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Four-dimensional matrix transformation and the rate of A-statistical convergence of continuous functions

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A B S T R A C T

Motivated by our earlier work on the statistical approximation of continuous functions by positive linear operators of two variables, we study rates of A-statistical convergence of a sequence of positive linear operators acting on the space of all continuous real valued functions on any D compact subset of the real two-dimensional space. Furthermore, displaying an example, it is shown that our statistical rates are more efficient than the classical aspects in the approximation theory.

1. Introduction

For a sequence (L_n) of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [1] established first the necessary and sufficient conditions for the uniform convergence of $L_n(f)$ to a function f by using the test function f_i defined by $f_i(x) = x^i$, $(i = 0, 1, 2)$ (see, for instance, [2,3]). Later many researchers investigated these conditions for various operators defined on different spaces. Using the concept of statistical convergence in approximation theory provides us with many advantages. In particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of linear operators such as the interpolation operator of Hermite and Fejér [4], because such operators do not converge at points of simple discontinuity. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results have been proved [5–10]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem. In this paper, we study rates of A-statistical convergence of a sequence of positive linear operators acting on the space of all continuous real valued functions on any D compact subset of the real two-dimensional space. Furthermore, displaying an example, it is shown that our statistical rates are more efficient than the classical aspects in the approximation theory.

We now recall some basic definitions and notation used in the paper.

A double sequence $x = \{x_{m,n}\}$, $m, n \in \mathbb{N}$, is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. Then, L is called the Pringsheim limit of x and is denoted by $P\text{-}\lim x = L$ (see [11]). In this case, we say that $x = \{x_{m,n}\}$ is "P-convergent to L". Also, if there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Now let $A = [a_{j,k,m,n}]$, $j, k, m, n \in \mathbb{N}$, be a four-dimensional summability matrix. For a given double sequence $x = \{x_{m,n}\}$, the A-transform of x, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

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$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j,k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(m,n) \in \mathbb{N}^2$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two-dimensional matrix transformations is known as the Silverman–Toeplitz conditions (see, for instance, [12]). In 1926, Robison [13] presented a four-dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four-dimensional matrices is known as the Robison–Hamilton conditions, or for short, RH-regularity (see [14,13]).

Recall that a four-dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison–Hamilton conditions state that a four-dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

(i) $P\text{-}\lim_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$,
(ii) $P\text{-}\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,
(iii) $P\text{-}\lim_{j,k} \sum_{m\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,
(iv) $P\text{-}\lim_{j,k} \sum_{n\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,
(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is P-convergent,
(vi) there exist finite positive integers A and B such that $\sum_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j,k) \in \mathbb{N}^2$.

Now let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a real double sequence $x = \{x_{m,n}\}$ is said to be A-statistically convergent to a number L if, for every $\varepsilon > 0$,

$$P\text{-}\lim_{j,k} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}.$$

In this case we write $st^2_{(A)} - \lim_{m,n} x_{m,n} = L$. Observe that a P-convergent double sequence is A-statistically convergent to the same value but the converse does not hold true. For example, consider the double sequence $x = \{x_{m,n}\}$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares}, \\ 1, & \text{otherwise}. \end{cases}$$

We should note that if we take $A = C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for a double sequence, which was introduced in [15,16]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

By $C(D)$, we denote the space of all continuous real valued functions on any D compact subset of the real two-dimensional space. This space is equipped with the supremum norm

$$\|f\|_{C(D)} = \sup_{(x,y)\in D} |f(x,y)|, \quad (f \in C(D)).$$

Let L be a linear operator from $C(D)$ into $C(D)$. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $(x,y) \in D$ by $L(f; x, y)$.

With this terminology the authors [17] proved the following theorem, which corresponds to the A-statistical sense of the problem studied by Volkov [18].

Theorem 1 *([17]). Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix. Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,*

$$st^2_{(A)} - \lim_{m,n} \left\| L_{m,n}(f) - f \right\|_{C(D)} = 0 \tag{1}$$

if and only if

$$st^2_{(A)} - \lim_{m,n} \left\| L_{m,n}(f_i) - f_i \right\|_{C(D)} = 0, \quad (i = 0,1,2,3), \tag{2}$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$.

The aim of the present paper is to compute the rates of A-statistical approximation in Theorem 1 with the help of the modulus of continuity.

2. The rate of A-statistical convergence

Various ways of defining rates of convergence in the A-statistical sense for two-dimensional summability matrices were introduced in [6]. In a similar way, for four-dimensional summability matrices, we present four different ways to compute the corresponding rates of A-statistical convergence in Theorem 1.

Definition 2. Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ be a positive non-increasing double sequence. A double sequence $x = \{x_{m,n}\}$ is A-statistically convergent to a number L with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P\text{-}\lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \left\{(m, n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\right\}.$$

In this case, we write

$$x_{m,n} - L = st^2_{(A)} - o(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

Definition 3. Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistically bounded with the rate of $O(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$\sup_{j,k} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in L(\varepsilon)} a_{j,k,m,n} < \infty,$$

where

$$L(\varepsilon) := \left\{(m, n) \in \mathbb{N}^2 : \left|x_{m,n}\right| \geq \varepsilon\right\}.$$

In this case, we write

$$x_{m,n} = st^2_{(A)} - O(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

Definition 4. Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistically convergent to a number L with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P\text{-}\lim_{j,k\to\infty} \sum_{(m,n)\in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \left\{(m, n) \in \mathbb{N}^2 : \left|x_{m,n} - L\right| \geq \varepsilon\alpha_{m,n}\right\}.$$

In this case, we write

$$x_{m,n} - L = st^2_{(A)} - o_{m,n}(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

Definition 5. Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistically bounded with the rate of $O_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P\text{-}\lim_{j,k} \sum_{(m,n)\in N(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$N(\varepsilon) := \left\{(m, n) \in \mathbb{N}^2 : \left|x_{m,n}\right| \geq \varepsilon\alpha_{m,n}\right\}.$$

In this case, we write

$$x_{m,n} - L = st^2_{(A)} - O_{m,n}(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

As a tool, we use the modulus of continuity $\omega(f; \delta)$ defined as follows:

$$\omega(f; \delta) := \sup\left\{|f(u, v) - f(x, y)| : (u, v), (x, y) \in D, \sqrt{(u - x)^2 + (v - y)^2} \leq \delta\right\},$$

where $f \in C(D)$ and $\delta > 0$. In order to obtain our result, we will make use of the elementary inequality, for all $f \in C(D)$ and for $\lambda, \delta > 0$,

$$\omega(f; \lambda\delta) \leq (1 + [\lambda])\, \omega(f; \delta) \tag{3}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.

Then we have the following result.

Theorem 6. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself and let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix. Let $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ be a positive non-increasing double sequence. Then, for all $f \in C(D)$,*

$$\left\|L_{m,n}(f) - f\right\|_{C(D)} = st^2_{(A)} - o(\gamma_{m,n}) \quad \text{as } m, n \to \infty, \text{ with } \gamma_{m,n} := \max\left\{\alpha_{m,n}, \beta_{m,n}\right\}$$

provided that the following conditions hold:

(i) $\|L_{m,n}(f_0) - f_0\|_{C(D)} = st^2_{(A)} - o(\alpha_{m,n})$ *as $m, n \to \infty$, with $f_0(u, v) = 1$,*

(ii) $\omega(f; \delta_{m,n}) = st^2_{(A)} - o(\beta_{m,n})$ *as $m, n \to \infty$, where $\delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|_{C(D)}}$ with $\Psi(u, v) = \Psi_{x,y}(u, v) = (u-x)^2 + (v-y)^2$ for each $(x, y), (u, v) \in D$; furthermore, similar results holds when the symbol "o" is replaced by "O".*

Proof. To see this, we first assume that $(x, y) \in D$ and $f \in C(D)$ are fixed, and that (i) and (ii) hold. Using the definition of the modulus of continuity and the linearity and the positivity of the operators $L_{m,n}$, for all $(m, n) \in \mathbb{N}^2$, we have

$$\begin{aligned}\left|L_{m,n}(f; x, y) - f(x, y)\right| &\leq \omega(f; \delta)\left|L_{m,n}(f_0, x, y) - f_0(x, y)\right| + \frac{\omega(f; \delta)}{\delta^2} L_{m,n}(\Psi; x, y) \\ &\quad + \omega(f; \delta) + |f(x, y)|\left|L_{m,n}(f_0, x, y) - f_0(x, y)\right|.\end{aligned}$$

Taking the supremum over $(x, y) \in D$ on the both sides of the above inequality and $\delta := \delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|_{C(D)}}$, we then obtain

$$\left\|L_{m,n}(f) - f\right\|_{C(D)} \leq \omega(f; \delta_{m,n})\left\|L_{m,n}(f_0) - f_0\right\|_{C(D)} + 2\omega(f; \delta_{m,n}) + \|f\|_{C(D)}\left\|L_{m,n}(f_0) - f_0\right\|_{C(D)}$$

where the quantity $\|f\|_{C(D)}$ is a finite number since $f \in C(D)$. Hence, we get

$$\left\|L_{m,n}(f) - f\right\|_{C(D)} \leq B\left\{\omega(f; \delta_{m,n})\left\|L_{m,n}(f_0) - f_0\right\|_{C(D)} + \omega\left(f; \delta_{m,n}\right) + \left\|L_{m,n}(f_0) - f_0\right\|_{C(D)}\right\} \tag{4}$$

where $B = \max\{2, \|f\|_{C(D)}\}$. Now, given $\varepsilon > 0$, define the following sets:

$$\begin{aligned}
D &:= \left\{(m, n) \in \mathbb{N}^2 : \left\|L_{m,n}(f) - f\right\|_{C(D)} \geq \varepsilon\right\}, \\
D_1 &:= \left\{(m, n) \in \mathbb{N}^2 : \omega\left(f; \delta_{m,n}\right)\left\|L_{m,n}(f_0) - f_0\right\|_{C(D)} \geq \frac{\varepsilon}{3B}\right\}, \\
D_2 &:= \left\{(m, n) \in \mathbb{N}^2 : \omega\left(f; \delta_{m,n}\right) \geq \frac{\varepsilon}{3B}\right\}, \\
D_3 &:= \left\{(m, n) \in \mathbb{N}^2 : \left\|L_{m,n}(f_0) - f_0\right\|_{C(D)} \geq \frac{\varepsilon}{3B}\right\}.
\end{aligned}$$

Then, it follows from (4) that $D \subset D_1 \cup D_2 \cup D_3$. Also, defining

$$\begin{aligned}
D_4 &:= \left\{(m, n) \in \mathbb{N}^2 : \omega\left(f; \delta_{m,n}\right) \geq \sqrt{\frac{\varepsilon}{3B}}\right\}, \\
D_5 &:= \left\{(m, n) \in \mathbb{N}^2 : \left\|L_{m,n}(f_0) - f_0\right\|_{C(D)} \geq \sqrt{\frac{\varepsilon}{3B}}\right\},
\end{aligned}$$

we have $D_1 \subset D_4 \cup D_5$, which yields

$$D \subseteq \bigcup_{i=2}^{5} D_i.$$

Therefore, since $\gamma_{m,n} = \max\{\alpha_{m,n}, \beta_{m,n}\}$, we conclude that, for all $(j, k) \in \mathbb{N}^2$,

$$\frac{1}{\gamma_{j,k}} \sum_{(m,n)\in D} a_{j,k,m,n} \leq \frac{1}{\beta_{j,k}} \sum_{(m,n)\in D_2} a_{j,k,m,n} + \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in D_3} a_{j,k,m,n} + \frac{1}{\beta_{j,k}} \sum_{(m,n)\in D_4} a_{j,k,m,n} + \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in D_5} a_{j,k,m,n}. \tag{5}$$

Letting $j, k \to \infty$ (in any manner) on both sides of (5), we get

$$P\text{-}\lim_{j,k\to\infty} \frac{1}{\gamma_{j,k}} \sum_{(m,n)\in D} a_{j,k,m,n} = 0.$$

Therefore, the proof is completed. $\square$

Now, specializing Theorem 6, we can give the ordinary rates of convergence of a sequence of positive linear operators defined on the space $C(D)$. We first note that, if we choose $\alpha_{m,n} = \beta_{m,n} = 1$ for all $m, n \in \mathbb{N}$, then Theorem 1 is obtained from Theorem 6 at once. So our theorem gives us the rate of A-statistical convergence in Theorem 1. Furthermore, if one replaces the matrix $A = [a_{j,k,m,n}]$ by the double identity matrix, then Theorem 6 immediately gives the following classical rates of convergence of a sequence of positive linear operators defined on $C(D)$.

Corollary 7. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,*

$$P\text{-}\lim_{m,n} \left\| L_{m,n}(f) - f \right\|_{C(D)} = 0,$$

provided that the following conditions hold:

(i) $P\text{-}\lim_{m,n}\|L_{m,n}(f_0) - f_0\|_{C(D)} = 0,$
(ii) $P\text{-}\lim_{m,n}\omega(f; \delta_{m,n}) = 0$

where f_0 and $\{\delta_{m,n}\}$ are the same as in Theorem 6.

One can immediately obtain the next result using a technique similar to that used in the proof of Theorem 6.

Theorem 8. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself and let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix. Let $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ be a positive non-increasing double sequence. Then, for all $f \in C(D)$,*

$$\left\| L_{m,n}(f) - f \right\|_{C(D)} = st^2_{(A)} - o_{m,n}(\gamma_{m,n}) \quad \text{as } m, n \to \infty,$$

with $\gamma_{m,n} := \max\{\alpha_{m,n}, \beta_{m,n}, \alpha_{m,n}\beta_{m,n}\}$, provided that the following conditions hold:

(i) $\|L_{m,n}(f_0) - f_0\|_{C(D)} = st^2_{(A)} - o_{m,n}(\alpha_{m,n})$ *as* $m, n \to \infty$, *with* $f_0(u, v) = 1$,
(ii) $\omega(f; \delta_{m,n}) = st^2_{(A)} - o_{m,n}(\beta_{m,n})$ *as* $m, n \to \infty$, *where* $\delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|_{C(D)}}$ *with* $\Psi(u, v) = \Psi_{x,y}(u, v) = (u - x)^2 + (v - y)^2$ *for each* $(x, y), (u, v) \in D$.

Similar results hold when little "$o_{m,n}$" is replaced by big "$O_{m,n}$".

3. An application to Theorem 6

In this section, we display an example of positive linear operators, which satisfies Theorem 6 but not Corollary 7.

Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix. We know that a P-convergent double sequence is A-statistically convergent to the same value but the converse does not hold true. So, we can choose a non-negative double sequence $\{u_{m,n}\}$ that converges A-statistically to 0 but is not P-convergent. Then, the authors in their study considered the following Bernstein-type operators for two variables:

$$L_{m,n}(f; x, y) = \left(1 + u_{m,n}\right) \sum_{s=0}^{m} \sum_{t=0}^{n} f\left(\frac{s}{m}, \frac{t}{n}\right) \binom{m}{s} \binom{n}{t} x^s y^t (1 - x)^{m-s} (1 - y)^{n-t}, \tag{6}$$

where $(x, y) \in I^2 = [0, 1] \times [0, 1]$; $f \in C(I^2)$. Now, we take $A = C(1, 1) := [c_{j,k,m,n}]$, the double Cesáro matrix, defined by

$$c_{j,k,m,n} = \begin{cases} \dfrac{1}{jk}, & \text{if } 1 \le m \le j \text{ and } 1 \le n \le k, \\ 0, & \text{otherwise,} \end{cases}$$

and also replace the double sequence $\{u_{m,n}\}$ by

$$u_{m,n} = \begin{cases} \sqrt{mn}, & \text{if } m \text{ and } n \text{ are square,} \\ 0, & \text{otherwise.} \end{cases}$$

Now, setting $\{\alpha_{m,n}\} = \left\{\frac{1}{\sqrt[5]{mn}}\right\}$, we have, for any $\varepsilon > 0$,

$$\frac{1}{\alpha_{j,k}} \sum_{(m,n):|u_{m,n}|\ge\varepsilon} c_{j,k,m,n} = \sqrt[5]{jk} \sum_{(m,n):|u_{m,n}|\ge\varepsilon} \frac{1}{jk} \le \frac{\sqrt[5]{jk}\sqrt{jk}}{jk} = \frac{1}{\sqrt[10]{(jk)^3}}. \tag{7}$$

Taking the limit as $j,k\to\infty$ (in any manner) in (7), we get, for any $\varepsilon>0$,

$$P\text{-}\lim_{j,k}\frac{1}{\alpha_{j,k}}\sum_{(m,n):|u_{m,n}|\geq\varepsilon}c_{j,k,m,n}=0$$

which gives

$$u_{m,n}=st^2_{(C(1,1))}-o\left(\frac{1}{\sqrt[5]{mn}}\right)\quad\text{as } m,n\to\infty. \tag{8}$$

Also, observe that

$$\begin{aligned}
L_{m,n}(f_0;x,y)&=1+u_{m,n},\\
L_{m,n}(f_1;x,y)&=\left(1+u_{m,n}\right)f_1(x,y),\\
L_{m,n}(f_2;x,y)&=\left(1+u_{m,n}\right)f_2(x,y),\\
L_{m,n}(f_3;x,y)&=\left(1+u_{m,n}\right)\left(f_3(x,y)+\frac{x-x^2}{m}+\frac{y-y^2}{n}\right),
\end{aligned}$$

where $f_0(x,y)=1, f_1(x,y)=x, f_2(x,y)=y$ and $f_3(x,y)=x^2+y^2$. Since $\|L_{m,n}(f_0)-f_0\|_{C(I^2)}=u_{m,n}$, we obtain from (8)

$$\left\|L_{m,n}(f_0)-f_0\right\|_{C(I^2)}=st^2_{(C(1,1))}-o(\alpha_{m,n})\quad\text{as } m,n\to\infty. \tag{9}$$

Now, we compute the quantity $L_{m,n}(\Psi;x,y)$, where $\Psi(u,v)=(u-x)^2+(v-y)^2$. After some calculations, we get

$$L_{m,n}(\Psi;x,y)=\left(1+u_{m,n}\right)\left(\frac{x-x^2}{m}+\frac{y-y^2}{n}\right).$$

Then, we obtain $\delta_{m,n}:=\sqrt{\|L_{m,n}(\Psi)\|_{C(I^2)}}=\sqrt{\frac{(1+u_{m,n})}{2}\left(\frac{1}{m}+\frac{1}{n}\right)}$. In this case, setting $\{\beta_{m,n}\}=\left\{\frac{1}{\sqrt[3]{mn}}\right\}$, we have, for any $\varepsilon>0$,

$$\frac{1}{\beta_{j,k}}\sum_{(m,n):|\delta_{m,n}|\geq\varepsilon}c_{j,k,m,n}=\sqrt[3]{jk}\sum_{(m,n):|\delta_{m,n}|\geq\varepsilon}\frac{1}{jk}\leq\frac{\sqrt[3]{jk}\sqrt{jk}}{jk}=\frac{1}{\sqrt[6]{jk}}$$

which gives that $P\text{-}\lim_{j,k}\frac{1}{\beta_{j,k}}\sum_{(m,n):|\delta_{m,n}|\geq\varepsilon}c_{j,k,m,n}=0$. Hence, we obtain $\delta_{m,n}=st^2_{(C(1,1))}-o\left(\frac{1}{\sqrt[3]{mn}}\right)$ as $m,n\to\infty$. By the uniform continuity of f on I^2, we write that

$$\omega(f;\delta_{m,n})=st^2_{(C(1,1))}-o\left(\frac{1}{\sqrt[3]{mn}}\right)\quad\text{as } m,n\to\infty. \tag{10}$$

Then, the sequence of positive linear operators $\{L_{m,n}\}$ satisfy all hypotheses of Theorem 6 from (9) and (10). So, we have, for all $f\in C(I^2)$,

$$\left\|L_{m,n}(f)-f\right\|_{C(I^2)}=st^2_{(C(1,1))}-o\left(\frac{1}{\sqrt[5]{mn}}\right)\quad\text{as } m,n\to\infty.$$

However, since $\{u_{m,n}\}$ is not P-convergent, the sequence $\{L_{m,n}\}$ given by (6) does not converge uniformly to the function $f\in C(I^2)$.

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