4) holds. If*

$$\lim_k \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \int_{-\delta}^{\delta} K_j(y)\,dy \right\} = 1, \text{ uniformly in } n, \tag{19}$$

and

$$\lim_k \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \sup_{|y| \geq \delta} K_j(y) \right\} = 0, \text{ uniformly in } n, \tag{20}$$

then we have

$$\lim_k \left\| B_k^{(n)}(f) - f \right\|_\delta = 0, \text{ uniformly in } n. \tag{21}$$

Let us now present an example that both satisfies our main theorem and shows that our new method is incomparable with the Theorem 2.4 in [14] .

Example 2 Let p_j, $\alpha = \left(\alpha_j\right)$ and $\beta = \left(\beta_j\right)$ be the same as in Example 1. Also, let $\mathcal{A} = \left(A^{(n)}\right) = \left(a_{k,j}^{(n)}\right)$ be a sequence of infinite matrices defined by $a_{k,j}^{(n)} = \frac{1}{k}$, if $n \leq j \leq k+n-1$, and $a_{k,j}^{(n)} = 0$ otherwise. Now let the operators L_j on $C[a,b]$ be defined by

$$L_j(f;x) = \frac{j\left(1+\alpha_j\right)}{\sqrt{\pi}} \int_a^b f(y)\, e^{-j^2(y-x)^2}\,dy. \tag{22}$$

If we choose

$$K_j(y) = \frac{j\left(1+\alpha_j\right)}{\sqrt{\pi}} e^{-j^2 y^2}, \tag{23}$$

then the operators L_j given by (22) have form of the convolution operators as in (3). For every $\delta > 0$ such that $\delta < \frac{b-a}{2}$, we have

$$\begin{aligned} \int_{-\delta}^{\delta} K_j(y)\,dy &= \frac{j\left(1+\alpha_j\right)}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} e^{-j^2y^2}dy - \int_{|y|\geq\delta} e^{-j^2y^2}dy \right) \\ &= \frac{2\left(1+\alpha_j\right)}{\sqrt{\pi}} \left(\int_{0}^{\infty} e^{-y^2}dy - \int_{j\delta}^{\infty} e^{-y^2}dy \right). \end{aligned}$$

Since $\int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2} < \infty$, it is clear that $\lim_j \int_{j\delta}^\infty e^{-y^2}dy = 0$. Then we get

$$\begin{aligned} \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-\delta}^{\delta} K_j(y)\,dy &= \frac{1}{k} \sum_{j=n}^{k+n-1} \left(1+\alpha_j\right) \\ &= 1 + \frac{1}{k} \sum_{j=n}^{k+n-1} \alpha_j. \end{aligned}$$

Also, since $st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \alpha_j = 0$, uniformly in n, we have

$$st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \int_{-\delta}^{\delta} K_j(y)\,dy = 1, \text{ uniformly in } n,$$

which gives (16). On the other hand, we get

$$\begin{aligned} \sup_{|y|\geq\delta} K_j(y) &= \frac{j\left(1+\alpha_j\right)}{\sqrt{\pi}} \sup_{|y|\geq\delta} e^{-j^2y^2} \\ &\leq \frac{j\left(1+\alpha_j\right)}{e^{j^2\delta^2}}. \end{aligned}$$

Since $\lim_j \frac{j}{e^{j^2\delta^2}} = 0$ and $st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left(1+\alpha_j\right) = 1$, we conclude that

$$st_P - \lim \sum_{j=1}^{\infty} a_{k,j}^{(n)} \left\{ \sup_{|y|\geq\delta} K_j(y) \right\} = 0, \text{ uniformly in } n,$$

hence (17) holds. Hence the conditions in Theorem 2 are satisfied. Also, since $\left(\alpha_j\right)$ is not statistical convergent and classical convergent, the function K_j given by (23) does not satisfy the hypotheses of Corollary 2 and Theorem 2.4 in [14]. Conversely, if $\left(\beta_j\right)$ is taken instead of $\left(\alpha_j\right)$ in the K_j convolution operators given by (23) and since $\left(\beta_j\right)$ is statistically convergent, observe that the functions K_j given by (23) satisfies the hypotheses of Theorem 2.4 in [14]. Moreover, it does not provide our main theorem. Therefore these two methods cannot be compared depending on the choice of $\left(p_j\right)$.

3 Rate of convergence

We give some estimates of rate for $P-$statistical Korovkin theorem via $\mathcal{A}-$summation process for a sequence of positive linear convolution operators with the help of modulus of continuity.

Let $f \in C[a,b]$. The modulus of continuity, denoted by $\omega(f,\alpha)$ is defined by

$$\omega(f,\alpha) = \sup_{|y-x|\leq\alpha} |f(y) - f(x)|.$$

As is well known, if $f \in C[a,b]$, then

$$\lim_{\alpha\to 0} \omega(f,\alpha) = 0, \tag{24}$$

and that for any constant $\lambda > 0$, $\alpha > 0$,

$$\omega(f;\alpha) \leq (1+[\lambda])\omega(f;\alpha) \tag{25}$$

where the symbol $[\lambda]$ stands for the greatest integer less than or equal to λ (see also [1]).

Let's now present the following.

Theorem 3 *Let $A = \left(A^{(n)}\right)$ be a sequence of infinite matrices with non-negative real entries and δ be a fixed positive number such that $\delta < \frac{b-a}{2}$. Suppose that $\left(L_j\right)$ is a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ for which (4) holds. The following conditions hold:*

i) $st_P - \lim \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta = 0$, uniformly in n,

ii) $st_P - \lim \left\| w\left(f;\alpha_k^{(n)}\right) \right\|_\delta = 0$, uniformly in n, where $\alpha := \alpha_k^{(n)} = \sqrt{\left\| B_k^{(n)}\left((y-.)^2\right) \right\|_\delta}$, then for all $f \in C[a,b]$, we get

$$st_P - \lim \left\| B_k^{(n)}(f) - f \right\|_\delta = 0, \text{ uniformly in } n.$$

Proof Let $0 < \delta < \frac{b-a}{2}$, $f \in C[a,b]$ and $x \in [a+\delta, b-\delta]$. By positivity and linearity of operators L_j and in view of (25), we get for any $\alpha > 0$, that

$$\begin{aligned}
\left| B_k^{(n)}(f;x) - f(x) \right| &\leq B_k^{(n)}(|f(y) - f(x)|;x) + |f(x)| \left| B_k^{(n)}(f_0) - f_0 \right| \\
&\leq B_k^{(n)}\left(w\left(f; \alpha \frac{|y-x|}{\alpha}; x\right)\right) + |f(x)| \left| B_k^{(n)}(f_0) - f_0 \right| \\
&\leq w(f;\alpha) B_k^{(n)}\left(1 + \frac{(y-x)^2}{\alpha^2}; x\right) + |f(x)| \left| B_k^{(n)}(f_0) - f_0 \right| \\
&\leq w(f;\alpha) \left\{ B_k^{(n)}(f_0;x) + \frac{1}{\alpha^2} B_k^{(n)}\left((y-x)^2;x\right)\right\} + |f(x)| \left| B_k^{(n)}(f_0) - f_0 \right|.
\end{aligned}$$

In the sequel for all $n \in \mathbb{N}$,

$$\left\| B_k^{(n)}(f) - f \right\|_\delta \leq w(f;\alpha) \left\{ \left\| B_k^{(n)} f_0 \right\|_\delta + \left\| B_k^{(n)}\left((y-.)^2\right) \right\|_\delta \right\} + M_1 \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta \tag{26}$$

where $M_1 := \|f\|_\delta$. Now letting $\alpha = \alpha_k^{(n)} = \sqrt{\left\| B_k^{(n)}\left((y-.)^2\right) \right\|_\delta}$, we have

$$\begin{aligned}
\left\| B_k^{(n)}(f) - f \right\|_\delta &\leq w\left(f;\alpha_k^{(n)}\right) \left\{ \left\| B_k^{(n)} f_0 \right\|_\delta + 1 \right\} + M_1 \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta \\
&\leq 2w\left(f;\alpha_k^{(n)}\right) + w\left(f;\alpha_k^{(n)}\right) \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta + M_1 \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta.
\end{aligned}$$

Let $M := \max\{2, M_1\}$. Then we can express for all $n \in \mathbb{N}$,

$$\left\| B_k^{(n)}(f) - f \right\|_\delta \leq M \left\{ w\left(f;\alpha_k^{(n)}\right) + w\left(f;\alpha_k^{(n)}\right) \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta + \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta \right\}. \tag{27}$$

Given $\varepsilon > 0$, define the following sets:

$$E := \left\{ k : \left\| B_k^{(n)}(f) - f \right\|_\delta \geq \varepsilon \right\},$$
$$E_1 := \left\{ k : w\left(f; \alpha_k^{(n)} \right) \geq \frac{\varepsilon}{3M} \right\},$$
$$E_2 := \left\{ k : w\left(f; \alpha_k^{(n)} \right) \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta \geq \frac{\varepsilon}{3M} \right\},$$
$$E_3 := \left\{ k : \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta \geq \frac{\varepsilon}{3M} \right\},$$

Then we easily see from (27) that $E \subseteq E_1 \cup E_2 \cup E_3$. Also defining

$$E_2^{'} := \left\{ k : w\left(f; \alpha_k^{(n)} \right) \geq \sqrt{\frac{\varepsilon}{3M}} \right\},$$
$$E_2^{''} := \left\{ k : \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta \geq \sqrt{\frac{\varepsilon}{3M}} \right\},$$

one can deduce that $E_2 \subseteq E_2^{'} \cup E_2^{''}$. Hence we get $E \subseteq E_1 \cup E_2^{'} \cup E_2^{''} \cup E_3$. So we get

$$\frac{1}{p(s)} \sum_{k \in E} p_k s^k \leq \frac{1}{p(s)} \sum_{k \in E_1} p_k s^k + \frac{1}{p(s)} \sum_{k \in E_2^{'}} p_k s^k + \frac{1}{p(s)} \sum_{k \in E_2^{''}} p_k s^k + \frac{1}{p(s)} \sum_{k \in E_3} p_k s^k. \tag{28}$$

Letting $0 < s \to R^-$ in (28) we have $st_P - \lim \left\| B_k^{(n)}(f) - f \right\|_\delta = 0$, uniformly in n, and the proof is complete. □

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