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Statistical approximation by positive linear operators on modular spaces

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Abstract In this paper, we investigate the problem of statistical approximation to a function by means of positive linear operators defined on a modular space. Especially, in order to get more powerful results than the classical aspects we mainly use the concept of statistical convergence. A non-trivial application is also presented.

Keywords Positive linear operators · Modular space · Regular matrix · Statistical convergence · Korovkin theorem

Mathematics Subject Classification (2000) 41A25 · 41A36

1 Introduction

In this paper, we investigate the problem of statistical approximation to a function f by means of positive linear operators defined on a modular space. So, we obtain various Korovkin-type approximation theorems in modular spaces via the concept of A-statistical convergence, where A is a non-negative regular summability matrix.

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With the help of the notions of modular convergence and strong convergence, some approximation theorems have recently been introduced by Bardaro and Mantellini [3]. However, in this paper, we replace the strong (or modular) convergence by the more general A-statistical convergence with respect to a fixed summability matrix A. When A is the identity matrix, our results reduce to the results in [3]. Also, we display an example satisfying all conditions of our statistical approximation result in a modular space but not the classical one in [3].

We now recall some basic definitions and notations used in the paper.

Let

$$A := [a_{jn}] \quad (j, n \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

be an infinite summability matrix. For a given sequence $x := \{x_n\}$, the *A-transform of x*, denoted by

$$Ax := \{(Ax)_j\},$$

is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n,$$

provided the series converges for each $j \in \mathbb{N}$. We say that A is *regular* (see [12]) if

$$\lim Ax = L \quad \text{whenever} \quad \lim x = L.$$

Assume that A is a non-negative regular summability matrix. Then the sequence $x = \{x_n\}$ is called *A-statistically convergent* to L provided that, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:\ |x_n - L| \geq \varepsilon} a_{jn} = 0. \tag{1.1}$$

We denote this limit as follows (cf. [8]; see also [5,6,13])

$$st_A - \lim x = L.$$

Actually, this convergence method is based on the concept of A-density. Recall the *A-density* of a subset $K \subset \mathbb{N}$, denoted by

$$\delta_A\{K\},$$

is given by

$$\delta_A\{K\} = \lim_j \sum_{n=1}^{\infty} a_{jn} \chi_K(n),$$

provided the limit exists, where χ_K is the characteristic function of K; or equivalently

$$\delta_A\{K\} = \lim_n \sum_{n \in K} a_{jn}.$$

So, by (1.1), we easily see that

$$st_A - \lim x = L \quad \text{iff} \quad \delta_A\{n : |x_n - L| \geq \varepsilon\} = 0 \quad \text{for every } \varepsilon > 0.$$

We should note that if we take $A = C_1 := [c_{jn}]$, the *Cesáro matrix* defined by

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

then A-statistical convergence reduces to the concept of *statistical convergence* (cf. [7]; see also [9–11]). In this case, we write

$$st - \lim x = L \quad \text{instead of} \quad st_{C_1} - \lim x = L.$$

Further, taking $A = \mathcal{I}$, the identity matrix, A-statistical convergence coincides with the ordinary convergence, i.e.,

$$st_{\mathcal{I}} - \lim x = \lim x = L.$$

Observe that every convergent sequence (in the usual sense) is A-statistically convergent to the same value for any non-negative regular matrix A, but its converse is not always true. Actually, in Kolk [13], proved that A-statistical convergence is stronger than convergence when $A = [a_{jn}]$ is a non-negative regular summability matrix such that

$$\lim_j \max_n \{a_{jn}\} = 0.$$

The concepts of *statistical limit superior* and *limit inferior* have been introduced by Fridy and Orhan [11]. A-statistical analogs of these concepts have been examined by Connor and Kline [5], and Demirci [6] as follows. The A*-statistical limit superior* of a number sequence $x = \{x_n\}$, denoted by

$$st_A - \lim\sup x,$$

is defined by

$$st_A - \lim\sup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi, \\ -\infty, & \text{if } B_x = \phi, \end{cases}$$

where $B_x := \{b \in \mathbb{R} : \delta_A\{n : x_n > b\} \neq 0\}$ and ϕ denotes the empty set. We note that by $\delta_A\{K\} \neq 0$ we mean either $\delta_A\{K\} > 0$ or K fails to have A-density. Similarly, the *A -statistical limit inferior* of $\{x_n\}$, denoted by

$$st_A - \liminf x,$$

is defined by

$$st_A - \liminf x = \begin{cases} \inf C_x, & \text{if } C_x \neq \phi, \\ +\infty, & \text{if } C_x = \phi, \end{cases}$$

where $C_x := \{c \in \mathbb{R} : \delta_A\{n : x_n < c\} \neq 0\}$. Of course, if we take $A = C_1$, then the above definitions reduce to the concepts of $st - \limsup x$ and $st - \liminf x$ given in [11], respectively. As in the ordinary limit superior or inferior, it was proved that

$$st_A - \liminf x \leq st_A - \limsup x$$

and also that, for any sequence $x = \{x_n\}$ satisfying $\delta_A\{n : |x_n| > M\} = 0$ for some $M > 0$,

$$st_A - \lim x = L \quad \text{iff} \quad st_A - \liminf x = st_A - \limsup x = L.$$

We now focus on modular spaces.

Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X(I)$ we denote the space of all real-valued measurable functions on I provided with equality a.e. As usual, let $C(I)$ denote the space of all continuous real-valued functions, and $C^\infty(I)$ denote the space of all infinitely differentiable functions on I. In this case, we say that a functional $\rho : X(I) \to [0, +\infty]$ is a *modular* on $X(I)$ provided that the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ a.e. in I,
(ii) $\rho(-f) = \rho(f)$ for every $f \in X(I)$,
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(I)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Recall that a modular ρ is called *N-quasi convex* if the following is satisfied:

- there exists a constant $N \geq 1$ such that

$$\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng)$$

holds for every $f, g \in X(I)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*.

Furthermore, a modular ρ is called *N-quasi semiconvex* if it holds:

- there exists a constant $N \geq 1$ such that

$$\rho(af) \leq Na\rho(Nf)$$

holds for every $f \in X(I)$ and $a \in (0, 1]$.

It is clear that every N-quasi semiconvex modular is N-quasi convex. We should recall that the above two concepts were introduced and discussed in details by Bardaro et al. [4].

We now consider some appropriate vector subspaces of $X(I)$ by means of a modular ρ as follows:

$$L^{\rho}(I) := \left\{ f \in X(I) : \lim_{\lambda \to 0^{+}} \rho(\lambda f) = 0 \right\}$$

and

$$E^{\rho}(I) := \left\{ f \in L^{\rho}(I) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0 \right\}.$$

Here, $L^{\rho}(I)$ is called the *modular space* generated by ρ; and also $E^{\rho}(I)$ is called the space of the finite elements of $L^{\rho}(I)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\{ f \in X(I) : \rho[\lambda f] < +\infty \text{ for some } \lambda > 0 \}$$

coincides with $L^{\rho}(I)$. The notions about modulars are introduced in [18] and widely discussed in [4] (see also [15,17]).

Now we recall the convergence methods in modular spaces.

Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}(I)$. Then, $\{f_n\}$ is *modularly convergent* to a function $f \in L^{\rho}(I)$ iff

$$\lim_{n} \rho\left(\lambda_0 \left(f_n - f\right)\right) = 0 \quad \text{for some } \lambda_0 > 0. \tag{1.2}$$

Also, $\{f_n\}$ is *F-norm convergent* (or, *strongly convergent*) to f iff

$$\lim_{n} \rho\left(\lambda \left(f_n - f\right)\right) = 0 \quad \text{for every } \lambda > 0. \tag{1.3}$$

It is known from [17] that (1.2) and (1.3) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho[2f] \leq M\rho[f]$ for every $f \in X(I)$.

In this paper, we will need the following assumptions on a modular ρ:

- if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$, then ρ is called *monotone*,
- if the characteristic function χ_I of the interval I belongs to $L^{\rho}(I)$, ρ is called *finite*,
- if ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda \chi_B) < \varepsilon$ for any measurable subset $B \subset I$ with $|B| < \delta$, then ρ is called *absolutely finite*,
- if $\chi_I \in E^{\rho}(I)$, then ρ is called *strongly finite*,
- ρ is called *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X(I)$ with $\rho(f) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ whenever B is any measurable subset of I with $|B| < \delta$.

Observe now that (see [3]) if a modular ρ is monotone and finite, then we have $C(I) \subset L^{\rho}(I)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I) \subset E^{\rho}(I)$. Some important relations between the above properties may be found in [2,4,16,18].

2 Statistical Korovkin theorems in modular spaces

Let ρ be a monotone and finite modular on $X(I)$, and let $A = [a_{jn}]$ be a non-negative regular summability matrix. Assume that D is a set satisfying $C^{\infty}(I) \subset D \subset L^{\rho}(I)$. We can construct such a subset D since ρ is monotone and finite (see [3]). Assume further that $\mathbb{T} := \{T_n\}$ is a sequence of positive linear operators from D into $X(I)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ containing $C^{\infty}(I)$ such that

$$st_A - \limsup_{n} \rho\left(\lambda\left(T_n h\right)\right) \leq P\rho\left(\lambda h\right) \tag{2.1}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. We also use the test functions e_i defined by

$$e_i(x) = x^i \quad (i = 0, 1, 2, \ldots).$$

Theorem 2.1 *Let $A = [a_{jn}]$ be a non-negative regular summability matrix and let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T} := \{T_n\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (2.1). Assume that*

$$st_A - \lim_{n} \rho\left(\lambda\left(T_n e_i - e_i\right)\right) = 0 \;\; \textit{for every } \lambda > 0 \textit{ and } i = 0, 1, 2. \tag{2.2}$$

Now let f be any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$. Then, we have

$$st_A - \lim_{n} \rho\left(\lambda_0\left(T_n f - f\right)\right) = 0 \;\; \textit{for some } \lambda_0 > 0. \tag{2.3}$$

Proof We first claim that

$$st_A - \lim_{n} \rho\left(\mu(T_n g - g)\right) = 0 \;\; \text{for every } g \in C(I) \text{ and every } \mu > 0. \tag{2.4}$$

To see this assume that g belongs to $C(I)$ and μ is any positive number. Then, by the uniform continuity of g on I and by the linearity and positivity of the operators T_n, we easily see that (see, for instance, [1,14]), for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|T_n(g; x) - g(x)| \leq \varepsilon T_n(e_0; x) + M\left|T_n\left(e_0; x\right) - e_0(x)\right| + \frac{2M}{\delta^2} T_n\left((y - x)^2; x\right)$$

holds for every $x \in I$ and $n \in \mathbb{N}$, where $M := \sup_{x \in I} |g(x)|$. The above inequality implies that

$$|T_n(g; x) - g(x)| \leq \varepsilon + \left(\varepsilon + M + \frac{2Mc^2}{\delta^2}\right) |T_n(e_0; x) - e_0(x)| + \frac{4Mc}{\delta^2} |T_n(e_1; x) - e_1(x)| + \frac{2M}{\delta^2} |T_n(e_2; x) - e_2(x)|,$$

where $c := \max\{|a|, |b|\}$. So, the last inequality gives, for any $\mu > 0$, that

$$\mu |T_n(g; x) - g(x)| \leq \mu\varepsilon + K\mu\{|T_n(e_0; x) - e_0(x)| + |T_n(e_1; x) - e_1(x)| + |T_n(e_2; x) - e_2(x)|\},$$

where $K := \max\left\{\varepsilon + M + \frac{2Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$. Now, applying the modular ρ in the both-sides of the above inequality, since ρ is monotone, we have

$$\rho(\mu(T_n g - g)) \leq \rho(\mu\varepsilon + \mu K |T_n e_0 - e_0| + \mu K |T_n e_1 - e_1| + \mu K |T_n e_2 - e_2|).$$

Hence, we may write that

$$\rho(\mu(T_n g - g)) \leq \rho(4\mu\varepsilon) + \rho(4\mu K(T_n e_0 - e_0)) + \rho(4\mu K(T_n e_1 - e_1)) + \rho(4\mu K(T_n e_2 - e_2))$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0 < \varepsilon \leq 1$

$$\rho(\mu(T_n g - g)) \leq N\varepsilon\rho(4\mu N) + \rho(4\mu K(T_n e_0 - e_0)) + \rho(4\mu K(T_n e_1 - e_1)) + \rho(4\mu K(T_n e_2 - e_2)).$$

For a given $r > 0$, choose an $\varepsilon \in (0, 1]$ such that $N\varepsilon\rho(4\mu N) < r$. Now define the following sets:

$$\begin{aligned} G_\mu &:= \{n : \rho(\mu(T_n g - g)) \geq r\} \\ G_{\mu,1} &:= \left\{n : \rho(4\mu K(T_n e_0 - e_0)) \geq \frac{r - N\varepsilon\rho(4\mu N)}{3}\right\} \\ G_{\mu,2} &:= \left\{n : \rho(4\mu K(T_n e_1 - e_1)) \geq \frac{r - N\varepsilon\rho(4\mu N)}{3}\right\} \\ G_{\mu,3} &:= \left\{n : \rho(4\mu K(T_n e_2 - e_2)) \geq \frac{r - N\varepsilon\rho(4\mu N)}{3}\right\}. \end{aligned}$$

Then, it is easy to see that $G_\mu \subseteq G_{\mu,1} \cup G_{\mu,2} \cup G_{\mu,3}$. So, we can write, for all $j \in \mathbb{N}$, that

$$\sum_{n \in G_\mu} a_{jn} \leq \sum_{n \in G_{\mu,1}} a_{jn} + \sum_{n \in G_{\mu,2}} a_{jn} + \sum_{n \in G_{\mu,3}} a_{jn}. \tag{2.5}$$

Taking limit as $j \to \infty$ in (2.5) and using the hypothesis (2.2), we get

$$\lim_j \sum_{n \in G_\mu} a_{jn} = 0,$$

which proves our claim (2.4). Observe that (2.4) also holds for every $g \in C^\infty(I)$ because of $C^\infty(I) \subset C(I)$. Now let $f \in L^\rho(I)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty(I)$. Since $|I| < \infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I)$ (see [2]). Using these properties of the modular ρ, it is known from [4,16] that the space $C^\infty(I)$ is modularly dense in $L^\rho(I)$, i.e., there exists a sequence $\{g_k\} \subset C^\infty(I)$ such that

$$\lim_k \rho\,[3\lambda_0\,(g_k - f)] = 0 \;\text{ for some } \lambda_0 > 0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\rho\,[3\lambda_0\,(g_k - f)] < \varepsilon \quad \text{for every } k \geq k_0. \tag{2.6}$$

On the other hand, by the linearity and positivity of the operators T_n, we may write that

$$\begin{aligned}\lambda_0|T_n(f;x) - f(x)| \leq \lambda_0|T_n(f - g_{k_0};x)| &+ \lambda_0|T_n(g_{k_0};x) - g_{k_0}(x)| \\ &+ \lambda_0|g_{k_0}(x) - f(x)|\end{aligned}$$

holds for every $x \in I$ and $n \in \mathbb{N}$. Applying the modular ρ in the last inequality and using the monotonicity of ρ, we have

$$\begin{aligned}\rho\,(\lambda_0(T_n f - f)) \leq \rho\left(3\lambda_0 T_n\left(f - g_{k_0}\right)\right) &+ \rho\left(3\lambda_0\left(T_n g_{k_0} - g_{k_0}\right)\right) \\ &+ \rho\left(3\lambda_0\left(g_{k_0} - f\right)\right).\end{aligned} \tag{2.7}$$

Then, it follows from (2.6) and (2.7) that

$$\rho\,(\lambda_0\,(T_n f - f)) \leq \varepsilon + \rho\left(3\lambda_0 T_n\left(f - g_{k_0}\right)\right) + \rho\left(3\lambda_0\left(T_n g_{k_0} - g_{k_0}\right)\right). \tag{2.8}$$

So, taking A-statistical limit superior as $n \to \infty$ in the both-sides of (2.8) and also using the facts that $g_{k_0} \in C^\infty(I)$ and $f - g_{k_0} \in X_{\mathbb{T}}$, we obtained from (2.1) that

$$\begin{aligned}st_A - \limsup_n \rho\,(\lambda_0\,(T_n f - f)) \leq \varepsilon + P\rho\left(3\lambda_0(f - g_{k_0})\right) &+ st_A \\ - \limsup_n \rho\left(3\lambda_0\left(T_n g_{k_0} - g_{k_0}\right)\right),&\end{aligned}$$

which gives

$$st_A - \limsup_n \rho\,(\lambda_0\,(T_n f - f)) \leq \varepsilon(P+1) + st_A - \limsup_n \rho\left(3\lambda_0\left(T_n g_{k_0} - g_{k_0}\right)\right). \tag{2.9}$$

By (2.4), since

$$st_A - \lim_n \rho\left(3\lambda_0\left(T_n g_{k_0} - g_{k_0}\right)\right) = 0,$$

we get

$$st_A - \limsup_n \rho\left(3\lambda_0\left(T_n g_{k_0} - g_{k_0}\right)\right) = 0. \tag{2.10}$$

Combining (2.9) with (2.10), we conclude that

$$st_A - \limsup_n \rho\left(\lambda_0\left(T_n f - f\right)\right) \leq \varepsilon(P+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$st_A - \limsup_n \rho\left(\lambda_0\left(T_n f - f\right)\right) = 0.$$

Furthermore, since $\rho(\lambda_0(T_n f - f))$ is non-negative for all $n \in \mathbb{N}$, we can easily show that

$$st_A - \lim_n \rho\left(\lambda_0\left(T_n f - f\right)\right) = 0,$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition, then one can get the following result from Theorem 2.1 at once.

Theorem 2.2 *Let $A = [a_{jn}]$, $\mathbb{T} := \{T_n\}$, ρ be the same as in Theorem 2.1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent*:

(a) $st_A - \lim_n \rho(\lambda(T_n e_i - e_i)) = 0$ *for every $\lambda > 0$ and $i = 0, 1, 2,$*

(b) $st_A - \lim_n \rho(\lambda(T_n f - f)) = 0$ *for every $\lambda > 0$ provided that f is any function belonging to $L^\rho(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty(I)$.*

If one replaces the matrix A by the identity matrix, then the condition (2.1) reduces to

$$\limsup_n \rho\left(\lambda\left(T_n h\right)\right) \leq P\rho(\lambda h) \tag{2.11}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. In this case, the next results which were obtained by Bardaro and Mantellini [3] immediately follows from our Theorems 2.1 and 2.2.

Corollary 2.3 [3] *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T} := \{T_n\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (2.11). If $\{T_n e_i\}$ is strongly convergent to e_i for each $i = 0, 1, 2$, then $\{T_n f\}$ is modularly convergent to f provided that f is any function belonging to $L^\rho(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty(I)$.*

Corollary 2.4 [3] $\mathbb{T} := \{T_n\}$ *and* ρ *be the same as in Corollary* 2.3. *If* ρ *satisfies the* Δ_2*-condition, then the following statements are equivalent*:

(a) $\{T_n e_i\}$ *is strongly convergent to* e_i *for each* $i = 0, 1, 2$,
(b) $\{T_n f\}$ *is strongly convergent to* f *provided that* f *is any function belonging to* $L^{\rho}(I)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}(I)$.

3 Concluding remarks

In this section, we display an example such that our Korovkin-type statistical approximation results in modular spaces are stronger than the classical ones in [3].

Example Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{n\to\infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^{φ} on $X(I)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \varphi\left(|f(x)|\right) dx \quad \text{for} f \in X(I). \tag{3.1}$$

In this case, ρ^{φ} is a convex modular on $X(I)$, which satisfies all assumptions listed in Sect. 1 (see [3]). Consider the Orlicz space generated by φ as follows:

$$L^{\rho}_{\varphi}(I) := \left\{ f \in X(I) : \rho^{\varphi}(\lambda f) < +\infty \quad \text{for some } \lambda > 0 \right\}.$$

Then consider the following classical Bernstein-Kantorovich operator $\mathbb{U} := \{U_n\}$ on the space $L^{\rho}_{\varphi}(I)$ (see [3]) which is defined by:

$$U_n(f; x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt \quad \text{for } x \in I. \tag{3.2}$$

Observe that the operators U_n map the Orlicz space $L^{\rho}_{\varphi}(I)$ into itself. Moreover, the property (2.11) is satisfied with the choice of $X_{\mathbf{U}} := L^{\rho}_{\varphi}(I)$. Then, by Corollary 2.3, we know that, for any function $f \in L^{\rho}_{\varphi}(I)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}(I)$, $\{U_n f\}$ is modularly convergent to f.

Now take $A = C_1$, the Cesáro matrix of order one, and define a sequence $\{s_n\}$ by

$$s_n = \begin{cases} 0, & \text{if } n \text{ is square} \\ 1, & \text{otherwise.} \end{cases} \tag{3.3}$$

In this case, we know that C_1-statistical convergence coincides with statistical convergence, and its limit is denoted by $st-\lim$. Observe now that, for the sequence $\{s_n\}$

given by (3.3),

$$st - \lim_n s_n = 1.$$

However, the sequence $\{s_n\}$ does not convergent in the usual sense. Then, using the operators U_n, we define the sequence of positive linear operators $\mathbb{V} := \{V_n\}$ on $L_\varphi^\rho(I)$ as follows:

$$V_n(f; x) = s_n U_n(f; x) \quad \text{for } f \in L_\varphi^\rho(I), x \in [0, 1] \text{ and } n \in \mathbb{N}, \tag{3.4}$$

where $s = \{s_n\}$ is the same as in (3.3). By Lemma 5.1 of [3], we get, for every $h \in X_{\mathbb{V}} := L_\varphi^\rho(I)$, $\lambda > 0$ and for an absolute positive constant P, that

$$\begin{aligned}\rho^\varphi(\lambda V_n h) &= \rho^\varphi(\lambda s_n U_n h)\\ &= s_n \rho^\varphi(\lambda U_n h)\\ &\leq P s_n \rho^\varphi(\lambda h).\end{aligned}$$

Now taking $st - \limsup$ as $n \to \infty$ on the both-sides of the above inequality and using the fact that $st - \limsup_n s_n = st - \lim_n s_n = 1$, we obtain

$$st - \limsup_n \rho^\varphi(\lambda V_n h) \leq P \rho^\varphi(\lambda h). \tag{3.5}$$

Therefore, the condition (2.1) works for our operators V_n given by (3.4) with the choice of $X_{\mathbb{V}} = X_{\mathbb{U}} = L_\varphi^\rho(I)$.

We now claim that

$$st - \lim_n \rho^\varphi(\lambda(V_n e_i - e_i)) = 0 \quad \text{for every } \lambda > 0 \text{ and } i = 0, 1, 2. \tag{3.6}$$

Indeed, first observe that (see [1])

$$\begin{aligned}V_n(e_0; x) &= s_n,\\ V_n(e_1; x) &= s_n\left(\frac{nx}{n+1} + \frac{1}{2(n+1)}\right),\\ V_n(e_2; x) &= s_n\left(\frac{n(n-1)x^2}{(n+1)^2} + \frac{2nx}{(n+1)^2} + \frac{1}{3(n+1)^2}\right).\end{aligned}$$

So, we can see, for any $\lambda > 0$, that

$$\lambda\,|V_n(e_0; x) - e_0(x)| = \lambda(1 - s_n),$$

which implies

$$\begin{aligned}\rho^{\varphi}\left(\lambda\left(V_n e_0 - e_0\right)\right) &= \rho^{\varphi}\left(\lambda(1-s_n)\right) \\ &= \int_0^1 \varphi\left(\lambda(1-s_n)\right) dx \\ &= \varphi\left(\lambda(1-s_n)\right) \\ &= (1-s_n)\varphi(\lambda)\end{aligned}$$

because of the definition of (s_n). Now, since $st - \lim_n (1-s_n) = 0$, we get

$$st - \lim_n \rho^{\varphi}\left(\lambda\left(V_n e_0 - e_0\right)\right) = 0 \quad \text{for every } \lambda > 0,$$

which guarantees that (3.6) holds true for $i = 0$. Also, since

$$\begin{aligned}\lambda\left|V_n(e_1; x) - e_1(x)\right| &= \lambda\left\{x\left(1 - \frac{ns_n}{n+1}\right) + \frac{s_n}{2(n+1)}\right\} \\ &\leq \lambda\left\{\left(1 - \frac{ns_n}{n+1}\right) + \frac{s_n}{2(n+1)}\right\} \\ &= \lambda\left\{(1-s_n) + \frac{3s_n}{2(n+1)}\right\},\end{aligned}$$

by the definitions of (s_n) and ρ^{φ}, we may write that

$$\begin{aligned}\rho^{\varphi}\left(\lambda\left(V_n e_1 - e_1\right)\right) &\leq \rho^{\varphi}\left(\lambda\left\{(1-s_n) + \frac{3s_n}{2(n+1)}\right\}\right) \\ &\leq \rho^{\varphi}\left(2\lambda\left(1-s_n\right)\right) + \rho^{\varphi}\left(\frac{3\lambda s_n}{n+1}\right) \\ &= \varphi\left(2\lambda\left(1-s_n\right)\right) + \varphi\left(\frac{3\lambda s_n}{n+1}\right),\end{aligned}$$

which implies, for any $\lambda > 0$, that

$$\rho^{\varphi}\left(\lambda\left(V_n e_1 - e_1\right)\right) \leq (1-s_n)\varphi(2\lambda) + s_n\varphi\left(\frac{3\lambda}{n+1}\right) \tag{3.7}$$

Since φ is continuous, we have $\lim_n \varphi(\frac{3\lambda}{n+1}) = \varphi(\lim_n \frac{3\lambda}{n+1}) = \varphi(0) = 0$. Also considering $st - \lim_n s_n = 1$, we obtain from (3.7) that

$$st - \lim_n \rho^{\varphi}\left(\lambda\left(V_n e_1 - e_1\right)\right) = 0 \quad \text{for every } \lambda > 0.$$

Hence (3.6) is valid for $i = 1$. Finally, since

$$\begin{aligned}\lambda\,|V_n(e_2;x)-e_2(x)| &= \lambda\left\{x^2\left(1-\frac{n(n-1)s_n}{(n+1)^2}\right)+x\left(\frac{2ns_n}{(n+1)^2}\right)+\frac{s_n}{3(n+1)^2}\right\}\\ &\leq \lambda\left\{\left(1-\frac{n\,(n-1)\,s_n}{(n+1)^2}\right)+\left(\frac{2ns_n}{(n+1)^2}\right)+\frac{s_n}{3(n+1)^2}\right\}\\ &= \lambda\left\{(1-s_n)+s_n\left(\frac{15n+4}{3(n+1)^2}\right)\right\}\end{aligned}$$

we get

$$\begin{aligned}\rho^{\varphi}\left(\lambda\left(V_ne_2-e_2\right)\right) &\leq \rho^{\varphi}\left(2\lambda\left(1-s_n\right)\right)+\rho^{\varphi}\left(2\lambda s_n\left(\frac{15n+4}{3(n+1)^2}\right)\right)\\ &= \varphi\left(2\lambda(1-s_n)\right)+\varphi\left(2\lambda s_n\left(\frac{15n+4}{3(n+1)^2}\right)\right),\end{aligned}$$

which yields

$$\rho^{\varphi}\left(\lambda\left(V_ne_2-e_2\right)\right) \leq (1-s_n)\varphi\left(2\lambda\right)+s_n\varphi\left(2\lambda\left(\frac{15n+4}{3(n+1)^2}\right)\right) \tag{3.8}$$

Considering the continuity of φ, it follows from (3.8) that

$$st-\lim_n \rho^{\varphi}\left(\lambda\left(V_ne_2-e_2\right)\right)=0 \quad \text{for every } \lambda>0.$$

So, our claim (3.6) holds true for each $i = 0, 1, 2$ and for any $\lambda > 0$. Now, from (3.5) and (3.6), we can say that our sequence $\mathbb{V} := \{V_n\}$ defined by (3.4) satisfy all assumptions of Theorem 2.1. Therefore, we conclude that

$$st-\lim_n \rho^{\varphi}\left(\lambda_0\left(V_nf-f\right)\right)=0 \;\text{ for some } \lambda_0>0$$

holds for any $f \in L^{\rho}_{\varphi}(I)$ such that $f - g \in X_{\mathbb{V}} = L^{\rho}_{\varphi}(I)$ for every $g \in C^{\infty}(I)$.

However, for the same f's and for any $\lambda > 0$, since

$$\begin{aligned}\rho^{\varphi}\left(\lambda\left(V_nf-f\right)\right) &= \rho^{\varphi}\left(\lambda\left(s_nU_nf-f\right)\right)\\ &= \begin{cases}\rho^{\varphi}(\lambda f), & \text{if } n \text{ is square}\\ \rho^{\varphi}\left(\lambda\left(U_nf-f\right)\right), & \text{otherwise,}\end{cases}\end{aligned}$$

the sequence $\{\rho^{\varphi}(\lambda(V_nf - f))\}$ has two subsequences with different limit points. In other words, there is not any $\lambda > 0$ such that the sequence $\{\rho^{\varphi}(\lambda(V_nf - f))\}$ goes to zero as $n \to \infty$. This clearly shows that $\{V_nf\}$ is not modularly convergent to f. So, Corollary 2.3 does not work for the sequence $\mathbb{V} := \{V_n\}$.

With this example, we can immediately say that our Theorem 2.1 is a non-trivial generalization of the classical Korovkin-type results in the modular spaces.

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