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Statistical σ approximation to max-product operators

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ABSTRACT

In this paper, using the concept of statistical σ-convergence which is stronger than statistical convergence, we obtain a statistical σ approximation theorem for a general sequence of max-product operators, including Shepard type operators, although its classical limit fails. We also compute the corresponding statistical σ rates of the approximation.

1. Introduction

Approximation theory, which has a close relationship with other branches of mathematics, has been used in the theory of polynomial approximation and various domains of functional analysis [1], and in numerical studies of differential and integral operators [2]. In the classical approximation theory, many well-known approximating operators obey the linearity condition. In recent years, Bede et al. [3] have shown that it is possible to find some approximating operators that are not linear, such as the max-product and max–min Shepard type approximating operators. Actually, these operators are pseudo-linear which is a quite effective structure in solving the problems in many branches of applied mathematics, such as image processing [4], differential equations [5,6], idempotent analysis [7] and approximation theory [3,8]. However, so far almost all results regarding approximations by pseudo linear operators are based on the validity of the classical limit of the operators. Using the notion of statistical convergence Duman [9] obtained various statistical approximation theorems for a general sequence of max-product approximating operators, including Shepard type operators, although its classical limit fails. Recently a kind of statistical convergence (statistical σ-convergence) which is stronger than statistical convergence has been introduced by Mursaleen and Edely [10]. In this paper, using statistical σ-convergence, we improve the Duman's results in [9].

Let K be the subset of positive integers. Then the natural density of K is given by $\delta(K) = \lim_n \frac{1}{n}|K_n|$ if it exists, where $K_n := \{k \in K : k \leq n\}$ and the vertical bars denote the cardinality of the set K_n. A sequence $x = \{x_k\}$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

i.e. if the set $K = K(\varepsilon) := \{k \leq n : |x_k - L| \geq \varepsilon\}$ has natural density zero [11–13]. In this case we write $\text{st} - \lim x_k = L$.

Let σ be a mapping of the set of $\mathbb{N}$ into itself. A continuous linear functional φ defined on the space l_∞ of all bounded sequences is called an invariant mean (or σ-mean) [14] if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = \{x_k\}$ has $x_k \geq 0$ for all k,

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(ii) $\varphi(e) = 1$, where $e = (1, 1, \ldots)$,
(iii) $\varphi(x) = \varphi((x_{\sigma(n)}))$ for all $x \in l_\infty$.

Thus, the σ-mean extends the limit functional on c of all convergent sequences in the sense that $\varphi(x) = \lim x$ for all $x \in c$ [15]. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal. It is known [16] that

$$V_\sigma = \{x \in l_\infty : \lim_p t_{pm}(x) = L \text{ uniformly in } m, L = \sigma - \lim x\}$$

where

$$t_{pm}(x) := \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^p(m)}}{p+1}.$$

We say that a bounded sequence $x = \{x_k\}$ is σ-convergent if and only if $x \in V_\sigma$. Let

$$V_\sigma^s = \{x \in l_\infty : \text{st} - \lim_p t_{pm}(x) = L \text{ uniformly in } m, L = \sigma - \lim x\}.$$

A sequence $x = \{x_k\}$ is said to be statistically σ- convergent to L if and only if $x \in V_\sigma^s$ (see, for details, [10]). In this case we write $\delta(\sigma) - \lim x_k = L$. That is,

$$\lim_n \frac{1}{n} |\{p \le n : |t_{pm}(x) - L| \ge \varepsilon\}| = 0, \quad \text{uniformly in } m.$$

Using the above definitions, the next result follows immediately.

Lemma 1. *Statistical convergence implies statistical σ-convergence.*

However, one can construct an example which guarantees that the converse of Lemma 1 is not always true. Such an example was given in [10] as follows:

Example 2. Consider the case $\sigma(n) = n + 1$ and the sequence $u = \{u_m\}$ defined as

$$u_m = \begin{cases} 1, & \text{if } m \text{ is odd}, \\ -1, & \text{if } m \text{ is even}, \end{cases} \tag{1}$$

is statistically σ-convergence ($\delta(\sigma) - \lim u_m = 0$) but it is neither convergent nor statistically convergent.

2. Statistical σ-approximation of max-product operators

Let (X, d) be an arbitrary compact metric space. By $C(X, [0, \infty))$ we denote the space of all non-negative continuous functions on X. Then we consider the following max-product operators:

$$L_n(f; x) = \bigvee_{k=0}^{n} K_n(x, x_k) \cdot f(x_k), \quad x \in X \text{ and } f \in C(X, [0, \infty)), \tag{2}$$

where $x_k \in X$, $k = 0, 1, \ldots, n$, are the knots; and $K_n(x, x_k)$ are non-negative continuous functions on X having relatively simple expression (algebraic or trigonometric polynomials, rational functions, wavelets, etc.) such that, for any $x \in X$,

$$\delta\left(\left\{p \in \mathbb{N} : \bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k) = 1\right\}\right) = 1 \quad \text{for every } m \in \mathbb{N}. \tag{3}$$

holds. Observe that the operators mapping $C(X, [0, \infty))$ into $C(X, [0, \infty))$ are pseudo-linear, i.e., for every $f, g \in C(X, [0, \infty))$ and for any non-negative numbers α, β,

$$L_n\left(\alpha \cdot f \bigvee \beta \cdot g; x\right) = \alpha \cdot L_n(f; x) \bigvee \beta \cdot L_n(g; x)$$

is satisfied (see [3]).

We first recall the following lemma.

Lemma 3 (*[3]*)**.** *For any $a_k, b_k \in [0, \infty)$, $k = 0, 1, \ldots, n$, we have*

$$\left| \bigvee_{k=0}^{n} a_k - \bigvee_{k=0}^{n} b_k \right| \le \bigvee_{k=0}^{n} |a_k - b_k|.$$

Now we get the following result for the max-product operators.

Theorem 4. *Let (X, d) be an arbitrary compact metric space. If, for the operators $L := \{L_n\}$ given by (2) and (3),*

$$\delta(\sigma) - \lim \left\{ \bigvee \{|L_n(\varphi_x; x)| : x \in X\} \right\} = 0 \quad \text{with } \varphi_x(y) = d^2(y, x) \tag{4}$$

then, for all $f \in C(X, [0, \infty))$, we have

$$\delta(\sigma) - \lim \left\{ \bigvee \{|L_n(f; x) - f(x)| : x \in X\} \right\} = 0.$$

Proof. Let $f \in C(X, [0, \infty))$ and $x \in X$ be fixed. Then, using the continuity of f and also considering the compactness of X, we immediately see that, for a given $\varepsilon > 0$, there exists a positive number δ such that

$$|f(y) - f(x)| \le \varepsilon + \frac{2M_f}{\delta^2}\varphi_x(y) \tag{5}$$

holds for all $y \in X$, where $M_f := \bigvee \{|f(y)| : y \in X\}$. Now put

$$K := \left\{ p \in \mathbb{N} : \bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k) = 1 \right\} \tag{6}$$

for every $m \in \mathbb{N}$. Then, by (3) we may write that

$$\lim_n \frac{|\{p \le n : p \in K\}|}{n} = 1, \qquad \lim_n \frac{|\{p \le n : p \in \mathbb{N}/K\}|}{n} = 0 \tag{7}$$

for every $m \in \mathbb{N}$. So, by (3) and (5) and Lemma 3, we get for all $p \in K$, that

$$\begin{aligned}
&\left| t_{pm}(L(f; x)) - f(x) \right| \\
&= \left| \frac{L_m(f; x) + L_{\sigma(m)}(f; x) + \cdots + L_{\sigma^p(m)}(f; x)}{p+1} - f(x) \right| \\
&= \left| \frac{(L_m(f; x) - f(x)) + \left(L_{\sigma(m)}(f; x) - f(x)\right) + \cdots + \left(L_{\sigma^p(m)}(f; x) - f(x)\right)}{p+1} \right| \\
&= \left| \frac{\left(\bigvee_{k=0}^{m} K_m(x, x_k) \cdot f(x_k) - \bigvee_{k=0}^{m} K_m(x, x_k) \cdot f(x) \right)}{p+1} + \frac{\left(\bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k) \cdot f(x_k) - \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k) \cdot f(x) \right)}{p+1} + \cdots \right. \\
&\qquad \left. + \frac{\left(\bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k) \cdot f(x_k) - \bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k) \cdot f(x) \right)}{p+1} \right| \\
&\le \frac{\bigvee_{k=0}^{m} K_m(x, x_k) \cdot |f(x_k) - f(x)| + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k) \cdot |f(x_k) - f(x)|}{p+1} + \cdots + \frac{\bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k) \cdot |f(x_k) - f(x)|}{p+1} \\
&\le \frac{\bigvee_{k=0}^{m} K_m(x, x_k) \cdot \left(\varepsilon + \frac{2M_f}{\delta^2}\varphi_x(x_k) \right) + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k) \cdot \left(\varepsilon + \frac{2M_f}{\delta^2}\varphi_x(x_k) \right)}{p+1} \\
&\qquad + \cdots + \frac{\bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k) \cdot \left(\varepsilon + \frac{2M_f}{\delta^2}\varphi_x(x_k) \right)}{p+1} \\
&\le \frac{\left(\varepsilon + \frac{2M_f}{\delta^2} \bigvee_{k=0}^{m} K_m(x, x_k) \cdot \varphi_x(x_k) \right) + \left(\varepsilon + \frac{2M_f}{\delta^2} \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k) \cdot \varphi_x(x_k) \right)}{p+1} + \cdots
\end{aligned}$$

$$+\frac{\left(\varepsilon+\frac{2M_f}{\delta^2}\bigvee_{k=0}^{\sigma^p(m)}K_{\sigma^p(m)}(x,x_k)\cdot\varphi_x(x_k)\right)}{p+1}$$

$$=\varepsilon+\frac{2M_f}{\delta^2}\left[\frac{\bigvee_{k=0}^{m}K_m(x,x_k)\cdot\varphi_x(x_k)+\bigvee_{k=0}^{\sigma(m)}K_{\sigma(m)}(x,x_k)\cdot\varphi_x(x_k)}{p+1}+\cdots+\frac{\bigvee_{k=0}^{\sigma^p(m)}K_{\sigma^p(m)}(x,x_k)\cdot\varphi_x(x_k)}{p+1}\right]$$

$$=\varepsilon+\frac{2M_f}{\delta^2}\left[\frac{L_m(\varphi_x;x)+L_{\sigma(m)}(\varphi_x;x)+\cdots+L_{\sigma^p(m)}(\varphi_x;x)}{p+1}\right]$$

$$=\varepsilon+\frac{2M_f}{\delta^2}t_{pm}\left(L(\varphi_x;x)\right).$$

Now, taking the maximum over $x\in X$, the last inequality gives, for all $p\in K$, that

$$\bigvee\left\{\left|t_{pm}\left(L(f;x)\right)-f(x)\right|:x\in X\right\}\le\varepsilon+\frac{2M_f}{\delta^2}\bigvee\left\{\left|t_{pm}\left(L(\varphi_x;x)\right)\right|:x\in X\right\}. \tag{8}$$

For a given $r>0$, choose an $\varepsilon>0$ such that $\varepsilon<r$. Then, it follows from (8) that

$$\frac{\left|\left\{p\le n:\left(\bigvee\left\{\left|t_{pm}\left(L(f;x)\right)-f(x)\right|:x\in X\right\}\right)\ge r\right\}\right|}{n}$$
$$=\frac{\left|\left\{p\le n:p\in K\text{ and }\left(\bigvee\left\{\left|t_{pm}\left(L(f;x)\right)-f(x)\right|:x\in X\right\}\right)\ge r\right\}\right|}{n}$$
$$+\frac{\left|\left\{p\le n:p\in\mathbb{N}/K\text{ and }\left(\bigvee\left\{\left|t_{pm}\left(L(f;x)\right)-f(x)\right|:x\in X\right\}\right)\ge r\right\}\right|}{n}$$
$$\le\frac{\left|\left\{p\le n:p\in K\text{ and }\left(\bigvee\left\{\left|t_{pm}\left(L(\varphi_x;x)\right)\right|:x\in X\right\}\right)\ge\frac{(r-\varepsilon)\delta^2}{2M_f}\right\}\right|}{n}$$
$$+\frac{\left|\left\{p\le n:p\in\mathbb{N}/K\text{ and }\left(\bigvee\left\{\left|t_{pm}\left(L(f;x)\right)-f(x)\right|:x\in X\right\}\right)\ge r\right\}\right|}{n}$$
$$\le\frac{\left|\left\{p\le n:\left(\bigvee\left\{\left|t_{pm}\left(L(\varphi_x;x)\right)\right|:x\in X\right\}\right)\ge\frac{(r-\varepsilon)\delta^2}{2M_f}\right\}\right|}{n}+\frac{\left|\left\{p\le n:p\in\mathbb{N}/K\right\}\right|}{n}.$$

Then, using (7) and the hypothesis (4), we have

$$\lim_n\frac{\left|\left\{p\le n:\left(\bigvee\left\{\left|t_{pm}\left(L(f;x)\right)-f(x)\right|:x\in X\right\}\right)\ge r\right\}\right|}{n},\quad\text{uniformly in } m$$

for every $r>0$. The proof is complete. □

The following theorems give the classical and the statistical approximation to a function $f\in C(X,[0,\infty))$ by means of the max-product operators L_n, respectively.

Theorem 5 ([9]). *Let (X,d) be an arbitrary compact metric space. Assume that the operators L_n given by (2) satisfy the condition*

$$\bigvee_{k=0}^{n}K_n(x,x_k)=1\quad(\text{for } n\in\mathbb{N}\text{ and } x\in X).$$

If the sequence $\{L_n(\varphi_x;x)\}_{n\in\mathbb{N}}$ converges uniformly to zero function with respect to $x\in X$, then, for all $f\in C(X,[0,\infty))$, $\{L_n(f;x)\}_{n\in\mathbb{N}}$ is also uniformly convergent to $f(x)$ with respect to $x\in X$.

Theorem 6 ([9]). *Let (X,d) be an arbitrary compact metric space. If, for the operators L_n given by (2) and*

$$\delta\left(\left\{n\in\mathbb{N}:\bigvee_{k=0}^{n}K_n(x,x_k)=1\right\}\right)=1,$$

$$st-\lim\left\{\bigvee\left\{\left|L_n(\varphi_x;x)\right|:x\in X\right\}\right\}=0\quad\textit{with } \varphi_x(y)=d^2(y,x),$$

then, for all $f \in C(X,[0,\infty))$, we have

$$st-\lim\left\{\bigvee\{|L_n(f;x)-f(x)|:x\in X\}\right\}=0.$$

Remark 7. We now show that our result Theorem 4 is stronger than its classical version (Theorem 5) and statistical version (Theorem 6).

Let (X,d) be an arbitrary compact metric space. Consider the Shepard-type max-product operators (see [8]) as follows:

$$S_m^{\lambda}(f;x)=\bigvee_{k=0}^{m}\left(\frac{\frac{1}{d^{\lambda}(x,x_k)}}{\bigvee_{j=0}^{m}\frac{1}{d^{\lambda}(x,x_j)}}\right)\cdot f(x_k)=\frac{\bigvee_{k=0}^{m}\frac{f(x_k)}{d^{\lambda}(x,x_k)}}{\bigvee_{j=0}^{m}\frac{1}{d^{\lambda}(x,x_j)}},\tag{9}$$

where $x\in X$, $\lambda,m\in\mathbb{N}$ and $f\in C(X,[0,\infty))$. We know from [8] that, for all $f\in C(X,[0,\infty))$, the sequence $\{S_m^{\lambda}(f;x)\}$ in (9) is uniformly convergent to f on X. Now, consider the case $\sigma(n)=n+1$. Then, we define the max-product operators on $C(X,[0,\infty))$ as

$$T_m(f;x)=(1+u_m)\,S_m^{\lambda}(f;x)\,,\quad x\in X\text{ and }f\in C(X,[0,\infty))\tag{10}$$

where the operators S_m are given by (9) and $u=\{u_m\}$ is given by (1). Since $\delta(\sigma)-\lim u_m=0$, we observe that the sequence of positive linear operators T_m defined by (10) satisfy all hypotheses of Theorem 4. Therefore, for all $f\in C(X,[0,\infty))$, we conclude that

$$\delta(\sigma)-\lim\left\{\bigvee\{|T_m(f;x)-f(x)|:x\in X\}\right\}=0.$$

However, since $\{u_m\}$ is not convergent and statistically convergent, we conclude that Theorems 5 and 6 do not work for the operators T_m in (10) while our Theorem 4 still works.

3. Rate of statistical σ-convergence

In this section we compute the rate of statistical σ-convergence of Theorem 4.

Definition 8. A bounded sequence $x=\{x_n\}$ is statistically σ-convergent to a number L with the rate of $\beta\in(0,1)$ if for every $\varepsilon>0$,

$$\lim_n\frac{\left|\left\{p\le n:\left|t_{pm}(x)-L\right|\ge\varepsilon\right\}\right|}{n^{1-\beta}}=0,\quad\text{uniformly in } m.$$

Then, this is denoted by

$$x_n-L=o(n^{-\beta})(\delta(\sigma)).$$

Using this definition, we obtain the following auxiliary result.

Lemma 9. *Let $x=\{x_n\}$ and $y=\{y_n\}$ be bounded sequences. Assume that $x_n-L_1=o(n^{-\beta_1})(\delta(\sigma))$ and $y_n-L_2=o(n^{-\beta_2})(\delta(\sigma))$. Then we have*

(i) $(x_n-L_1)\mp(y_n-L_2)=o(n^{-\beta})(\delta(\sigma))$, *where $\beta:=\min\{\beta_1,\beta_2\}$,*
(ii) $\lambda(x_n-L_1)=o(n^{-\beta_1})(\delta(\sigma))$, *for any real number λ.*

Proof. (i) Assume that $x_n-L_1=o(n^{-\beta_1})\,(\delta(\sigma))$ and $y_n-L_2=o(n^{-\beta_2})\,(\delta(\sigma))$. Then, for $\varepsilon>0$, observe that

$$\begin{aligned}&\frac{\left|\left\{p\le n:\left|\left(t_{pm}(x)-L_1\right)\mp\left(t_{pm}(y)-L_2\right)\right|\ge\varepsilon\right\}\right|}{n^{1-\beta}}\\&\quad\le\frac{\left|\left\{p\le n:\left|t_{pm}(x)-L_1\right|\ge\frac{\varepsilon}{2}\right\}\right|+\left|\left\{p\le n:\left|t_{pm}(y)-L_2\right|\ge\frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}}\\&\quad\le\frac{\left|\left\{p\le n:\left|t_{pm}(x)-L_1\right|\ge\frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_1}}+\frac{\left|\left\{p\le n:\left|t_{pm}(y)-L_2\right|\ge\frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_2}}.\end{aligned}\tag{11}$$

Now by taking the limit as $n\to\infty$ in (11) and using the hypotheses, we conclude that

$$\lim_n\frac{\left|\left\{p\le n:\left|\left(t_{pm}(x)-L_1\right)\mp\left(t_{pm}(y)-L_2\right)\right|\ge\varepsilon\right\}\right|}{n^{1-\beta}}=0,\quad\text{uniformly in } m,$$

which completes the proof of (i). Since the proof of (ii) is similar, we omit it. $\square$

We also need the following lemma.

Lemma 10 ([9]). *For every $a_k, b_k \geq 0$ $(k = 0, 1, \ldots, n)$ we have*

$$\bigvee_{k=0}^{n} a_k b_k = \sqrt{\bigvee_{k=0}^{n} a_k^2}\sqrt{\bigvee_{k=0}^{n} b_k^2}.$$

Now we recall the concept of the modulus of continuity. Let $f \in C(X, [0, \infty))$. Then the function $\omega(f, .) : [0, \infty) \to [0, \infty)$ defined by

$$\omega(f, \delta) = \bigvee \{|f(x) - f(y)| : x, y \in X, d(x, y) \leq \delta\}$$

is called the modulus of continuity of f. In order to obtain our result, we will make use of the elementary inequality, for all $f \in C(X, [0, \infty))$ and for $\lambda, \delta \in [0, \infty)$,

$$\omega(f, \lambda\delta) \leq (\lambda + 1)\,\omega(f, \delta).$$

Then we have the following result.

Theorem 11. *Let (X, d) be an arbitrary compact metric space. If the operators $L := \{L_n\}$ given by (2) and (3) satisfy that*

$$\omega\left(f, \alpha_{pm}\right) = o(n^{-\beta})\,(\delta(\sigma)) \quad \text{on } X \tag{12}$$

where $\alpha_{pm} := \sqrt{\bigvee \left\{t_{pm}\left(\sqrt{L(\varphi_x; x)}\right) : x \in X\right\}}$ with $\varphi_x(y) = d^2(y, x)$. Then we have for all $f \in C(X, [0, \infty))$,

$$\bigvee \{|L_n(f; x) - f(x)| : x \in X\} = o(n^{-\beta})\,(\delta(\sigma)) \quad \text{on } X.$$

Proof. Let $f \in C(X, [0, \infty))$ and $x \in X$ be fixed. Consider the set K given by (6), we can write for every $p \in K$ and for any $\delta > 0$, that

$$\begin{aligned}
&\left|t_{pm}(L(f; x)) - f(x)\right| \\
&= \left|\frac{L_m(f; x) + L_{\sigma(m)}(f; x) + \cdots + L_{\sigma^p(m)}(f; x)}{p+1} - f(x)\right| \\
&\leq \left|\frac{(L_m(f; x) - f(x)) + \left(L_{\sigma(m)}(f; x) - f(x)\right) + \cdots + \left(L_{\sigma^p(m)}(f; x) - f(x)\right)}{p+1}\right| \\
&\leq \frac{\bigvee_{k=0}^{m} K_m(x, x_k)\cdot|f(x_k) - f(x)| + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k)\cdot|f(x_k) - f(x)|}{p+1} + \cdots + \frac{\bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k)\cdot|f(x_k) - f(x)|}{p+1} \\
&\leq \frac{\bigvee_{k=0}^{m} K_m(x, x_k)\cdot\omega(f, d(x_k, x)) + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k)\cdot\omega(f, d(x_k, x))}{p+1} + \cdots + \frac{\bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k)\cdot\omega(f, d(x_k, x))}{p+1} \\
&\leq \frac{\omega(f, \delta)\bigvee_{k=0}^{m} K_m(x, x_k)\cdot\left(1 + \frac{d(x_k, x)}{\delta}\right)}{p+1} + \frac{\omega(f, \delta)\bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k)\cdot\left(1 + \frac{d(x_k, x)}{\delta}\right)}{p+1} + \cdots \\
&\quad + \frac{\omega(f, \delta)\bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k)\cdot\left(1 + \frac{d(x_k, x)}{\delta}\right)}{p+1} \\
&= \omega(f, \delta)\left\{\frac{\bigvee_{k=0}^{m} K_m(x, x_k) + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k) + \cdots + \bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k)}{p+1}\right. \\
&\quad \left. + \frac{1}{\delta}\left(\frac{\bigvee_{k=0}^{m} K_m(x, x_k)\cdot d(x_k, x) + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x, x_k)\cdot d(x_k, x)}{p+1} + \cdots + \frac{\bigvee_{k=0}^{\sigma^p(m)} K_{\sigma^p(m)}(x, x_k)\cdot d(x_k, x)}{p+1}\right)\right\}
\end{aligned}$$

$$= \omega(f,\delta)\left\{1+\frac{1}{\delta}\left(\frac{\bigvee_{k=0}^{m}\left[K_m^{1/2}(x,x_k)\right]\cdot\left[K_m^{1/2}(x,x_k)d(x_k,x)\right]}{p+1}+\frac{\bigvee_{k=0}^{\sigma(m)}\left[K_{\sigma(m)}^{1/2}(x,x_k)\right]\cdot\left[K_{\sigma(m)}^{1/2}(x,x_k)d(x_k,x)\right]}{p+1}+\cdots\right.\right.$$

$$\left.\left.+\frac{\bigvee_{k=0}^{\sigma^p(m)}\left[K_{\sigma^p(m)}^{1/2}(x,x_k)\right]\cdot\left[K_{\sigma^p(m)}^{1/2}(x,x_k)\,d(x_k,x)\right]}{p+1}\right)\right\}.$$

Now, by using Lemma 10, we immediately see that

$$\begin{aligned}
&\left|t_{pm}(L(f;x))-f(x)\right| \\
&\le \omega(f,\delta)\left\{1+\frac{1}{\delta}\frac{\sqrt{L_m\left(d^2(.,x);x\right)}+\sqrt{L_{\sigma(m)}\left(d^2(.,x);x\right)}+\cdots+\sqrt{L_{\sigma^p(m)}\left(d^2(.,x);x\right)}}{p+1}\right\} \\
&= \omega(f,\delta)\left\{1+\frac{1}{\delta}t_{pm}\left(\sqrt{L\left(d^2(.,x);x\right)}\right)\right\}
\end{aligned}$$

holds for every $p \in K$ and for any $\delta > 0$. Now taking the maximum over $x \in X$, the last inequality gives for all $p \in K$ and $\delta > 0$, that

$$\bigvee\left\{\left|t_{pm}(L(f;x))-f(x)\right| : x\in X\right\} \le \omega(f,\delta)\left\{1+\frac{1}{\delta}\left(\bigvee\left\{t_{pm}\left(\sqrt{L\left(d^2(.,x);x\right)}\right) : x\in X\right\}\right)\right\}.$$

So, we get

$$\bigvee\left\{\left|t_{pm}(L(f;x))-f(x)\right| : x\in X\right\} \le 2\omega(f,\delta) \tag{13}$$

where $\delta := \alpha_{pm} := \sqrt{\bigvee\left\{t_{pm}\left(\sqrt{L(\varphi_x;x)}\right) : x\in X\right\}}$. Hence, given $\varepsilon > 0$, it follows from (13) that

$$\begin{aligned}
&\frac{\left|\left\{p\le n : \left(\bigvee\left\{\left|t_{pm}(L(f;x))-f(x)\right| : x\in X\right\}\right)\ge\varepsilon\right\}\right|}{n^{1-\beta}} \\
&= \frac{\left|\left\{p\le n : p\in K \text{ and } \left(\bigvee\left\{\left|t_{pm}(L(f;x))-f(x)\right| : x\in X\right\}\right)\ge\varepsilon\right\}\right|}{n^{1-\beta}} \\
&\quad+\frac{\left|\left\{p\le n : p\in \mathbb{N}/K \text{ and } \left(\bigvee\left\{\left|t_{pm}(L(f;x))-f(x)\right| : x\in X\right\}\right)\ge\varepsilon\right\}\right|}{n^{1-\beta}} \\
&\le \frac{\left|\left\{p\le n : p\in K \text{ and } \left(\bigvee\left\{\left|t_{pm}(L(f;x))-f(x)\right| : x\in X\right\}\right)\ge\varepsilon\right\}\right|}{n^{1-\beta}}+\frac{\left|\left\{p\le n : p\in\mathbb{N}/K\right\}\right|}{n^{1-\beta}} \\
&\le \frac{\left|\left\{p\le n : p\in K \text{ and } \omega(f,\delta)\ge\frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}}+\frac{\left|\left\{p\le n : p\in\mathbb{N}/K\right\}\right|}{n} \\
&\le \frac{\left|\left\{p\le n : \omega(f,\delta)\ge\frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}}+\frac{\left|\left\{p\le n : p\in\mathbb{N}/K\right\}\right|}{n}.
\end{aligned}$$

Then using (7) and the hypothesis (12), we have

$$\bigvee\left\{\left|(L(f;x))-f(x)\right| : x\in X\right\} = o(n^{-\beta})\,(\delta(\sigma)) \quad \text{on } X.$$

The proof is complete. □

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