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# Bounded Multipliers of Bounded $A$-Statistically Convergent Sequences

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In this paper we provide an alternate proof of a result of Kline. It claims that the set of all $A$-statistically convergent sequences cannot be given a locally convex $FK$ topology where $A$ is a nonnegative regular matrix whose rows spread. We also study the bounded multiplier space of all bounded $A$-statistically convergent sequences, and using the "*βN program*" we give an analogue of a result of Fridy and Miller for bounded multipliers. © 1999 Academic Press

## 1. INTRODUCTION

Kline [20], using a result due to Bennett and Kalton [3], has shown that the space of $A$-statistically convergent sequences cannot be endowed with a locally convex $FK$ topology where $A$ is a nonnegative regular summability matrix whose rows spread. In this paper, we have provided an alternate proof of it via counting functions (see Section 2). Although the "$FK$ program" is not available when studying $A$-statistical convergence, there is another machine, the so-called "*βN program*" (see [20, 8]). It involves the Stone–Čech compactification of the integers. In the final section, we study the bounded multiplier space of bounded $A$-statistically convergent sequences, and using the "*βN program*" we give an analogue of Theorem 4 of Fridy and Miller [14].

If $K$ is a set of positive integers, $K_n$ will denote the set $\{k \in K: k \leq n\}$ and $|K_n|$ will denote the cardinality of $K_n$. The natural density of $K$ [23] is given by $\delta(K) := \lim_n \frac{1}{n}|K_n|$, if it exists. The real number sequence $x = (x_k)$ is statistically convergent to $L$ provided that for every $\varepsilon > 0$ the set $K := K(\varepsilon) := \{k \in N: |x_k - L| \geq \varepsilon\}$ has natural density zero [11, 13]. In



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this case, we write $st$-lim $x = L$. Hence $x$ is statistically convergent to $L$ iff $(C_1 \chi_{K(\varepsilon)})_n \to 0$ (as $n \to \infty$, for every $\varepsilon > 0$), where $C_1$ is the Cesàro mean of order one and $\chi_K$ is the characteristic function of the set $K$.

A matrix characterization and a measure theoretical subsequence characterization of statistical convergence may be found in [14, 22], respectively.

We denote by $s, s_b$, and $s_o$, respectively, the set of all statistically convergent sequences, the set of all bounded statistically convergent sequences, and the set of all statistically null sequences.

Statistical convergence has been studied in number theory [10], trigonometric series [28], summability theory [16], and in locally convex spaces [21]. Generalizations of statistical convergence have been made in the study of strong integral summability [6] and the structure of ideals of bounded continuous functions on locally compact spaces [7]. It is also connected with probability theory [17] and subsets of the Stone–Čech compactification of the natural numbers [8, 20].

We shall use the common notation for matrix summability: if $x = (x_k)$ is a number sequence and $A = (a_{nk})$ is an infinite matrix, then $Ax$ is the sequence whose $n$th term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$, where we assume that the series $\sum_{k=1}^{\infty} a_{nk} x_k$ is convergent for each $n \in N$. If $\lim_n (Ax)_n = L$, we then say that $x$ is $A$-summable to $L$. In this case, we write $A$-lim $x = L$.

Following Freedman and Sember [12], we say that a set $K \subseteq N$ has $A$-density if $\delta_A(K) := \lim_n \sum_{k \in K} a_{nk}$ exists where $A$ is a nonnegative regular matrix. The real number sequence $x = (x_k)$ is $A$-statistically convergent to $L$ provided that for every $\varepsilon > 0$ the set $K(\varepsilon)$ has $A$-density zero [5]. In this case we write $st_A$-lim $x = L$.

By $st_A$, $st_A^0$, $st_A(b)$ we denote the set of all $A$-statistically convergent sequences, the set of all $A$-statistically null sequences, and the set of all bounded $A$-statistically convergent sequences, respectively.

The case in which $A = C_1$ reduces to the usual definition of statistical convergence [11].

We now recall lacunary statistical convergence, introduced in [15, 16], that can be regarded as a special case of $A$-statistical convergence. A lacunary sequence is an increasing sequence of positive integers $\theta = (k_n)$ such that $k_0 = 0$ and $h_n := k_n - k_{n-1} \to \infty$ as $n \to \infty$; and we set $I_n := (k_{n-1}, k_n]$. Let $\theta$ be a lacunary sequence. The sequence $x = (x_k)$ is said to be lacunary statistically convergent to $L$ (see [15, 16]) provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{h_n} \left| \{ k \in I_n : |x_k - L| \geq \varepsilon \} \right| = 0.$$

This definition is included by the definition of $A$-statistical convergence above, where the matrix $C_\theta$ associated with a given lacunary sequence $\theta$ is determined by

$$C_\theta = (c_{nk}) = \begin{cases} \dfrac{1}{h_n}, & \text{if } k \in I_n \\ 0, & \text{otherwise.} \end{cases}$$

By $s_\theta$, $s_\theta^0$, $s_\theta(b)$ we denote the set of all lacunary statistically convergent sequences, the set of all lacunary statistically null sequences, and the set of all bounded lacunary statistically convergent sequences, respectively.

## 2. THE FAILURE OF *FK* TOPOLOGY FOR A-STATISTICALLY CONVERGENT SEQUENCES

Let $w$ denote the space of all real- (or complex-) valued sequences, topologized by means of coordinatewise convergence, and any linear subspace of $w$ is called a sequence space. A sequence space $E$ with a complete, metrizable, locally convex topology $H$ is an $FK$-space if the inclusion mapping $(E, H) \to w$ is continuous. The basic properties of such spaces may be found in [27].

In this section we show that the set of all $A$-statistically convergent sequences cannot be given a locally convex $FK$ topology where $A$ is a nonnegative regular summability matrix whose rows spread. Recall that the rows of $A$ spread provided that

$$\lim_n \max_k |a_{nk}| = 0.$$

For instance, the matrices $C_1$ and $C_\theta$ have rows which spread.

Let $r = (r_n)$ denote a nondecreasing, unbounded sequence of positive integers with $r_1 = 1$ and $r_n = o(n)$. By $c_n(x)$ we denote the number of nonzero elements in the set $\{x_1, x_2, \dots, x_n\}$. Given any subset $E$ of $w$, we consider

$$\sigma(E, r) := \{x \in E\colon c_n(x) \le r_n \text{ for } n = 1, 2, \dots\}.$$

$\sigma(E, r)$ is not, in general, a vector space, and we denote its linear span by $\Sigma(E, r)$. Such a space will be called a scarce copy of $E$ [2].

Let $r\colon N \to R$ be a monotone increasing function which is unbounded. If $A$-lim $x = 0$ for every bounded sequence, $x = (x_n)$, satisfying the condition $c_n(x) \le r_n$ $(n = 1, 2, \dots)$, then $r$ is said to be a counting function of the first kind for the matrix $A$ (see [24, p. 65]).

We recall that the condition

$$\lim_n \max_k |a_{nk}| = 0,$$

is necessary and sufficient for a regular matrix $A = (a_{nk})$ to have a counting function of the first kind (see [24; Theorem 3.3.3]).

Now we are ready to give the first result of this section.

THEOREM 1. *Let $A = (a_{nk})$ be a nonnegative regular matrix whose rows spread. Then $\Sigma(w,r) \subseteq st_A^o$.*

*Proof.* Let $z \in \Sigma(w,r)$. Then we have $z = \sum_{k=1}^n \alpha_k x^{(k)}$ where $\alpha_k$ are scalars and $x^{(k)} \in \sigma(w,r)$ with $x^{(k)} = (x_n^{(k)}) = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}, \dots)$.

For fixed $k$ $(1 \le k \le n)$, consider $x^{(k)} \in \sigma(w,r)$. It follows from the definition that

$$c_n(x^{(k)}) = \left|\left\{m \le n\colon\ x_m^{(k)} \ne 0\right\}\right| = \left|\left\{m \le n\colon\ \chi_K(m) \ne 0\right\}\right| \le r_n$$

where $K := \{m \le n\colon x_m^{(k)} \ne 0\}$. So we must have $A$-lim $\chi_K = 0$. Thus $x^{(k)} \in st_A^0$. Since $A$-statistical convergence is linear, we get $z \in st_A^0$ which completes the proof.

Recall that every scarce copy of $w$ is barrelled in $w$ (see [2; Theorem 7]) and $\Sigma(w,r)$ is dense in $w$. Hence we conclude the following immediately.

COROLLARY 2. *If $A$ is a nonnegative regular matrix whose rows spread, then $st_A^0$ is barrelled in $w$.*

In particular, $s_0$ and $s_\theta^0$ are barrelled in $w$.

If $st_A$ is given a locally convex $FK$ topology, then the corollary to Theorem 7 of Bennett [2] implies $st_A = w$, which is impossible. So we have the following result of [20].

THEOREM 3. *Let $A = (a_{nk})$ be a nonnegative regular matrix whose rows spread. Then $st_A$ and $st_A^0$ cannot be given a locally convex FK topology.*

Thus we also get the following result of [4].

COROLLARY 4. *$s$ cannot be given a locally convex FK topology.*

Of course, $s_0, s_\theta^0$ and $s_\theta$ cannot be given a locally convex FK topology either.

We close this section with the following remark. If A is a nonnegative regular matrix whose rows do not spread, then the set of all A-statistically convergent sequences may be given a locally convex FK topology. Such an example is provided in [20].

## 3. BOUNDED MULTIPLIER SPACE OF $st_A(b)$

Let $A$ be a nonnegative regular matrix and $\tau_A$ to be the collection of all nonnegative summability matrices $T$ satisfying

(a) $\sum_{k=1}^{\infty} t_{nk} = 1$ for every $n$,

(b) if $K \subseteq N$ such that $\delta_A(K) = 0$, then $\lim_n \sum_{k \in K} t_{nk} = 0$.

When $A = C_1$ and $A = C_\theta$ we shall simply write $\tau$ and $\tau_\theta$ instead of $\tau_{C_1}$ and $\tau_{C_\theta}$. From now on the bounded summability field of the matrix $T \in \tau_A$ will be denoted by $c_T(b)$, i.e.,

$$c_T(b) = \left\{ x \in l_\infty \colon \lim_n (Tx)_n \text{ exists} \right\}.$$

In the sequel we make extensive use of the following characterization of Fridy and Miller.

THEOREM A ([14; Theorem 4]). *The bounded sequence x satisfies $st_A$-$\lim x = L$ if and only if $x$ is $T$-summable to $L$ for every $T$ in $\tau_A$ (i.e., $st_A(b) = \bigcap_{T \in \tau_A} c_T(b)$).*

The bounded strong summability field of the matrix $T \in \tau_A$ is the set

$$W_b(T) = \left\{ x \in l_\infty \colon \lim_n \sum_{k=1}^{\infty} t_{nk} |x_k - a| = 0 \text{ for some } a \right\}.$$

We say that x is strongly T-summable to a real number $a$ if

$$\lim_n \sum_{k=1}^{\infty} t_{nk} |x_k - a| = 0.$$

Suppose that two sequence spaces, E and F, are given. We say that a sequence m is a bounded multiplier of E into F, and we write M(E, F), if m.x $\in$ F whenever x $\in$ E, i.e.,

$$M(E, F) = \{ m \in l_\infty \colon m.x \in F \text{ for all } x \in E \}$$

where the multiplication is coordinatewise. If F = E, then we write M(E) instead of M(E, E).

It is well known that

$$M(c_T(b)) = W_b(T) \tag{3.1}$$

provided that $t_{nk} \geq 0$ for all n and k (see [19; Theorem 3.3]).

We also need the following known result in the sequel.

THEOREM B ([19; Theorem 4.1]). *If $T$ is a regular matrix, then the bounded sequence $x$ is strongly $T$-summable to $a$ if and only if there exists a subset $Z$ of $N$ such that $\chi_{N \setminus z}$ is strongly $T$-summable to zero and* $\lim_{n \in Z} x_n = a$.

In this section, we study the bounded multiplier space of $st_A(b)$ and then give an analogue of Theorem A for bounded multipliers.

With this terminology we have

THEOREM 5. *$x \in M(st_A(b))$ if and only if $x \in st_A(b)$.*

*Proof.* Note that Theorem 5 may also be derived from Theorem 1 of [25]. But we present here another proof of it.

Necessity. Let $x \in M(st_A(b))$. Then we have $x \in M(\cap_{T \in \tau_A} c_T(b))$ by Theorem A. Since $\chi_N \in \cap_{T \in \tau_A} c_T(b)$ we get $M(\cap_{T \in \tau_A} c_T(b)) \subset \cap_{T \in \tau_A} c_T(b)$; hence $x \in \cap_{T \in \tau_A} c_T(b)$. Thus, Theorem A yields $x \in st_A(b)$.

Sufficiency. Let $x \in st_A(b)$. Then $xy \in st_A(b)$ for an arbitrary $y \in st_A(b)$. Hence $x \in M(st_A(b))$, so the proof is completed.

Now using the same technique as in Theorem 3 of [9], we give an analogue of Theorem A for bounded multiplier space, but first we need some properties of $\beta N$, the Stone–Čech compactification of positive integers $N$. For each $B \subseteq N$, let $cl_{\beta N} B$ be the closure of $B$ in $\beta N$ and let $B^* = (cl_{\beta N} B) \setminus B$. It is well known [18] that the sets $\{B^*\colon B \subseteq N\}$ form a basis for the topology of $\beta N \setminus N$.

Recall that, for a regular matrix $A$, the support set $K_A$ is defined by

$$K_A = \bigcap \{B^*\colon B \subseteq N \text{ and } \chi_B \text{ is } A\text{-summable to } 1\}$$

which is nonempty compact subset of $\beta N \setminus N$ [1]. Observe that if $B^* \supseteq K_A$, then $\chi_B$ is $A$-summable to 1. Furthermore, $K_A$ is a $P$-set in $\beta N \setminus N$; i.e., the intersection of any sequence of neighborhoods of $K_A$ is again a neighborhood of $K_A$. Also, if $B$ and $D$ are infinite subsets of $N$, then $D^*$ is contained in $B^*$ if and only if $D \setminus B$ is finite [26; Proposition 3.14]. We note that the "*$\beta N$ program*" is extensively used in [8, 20].

Now we are ready to give the main result.

THEOREM 6. *$M(st_A(b)) = \cap_{T \in \tau_A} M(c_T(b))$.*

*Proof.* Let $x \in \cap_{T \in \tau_A} M(c_T(b))$ and $y \in st_A(b)$. By Theorem A, $st_A(b) \subset c_T(b)$ for every $T \in \tau_A$. This yields that $xy \in c_T$. Hence $xy \in \cap_{T \in \tau_A} c_T(b)$. Now Theorem A implies that $xy \in st_A(b)$ so $x \in M(st_A(b))$. Thus $\cap_{T \in \tau_A} M(c_T(b)) \subseteq M(st_A(b))$.

Conversely, assume that $x \in M(st_A(b))$ and $T \in \tau_A$. Let $K(k) := \{n \in N: |x_n - \alpha| < \frac{1}{k}\}$. Then by Theorem 5 we have $\delta_A(K(k)) = 1$, $(k = 1, 2, \ldots)$. Hence for each $k$, $\chi_{K(k)}$ is $A$-statistically convergent to 1. Now, Theorem A implies that, for each $k$, $\chi_{K(k)}$ is $T$-summable to 1 for every $T$ in $\tau_A$. Thus $K^*(k) \supseteq K_T$ for each $k$ and $\cap_{k=1}^{\infty} K^*(k) \supseteq K_T$. Since $K_T$ is compact and the sets $\{B^*: B \subseteq N\}$ form a basis for the topology of $\beta N \setminus N$, there exists a set $K \subseteq N$ such that $\cap_{k=1}^{\infty} K^*(k) \supseteq K^* \supseteq K_T$. Also $\lim_n (T\chi_K)_n = 1$. Since $K^*(k) \supseteq K^*$ for each $k$, there are at most a finite number of members of $K$ not in $K(k)$. Hence $|x_n - \alpha| < \frac{1}{k}$ for all but a finite number of $n \in K$. Since $k$ is arbitrary, we conclude that

$$\lim_{n \in K} x_n = \alpha. \tag{3.2}$$

Now we write

$$\lim_n \sum_{j=1}^{\infty} t_{nj} = \lim_n \sum_{j \in K} t_{nj} + \lim_n \sum_{j \in N \setminus K} t_{nj}.$$

Since $T$ is regular and $\lim_n (T\chi_K)_n = 1$, we get $\lim_n \sum_{j \in N \setminus K} t_{nj} = 0$. Hence $\chi_{N \setminus K}$ is strongly $T$-summable to zero. Combining this with (3.2), we conclude by Theorem B that $x$ is strongly $T$-summable to $\alpha$. Appealing (3.1) we get $x \in M(c_T(b))$. Hence $x \in \cap_{T \in \tau_A} M(c_T(b))$ for every $T$ in $\tau_A$ which completes the proof.

Taking the matrix $A$ to be $C_1$ and $C_\theta$ matrices we immediately get the next two corollaries from Theorem 5 while the latter are from Theorem 6.

COROLLARY 7. *$x \in M(s_b)$ if and only if $x \in s_b$.*

COROLLARY 8. *$x \in M(s_\theta(b))$ if and only if $x \in s_\theta(b)$.*

COROLLARY 9. *$M(s_b) = \cap_{A \in \tau} M(c_A(b))$.*

COROLLARY 10. *$M(s_\theta(b)) = \cap_{A \in \tau_\theta} M(c_A(b))$*

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