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Statistical approximation by double sequences of positive linear operators on modular spaces

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Abstract In this paper, using the concept of statistical convergence, which is stronger than the Pringsheim convergence, we study the problem of approximation to a function by means of double sequences of positive linear operators defined on a modular space. Also, a non-trivial application is presented.

Keywords Positive linear operators · Double sequences · Modular space · Regular matrix · Statistical convergence · Korovkin theorem

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1 Introduction

Bardaro and Mantellini [3] have introduced some approximation theorems with the help of the notions of modular convergence and strong convergence. Later, they have studied a modular version of the classical Korovkin theorem in multivariate modular function spaces in [4]. In recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved [7–9]. Recall that every convergent sequence (in the usual sense) is statistically convergent but its

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converse is not always true. Also, statistical convergent sequences do not need to be bounded. So, the usage of this method of convergence in the approximation theory provides us many advantages. Using the concept of statistical convergence, Karakuş et al. [9] have investigated the Korovkin-type approximation theorem on modular space. Also, in [2], some versions of abstract Korovkin-type theo- rems in modular function spaces, with respect to filter convergence for linear positive operators, by considering several kinds of test functions have studied. In this work, we investigate the problem of statistical approximation to a function f by means of a double sequence of positive linear operators defined on a multivariate modular function spaces. Finally, we construct an example such that our new result works but its classical case does not work.

Let us first remind of the concept of statistical convergence for double sequences.

A double sequence $x = \{x_{m,n}\}$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$, where L is called the Pringsheim limit of x and denoted by $P - \lim_{m,n} x_{m,n} = L$ (see [16]). We shall call such an x, briefly, "P-convergent". A double sequence is called bounded if there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Statistical convergence of single sequences was introduced by Steinhaus [17] and studied by many authors. Recently, this concept was extended to the double sequences. If $E \subset \mathbb{N}^2$ is a two-dimensional subset of positive integers, then $E_{j,k}$ denotes the set $\{(m, n) \in E : m \leq j,\ n \leq k\}$ and $|E_{j,k}|$ denotes the cardinality of $E_{j,k}$. The double natural density of E [12,13] is given by

$$\delta_2(E) := P - \lim_{j,k} \frac{1}{jk} |E_{j,k}|,$$

if it exists. The number sequence $x = \{x_{m,n}\}$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$E := E_{j,k}(\varepsilon) := \left\{m \leq j, n \leq k : |x_{m,n} - L| \geq \varepsilon\right\}$$

has natural density zero; in that case we write $st_2 - \lim_{m,n} x_{m,n} = L$.

Clearly, a P-convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, note that a statistically convergent double sequence need not to be bounded. For example, consider the double sequence $x = \{x_{m,n}\}$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

Then, clearly $st_2 - \lim_{m,n} x_{m,n} = 1$. Nevertheless, x is neither convergent nor bounded.

The concepts of *the statistical superior limit* and *inferior limit* for double sequences have been introduced by Çakan and Altay [6]. For any real double sequence $x = \{x_{m,n}\}$, the the statistical superior limit of x is

$$st_2 - \limsup_{m,n} x_{m,n} = \begin{cases} \sup G_x, & \text{if } G_x \neq \varnothing, \\ -\infty, & \text{if } G_x = \varnothing, \end{cases}$$

where $G_x := \left\{C \in \mathbb{R} : \delta_2 \left\{(m,n) : x_{m,n} > C\right\} \neq 0\right\}$ and $\varnothing$ denotes the empty set. We note that by $\delta_2 \{K\} \neq 0$ we mean either $\delta_2 \{K\} > 0$ or K fails to have the double natural density. Similarly, the statistical inferior limit of x is

$$st_2 - \liminf_{m,n} x_{m,n} = \begin{cases} \inf F_x, & \text{if } F_x \neq \varnothing, \\ \infty, & \text{if } F_x = \varnothing, \end{cases}$$

where $F_x := \left\{D \in \mathbb{R} : \delta_2 \left\{(m,n) : x_{m,n} < D\right\} \neq 0\right\}$. As in the ordinary superior or inferior limit, it was proved that

$$st_2 - \liminf_{m,n} x_{m,n} \leq st_2 - \limsup_{m,n} x_{m,n}$$

and also that, for any double sequence $x = \{x_{m,n}\}$ satisfying
$\delta_2\{(m,n) : \left|x_{m,n}\right| > M\} = 0$ for some $M > 0$,

$$st_2 - \lim_{m,n} x_{m,n} = L \text{ iff } st_2 - \liminf_{m,n} x_{m,n} = st_2 - \limsup_{m,n} x_{m,n} = L.$$

Let $I = [a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X\left(I^2\right)$ we denote the space of all real-valued measurable functions on $I^2 = [a,b] \times [a,b]$ provided with equality *a.e.* As usual, let $C\left(I^2\right)$ denote the space of all continuous real-valued functions, and $C^\infty\left(I^2\right)$ denote the space of all infinitely differentiable functions on I^2. In this case, we say that a functional $\rho : X\left(I^2\right) \to [0, +\infty]$ is a *modular* on $X\left(I^2\right)$ provided that the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ *a.e.* in I^2,
(ii) $\rho(-f) = \rho(f)$ for every $f \in X\left(I^2\right)$,
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(I^2)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Recall that a modular ρ is called *N-quasi convex* if the following is satisfied:

- there exists a constant $N \geq 1$ such that

$$\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng)$$

holds for every $f, g \in X\left(I^2\right)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*.

Furthermore, a modular ρ is called *N-quasi semiconvex* if it holds:

- there exists a constant $N \geq 1$ such that

$$\rho(af) \leq Na\rho(Nf)$$

holds for every $f \in X\left(I^2\right)$ and $a \in (0,1]$.

It is clear that every N-quasi semiconvex modular is N-quasi convex. We should recall that the above two concepts were introduced and discussed in details by Bardaro et al. [4,5].

We now consider some appropriate vector subspaces of $X(I^2)$ by means of a modular ρ as follows:

$$L^{\rho}\left(I^2\right) := \left\{f \in X\left(I^2\right) : \lim_{\lambda \to 0^+} \rho\left(\lambda f\right) = 0\right\}$$

and

$$E^{\rho}\left(I^2\right) := \left\{f \in L^{\rho}\left(I^2\right) : \rho\left(\lambda f\right) < +\infty \quad \text{for all } \lambda > 0\right\}.$$

Here, $L^{\rho}\left(I^2\right)$ is called the *modular space* generated by ρ; and $E^{\rho}\left(I^2\right)$ is called the space of the finite elements of $L^{\rho}\left(I^2\right)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\left\{f \in X\left(I^2\right) : \rho\left(\lambda f\right) < +\infty \quad \text{for some } \lambda > 0\right\}$$

coincides with $L^{\rho}\left(I^2\right)$. The notions about modulars are introduced in [14] and widely discussed in [5] (see also [10,15]).

Now we recall the convergence methods in modular spaces.

Let $\{f_{m,n}\}$ be a double function sequence whose terms belong to $L^{\rho}\left(I^2\right)$. Then, $\{f_{m,n}\}$ is *modularly convergent* to a function $f \in L^{\rho}\left(I^2\right)$ iff

$$P - \lim_{m,n} \rho\left(\lambda_0\left(f_{m,n} - f\right)\right) = 0 \quad \text{for some } \lambda_0 > 0. \tag{1}$$

Also, $\{f_{m,n}\}$ is *F-norm convergent* (or, *strongly convergent*) to f iff

$$P - \lim_{m,n} \rho\left(\lambda\left(f_{m,n} - f\right)\right) = 0 \quad \text{for every } \lambda > 0. \tag{2}$$

It is known from [14] that (1) and (2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho\left(2f\right) \leq M\rho\left(f\right)$ for every $f \in X\left(I^2\right)$.

In this paper, we will need the following assumptions on a modular ρ :

- ρ is *monotone* if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$,
- ρ is *finite* if $\chi_A \in L^{\rho}\left(I^2\right)$ whenever A is measurable subset of I^2 such that $\mu\left(A\right) < \infty$,
- ρ is *absolutely finite* if ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho\left(\lambda\chi_B\right) < \varepsilon$ for any measurable subset $B \subset I^2$ with $\mu\left(B\right) < \delta$,
- ρ is *strongly finite* if $\chi_{I^2} \in E^{\rho}\left(I^2\right)$,
- ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X\left(I^2\right)$ with $\rho\left(f\right) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there

is $\delta > 0$ such that $\rho\left(\alpha f \chi_B\right) < \varepsilon$ whenever B is any measurable subset of I^2 with $\mu\left(B\right) < \delta$.

Observe now that (see [3,4]) if a modular ρ is monotone and finite, then we have $C(I^2) \subset L^{\rho}\left(I^2\right)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I^2) \subset E^{\rho}\left(I^2\right)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^{\infty}(I^2)} = L^{\rho}\left(I^2\right)$. Some important relations between the above properties may be found in [1,4,5,11,15].

2 Statistical Korovkin theorems in modular spaces

Let ρ be a monotone and finite modular on $X\left(I^2\right)$. Assume that D is a set satisfying $C^{\infty}\left(I^2\right) \subset D \subset L^{\rho}\left(I^2\right)$. We can construct such a subset D when ρ is monotone and finite (see [3]). Here, D is domain of the operator $\mathbb{T}$. Assume further that $\mathbb{T} := \{T_{m,n}\}$ is a sequence of positive linear operators from D into $X\left(I^2\right)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ containing $C^{\infty}\left(I^2\right)$ such that

$$st_2 - \limsup_{m,n} \rho\left(\lambda\left(T_{m,n}h\right)\right) \leq R\rho\left(\lambda h\right) \tag{3}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R.

Also, we denote the value of $T_{m,n}f$ at a point $(x, y) \in I^2$ by $T_{m,n}(f(u, v); x, y)$ or, briefly, $T_{m,n}(f; x, y)$.

Theorem 1 *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{m,n}\}$ be a double sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (3). Assume that*

$$st_2 - \lim_{m,n} \rho\left(\lambda\left(T_{m,n}\left(e_i\right) - e_i\right)\right) = 0 \quad \text{for every } \lambda > 0 \text{ and } i = 0, 1, 2, 3 \tag{4}$$

where $e_0\left(x, y\right) = 1$, $e_1\left(x, y\right) = x$, $e_2\left(x, y\right) = y$ and $e_3\left(x, y\right) = x^2 + y^2$. Now let f be any function belonging to $L^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$. Then, we have

$$st_2 - \lim_{m,n} \rho\left(\lambda_0\left(T_{m,n}f - f\right)\right) = 0 \quad \text{for some } \lambda_0 > 0. \tag{5}$$

Proof We first claim that

$$st_2 - \lim_{m,n} \rho\left(\eta(T_{m,n}g - g)\right) = 0 \quad \text{for every } g \in C(I^2) \cap D \text{ and every } \eta > 0. \tag{6}$$

To see this assume that g belongs to $C\left(I^2\right) \cap D$. By the continuity of g on I^2, given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $\left(u, v\right), \left(x, y\right) \in I^2$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$ we have

$$|g(u, v) - g(x, y)| < \varepsilon. \tag{7}$$

Also we get for all $(u, v), (x, y) \in I^2$ satisfying $|u - x| > \delta$ and $|v - y| > \delta$ that

$$|g(u, v) - g(x, y)| \leq \frac{2M}{\delta^2} \left\{ (u - x)^2 + (v - y)^2 \right\} \tag{8}$$

where $M := \sup_{(x,y) \in I^2} |g(x, y)|$. Combining (7) and (8) we have for $(u, v), (x, y) \in I^2$ that

$$|g(u, v) - g(x, y)| < \varepsilon + \frac{2M}{\delta^2} \left\{ (u - x)^2 + (v - y)^2 \right\}.$$

Since $T_{m,n}$ is linear and positive, we obatin

$$\begin{aligned} \left|T_{m,n}(g; x, y) - g(x, y)\right| &\leq \varepsilon T_{m,n}(e_0; x, y) + M \left|T_{m,n}(e_0; x, y) - e_0(x, y)\right| \\ &\quad + \frac{2M}{\delta^2} T_{m,n} \left((u - x)^2 + (v - y)^2 ; x, y \right) \end{aligned}$$

holds for every $x, y \in I$ and $m, n \in \mathbb{N}$. The above inequality implies that

$$\begin{aligned} \left|T_{m,n}(g; x, y) - g(x, y)\right| &\leq \varepsilon + \left\{ \varepsilon + M + \frac{4M}{\delta^2} E^2 \right\} \left|T_{m,n}(e_0; x, y) - e_0(x, y)\right| \\ &\quad + \frac{4M}{\delta^2} E \left|T_{m,n}(e_1; x, y) - e_1(x, y)\right| \\ &\quad + \frac{4M}{\delta^2} E \left|T_{m,n}(e_2; x, y) - e_2(x, y)\right| \\ &\quad + \frac{2M}{\delta^2} \left|T_{m,n}(e_3; x, y) - e_3(x, y)\right|. \end{aligned}$$

where $E := \max\{|x|, |y|\}$. So, the last inequality gives, for any $\eta > 0$, that

$$\begin{aligned} \eta \left|T_{m,n}(g; x, y) - g(x, y)\right| \leq \eta\varepsilon + K\eta \Big\{ &\left|T_{m,n}(e_0; x, y) - e_0(x, y)\right| \\ &+ \left|T_{m,n}(e_1; x, y) - e_1(x, y)\right| \\ &+ \left|T_{m,n}(e_2; x, y) - e_2(x, y)\right| \\ &+ \left|T_{m,n}(e_3; x, y) - e_3(x, y)\right| \Big\}, \end{aligned}$$

where $K := \max\left\{ \varepsilon + M + \frac{4M}{\delta^2} E^2, \frac{4M}{\delta^2} E, \frac{2M}{\delta^2} \right\}$. Now, applying the modular ρ in both-sides of the above inequality, since ρ is monotone, we have

$$\begin{aligned} \rho\left(\eta(T_{m,n}g - g)\right) \leq \rho\Big(&\eta\varepsilon + \eta K \left|T_{m,n}e_0 - e_0\right| + \eta K \left|T_{m,n}e_1 - e_1\right| \\ &+ \eta K \left|T_{m,n}e_2 - e_2\right| + \eta K \left|T_{m,n}e_3 - e_3\right|\Big). \end{aligned}$$

Hence, we may write that

$$\begin{aligned} \rho\left(\eta(T_{m,n}g - g)\right) \leq\; &\rho(5\eta\varepsilon) + \rho\left(5\eta K\left(T_{m,n}e_0 - e_0\right)\right) + \rho\left(5\eta K\left(T_{m,n}e_1 - e_1\right)\right) \\ &+ \rho\left(5\eta K\left(T_{m,n}e_2 - e_2\right)\right) + \rho\left(5\eta K\left(T_{m,n}e_3 - e_3\right)\right) \end{aligned}$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0 < \varepsilon \leq 1$,

$$\begin{aligned}\rho\left(\eta(T_{m,n}g-g)\right) \leq N\varepsilon\rho\left(5\eta N\right)+\rho\left(5\eta K\left(T_{m,n}e_0-e_0\right)\right)\\ +\rho\left(5\eta K\left(T_{m,n}e_1-e_1\right)\right)+\rho\left(5\eta K\left(T_{m,n}e_2-e_2\right)\right)\\ +\rho\left(5\eta K\left(T_{m,n}e_3-e_3\right)\right).\end{aligned}$$

For a given $r > 0$, choose an $\varepsilon \in (0, 1]$ such that $N\varepsilon\rho\left(5\eta N\right) < r$. Now define the following sets:

$$\begin{aligned}G_\eta &:= \left\{(m,n) : \rho\left(\eta(T_{m,n}g-g)\right) \geq r\right\}\\ G_{\eta,i} &:= \left\{(m,n) : \rho\left(5\eta K\left(T_{m,n}e_i-e_i\right)\right) \geq \frac{r-N\varepsilon\rho\left(5\eta N\right)}{4}\right\},\end{aligned}$$

where $i = 0, 1, 2, 3$. Then, it is easy to see that $G_\eta \subseteq \bigcup_{i=0}^{3} G_{\eta,i}$. So, we can write that

$$\delta_2\left\{G_\eta\right\} \leq \sum_{i=0}^{3}\delta_2\left\{G_{\eta,i}\right\}.$$

Using the hypothesis (4), we get

$$\delta_2\left\{G_\eta\right\} = 0,$$

which proves our claim (6). Observe that (6) also holds for every $g \in C^\infty(I^2)$ because of $C^\infty(I^2) \subset C(I^2) \cap D$. Now let $f \in L^\rho\left(I^2\right)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty\left(I^2\right)$. Since $\mu\left(I^2\right) < \infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I^2)$. Using these properties of the modular ρ, it is known from [5,11] that the space $C^\infty(I^2)$ is modularly dense in $L^\rho\left(I^2\right)$, i.e., there exists a sequence $\{g_{k,j}\} \subset C^\infty\left(I^2\right)$ such that

$$P-\lim_{k,j}\rho\left[3\lambda_0^*\left(g_{k,j}-f\right)\right] = 0 \quad \text{for some } \lambda_0^* > 0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\rho\left[3\lambda_0^*\left(g_{k,j}-f\right)\right] < \varepsilon \quad \text{for every } k, j \geq k_0. \tag{9}$$

On the other hand, by the linearity and positivity of the operators $T_{m,n}$, we may write that

$$\begin{aligned}\lambda_0^*\left|T_{m,n}(f;x,y)-f(x,y)\right| \leq \lambda_0^*\left|T_{m,n}(f-g_{k_0,k_0};x,y)\right|\\ +\lambda_0^*\left|T_{m,n}(g_{k_0,k_0};x,y)-g_{k_0,k_0}(x,y)\right|\\ +\lambda_0^*\left|g_{k_0,k_0}(x,y)-f(x,y)\right|\end{aligned}$$

holds for every $x, y \in I$ and $m, n \in \mathbb{N}$. Applying the modular ρ in the last inequality and using the monotonicity of ρ, we have

$$\begin{aligned}\rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right) \leq\ & \rho\left(3\lambda_0^* T_{m,n}\left(f - g_{k_0,k_0}\right)\right) \\ & + \rho\left(3\lambda_0^*\left(T_{m,n}g_{k_0,k_0} - g_{k_0,k_0}\right)\right) \\ & + \rho\left(3\lambda_0^*\left(g_{k_0,k_0} - f\right)\right).\end{aligned} \tag{10}$$

Then, it follows from (9) and (10) that

$$\begin{aligned}\rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right) \leq\ & \varepsilon + \rho\left(3\lambda_0^* T_{m,n}\left(f - g_{k_0,k_0}\right)\right) \\ & + \rho\left(3\lambda_0^*\left(T_{m,n}g_{k_0,k_0} - g_{k_0,k_0}\right)\right).\end{aligned} \tag{11}$$

So, taking statistical limit superior as $m, n \to \infty$ in both-sides of (11) and also using the facts that $g_{k_0,k_0} \in C^{\infty}(I^2)$ and $f - g_{k_0,k_0} \in X_{\mathbb{T}}$, we obtained from (3) that

$$\begin{aligned}st_2 - \limsup_{m,n} \rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right) \leq\ & \varepsilon + R\rho\left(3\lambda_0^*(f - g_{k_0,k_0})\right) \\ & + st_2 - \limsup_{m,n} \rho\left(3\lambda_0^*\left(T_{m,n}g_{k_0,k_0} - g_{k_0,k_0}\right)\right),\end{aligned}$$

which gives

$$\begin{aligned}& st_2 - \limsup_{m,n} \rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right) \\ & \leq \varepsilon(R+1) + st_2 - \limsup_{m,n} \rho\left(3\lambda_0^*\left(T_{m,n}g_{k_0,k_0} - g_{k_0,k_0}\right)\right).\end{aligned} \tag{12}$$

By (6), since

$$st_2 - \lim_{m,n} \rho\left(3\lambda_0^*\left(T_{m,n}g_{k_0,k_0} - g_{k_0,k_0}\right)\right) = 0,$$

we get

$$st_2 - \limsup_{m,n} \rho\left(3\lambda_0^*\left(T_{m,n}g_{k_0,k_0} - g_{k_0,k_0}\right)\right) = 0. \tag{13}$$

Combining (12) with (13), we conclude that

$$st_2 - \limsup_{m,n} \rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right) \leq \varepsilon(R+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$st_2 - \limsup_{m,n} \rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right) = 0.$$

Furthermore, since $\rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right)$ is non-negative for all $m, n \in \mathbb{N}$, we can easily show that

$$st_2 - \lim_{m,n} \rho\left(\lambda_0^*\left(T_{m,n}f - f\right)\right) = 0,$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition, then one can get the following result from Theorem 1 at once.

Theorem 2 *Let $\mathbb{T} := \{T_{m,n}\}$ and ρ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $st_2 - \lim_{m,n} \rho\left(\lambda\left(T_{m,n}e_i - e_i\right)\right) = 0$ *for every* $\lambda > 0$ *and* $i = 0, 1, 2, 3,$
(b) $st_2 - \lim_{m,n} \rho\left(\lambda\left(T_{m,n}f - f\right)\right) = 0$ *for every* $\lambda > 0$ *provided that* f *is any function belonging to* $L^{\rho}(I^2)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}\left(I^2\right)$.

If one replaces the statistical limit by the Pringsheim limit, then the condition (3) reduces to

$$P - \limsup_{m,n} \rho\left(\lambda\left(T_{m,n}h\right)\right) \leq R\rho\left(\lambda h\right) \tag{14}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the following results immediately follows from our Theorems 1 and 2.

Corollary 1 *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{m,n}\}$ be a sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (14). If $\{T_{m,n}e_i\}$ is strongly convergent to e_i for each $i = 0, 1, 2, 3,$ then $\{T_{m,n}f\}$ is modularly convergent to f provided that f is any function belonging to $L^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.*

Corollary 2 *$\mathbb{T} := \{T_{m,n}\}$ and ρ be the same as in Corollary 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) *$\{T_{m,n}e_i\}$ is strongly convergent to e_i for each $i = 0, 1, 2, 3,$*
(b) *$\{T_{m,n}f\}$ is strongly convergent to f provided that f is any function belonging to $L^{\rho}(I^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.*

3 Concluding remarks

In this section, we display an example such that our Korovkin-type statistical approximation results in modular spaces are stronger than the Corollary 1.

Example 1 Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^{φ} on $X(I^2)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \int_0^1 \varphi\left(|f(x, y)|\right) dxdy \quad \text{for } f \in X\left(I^2\right). \tag{15}$$

In this case, ρ^{φ} is a convex modular on $X\left(I^2\right)$, which satisfies all assumptions listed in Sect. 1. Consider the Orlicz space generated by φ as follows:

$$L_{\varphi}^{\rho}(I^2) := \left\{ f \in X\left(I^2\right) : \rho^{\varphi}\left(\lambda f\right) < +\infty \quad \text{for some } \lambda > 0 \right\}.$$

Then consider the following bivariate Bernstein–Kantorovich operator $\mathbb{U} := \{U_{m,n}\}$ on the space $L_{\varphi}^{\rho}\left(I^2\right)$ which is defined by:

$$U_{m,n}(f; x, y) = \sum_{i=0}^{m}\sum_{k=0}^{n} p_{i,k}^{(m,n)}\left(x, y\right)\left(m+1\right)\left(n+1\right) \int_{i/(m+1)}^{(i+1)/(m+1)} \int_{k/(n+1)}^{(k+1)/(n+1)} f\left(t, s\right) ds dt \tag{16}$$

for $x, y \in I$, where $p_{i,k}^{(m,n)}\left(x, y\right)$ defined by

$$p_{i,k}^{(m,n)}\left(x, y\right) = \binom{m}{i}\binom{n}{k} x^i y^k \left(1-x\right)^{m-i}\left(1-y\right)^{n-k}.$$

Also, it is clear that,

$$\sum_{i=0}^{m}\sum_{k=0}^{n} p_{i,k}^{(m,n)}\left(x, y\right) = 1. \tag{17}$$

Observe that the operator $U_{m,n}$ maps $L_{\varphi}^{\rho}\left(I^2\right)$ into itself. Because of (17), as in the proof of [3] Lemma 5.1, we can use Jensen inequality and obtain that for every $f \in L_{\varphi}^{\rho}\left(I^2\right)$ and $m, n \in \mathbb{N}$ there is an absolute constant $M > 0$ such that

$$\rho^{\varphi}(U_{m,n} f) \leq M \rho^{\varphi}(f).$$

Moreover, the property (14) is satisfied with the choice of $X_{\mathbf{U}} := L_{\varphi}^{\rho}(I^2)$. Then, by Corollary 1, we know that, for any function $f \in L_{\varphi}^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}\left(I^2\right)$, $\left\{U_{m,n} f\right\}$ is modularly convergent to f.

Now define a sequence $\{s_{m,n}\}$ by

$$s_{m,n} = \begin{cases} 0, & \text{if } m, n \text{ are squares} \\ 1, & \text{otherwise.} \end{cases} \tag{18}$$

Observe now that, for the sequence $\{s_{m,n}\}$ given by (18),

$$st_2 - \lim_{m,n} s_{m,n} = 1.$$

However, the sequence $\{s_{m,n}\}$ does not convergent in the Pringsheim's sense. Then, using the operators $U_{m,n}$, we define the sequence of positive linear operators $\mathbb{V} := \{V_{m,n}\}$ on $L_{\varphi}^{\rho}\left(I^2\right)$ as follows:

$$V_{m,n}(f;x,y) = s_{m,n}U_{m,n}\left(f;x,y\right) \text{ for } f \in L_{\varphi}^{\rho}\left(I^2\right), x, y \in [0,1] \text{ and } m, n \in \mathbb{N}, \tag{19}$$

where $s = \{s_{m,n}\}$ is the same as in (18). As Lemma 5.1 of [3], we get, for every $h \in X_{\mathbb{V}} := L_{\varphi}^{\rho}(I^2)$, $\lambda > 0$ and for an absolute positive constant R, that

$$\begin{aligned} \rho^{\varphi}\left(\lambda V_{m,n}h\right) &= \rho^{\varphi}\left(\lambda s_{m,n}U_{m,n}h\right) \\ &= s_{m,n}\rho^{\varphi}\left(\lambda U_{m,n}h\right) \\ &\le Rs_{m,n}\rho^{\varphi}\left(\lambda h\right). \end{aligned}$$

Now taking $st_2 - \limsup$ as $m, n \to \infty$ on the both-sides of the above inequality and using the fact that $st_2 - \limsup_{m,n} s_{m,n} = st_2 - \lim_{m,n} s_{m,n} = 1$, we obtain

$$st_2 - \limsup_{m,n} \rho^{\varphi}\left(\lambda V_{m,n}f\right) \le R\rho^{\varphi}(\lambda f). \tag{20}$$

Therefore, the condition (3) works for our operators $V_{m,n}$ given by (19) with the choice of $X_{\mathbb{V}} = X_{\mathbb{U}} = L_{\varphi}^{\rho}(I^2)$. We now claim that

$$st_2 - \lim_{m,n} \rho^{\varphi}\left(\lambda\left(V_{m,n}e_i - e_i\right)\right) = 0 \quad \text{for every } \lambda > 0 \text{ and } i = 0, 1, 2, 3. \tag{21}$$

Indeed, first observe that

$$\begin{aligned} V_{m,n}(e_0;x,y) &= s_{m,n}, \\ V_{m,n}(e_1;x,y) &= s_{m,n}\left(\frac{mx}{m+1} + \frac{1}{2\left(m+1\right)}\right), \\ V_{m,n}(e_2;x,y) &= s_{m,n}\left(\frac{ny}{n+1} + \frac{1}{2\left(n+1\right)}\right), \\ V_{m,n}(e_3;x,y) &= s_{m,n}\left(\frac{m\left(m-1\right)x^2}{\left(m+1\right)^2} + \frac{2mx}{\left(m+1\right)^2} + \frac{1}{3\left(m+1\right)^2}\right. \\ &\qquad \left. + \frac{n\left(n-1\right)y^2}{\left(n+1\right)^2} + \frac{2ny}{\left(n+1\right)^2} + \frac{1}{3\left(n+1\right)^2}\right). \end{aligned}$$

So, we can see, for any $\lambda > 0$, that

$$\lambda\left|V_{m,n}(e_0;x,y) - e_0\left(x,y\right)\right| = \lambda(1 - s_{m,n}),$$

which implies

$$\begin{aligned} \rho^{\varphi}\left(\lambda\left(V_{m,n}e_0 - e_0\right)\right) &= \rho^{\varphi}\left(\lambda(1 - s_{m,n})\right) \\ &= \int_0^1\int_0^1 \varphi\left(\lambda(1 - s_{m,n})\right) dxdy \end{aligned}$$

$$= \varphi\left(\lambda(1 - s_{m,n})\right)$$
$$= (1 - s_{m,n})\varphi(\lambda)$$

because of the definition of $\{s_{m,n}\}$. Now, since $st_2 - \lim_{m,n}(1 - s_{m,n}) = 0$, we get

$$st_2 - \lim_{m,n} \rho^{\varphi}\left(\lambda\left(V_{m,n}e_0 - e_0\right)\right) = 0 \quad \text{for every } \lambda > 0,$$

which guarantees that (21) holds true for $i = 0$. Also, since

$$\begin{aligned}
\lambda\left|V_{m,n}(e_1; x, y) - e_1(x, y)\right| &= \lambda\left\{x\left(1 - \frac{ms_{m,n}}{m+1}\right) + \frac{s_{m,n}}{2(m+1)}\right\} \\
&\le \lambda\left\{\left(1 - \frac{ms_{m,n}}{m+1}\right) + \frac{s_{m,n}}{2(m+1)}\right\} \\
&= \lambda\left\{\left(1 - s_{m,n}\right) + \frac{3s_{m,n}}{2(m+1)}\right\},
\end{aligned}$$

by the definitions of $\{s_{m,n}\}$ and ρ^{φ}, we may write that

$$\begin{aligned}
\rho^{\varphi}\left(\lambda\left(V_{m,n}e_1 - e_1\right)\right) &\le \rho^{\varphi}\left(\lambda\left\{\left(1 - s_{m,n}\right) + \frac{3s_{m,n}}{2(m+1)}\right\}\right) \\
&\le \rho^{\varphi}\left(2\lambda\left(1 - s_{m,n}\right)\right) + \rho^{\varphi}\left(\frac{3\lambda s_{m,n}}{m+1}\right) \\
&= \varphi\left(2\lambda\left(1 - s_{m,n}\right)\right) + \varphi\left(\frac{3\lambda s_{m,n}}{m+1}\right),
\end{aligned}$$

which implies, for any $\lambda > 0$, that

$$\rho^{\varphi}\left(\lambda\left(V_{m,n}e_1 - e_1\right)\right) \le (1 - s_{m,n})\varphi(2\lambda) + s_{m,n}\varphi\left(\frac{3\lambda}{n+1}\right) \tag{22}$$

Since φ is continuous, we have

$$P - \lim_{m,n} \varphi\left(\frac{3\lambda}{m+1}\right) = \varphi\left(P - \lim_{m,n} \frac{3\lambda}{m+1}\right) = \varphi(0) = 0.$$

Also considering $st_2 - \lim_{m,n} s_{m,n} = 1$, we obtain from (22) that

$$st_2 - \lim_{m,n} \rho^{\varphi}\left(\lambda\left(V_{m,n}e_1 - e_1\right)\right) = 0 \quad \text{for every } \lambda > 0.$$

Hence (21) is valid for $i = 1$. Similarly, we have

$$st_2 - \lim_{m,n} \rho^{\varphi}\left(\lambda\left(V_{m,n}e_2 - e_2\right)\right) = 0 \quad \text{for every } \lambda > 0.$$

Finally, since

$$
\begin{aligned}
&\lambda \left| V_{m,n}(e_3; x, y) - e_3\,(x, y) \right| \\
&= \lambda \left\{ \left\{ x^2 \left(1 - \frac{m\,(m-1)\,s_{m,n}}{(m+1)^2} \right) + x \left(\frac{2ms_{m,n}}{(m+1)^2} \right) + \frac{s_{m,n}}{3\,(m+1)^2} \right\} \right. \\
&\quad \left. \times \left\{ +y^2 \left(1 - \frac{n\,(n-1)\,s_{m,n}}{(n+1)^2} \right) + y \left(\frac{2ns_{m,n}}{(n+1)^2} \right) + \frac{s_{m,n}}{3\,(n+1)^2} \right\} \right\} \\
&\leq \lambda \left\{ \left(1 - \frac{m\,(m-1)\,s_{m,n}}{(m+1)^2} \right) + \left(\frac{2ms_{m,n}}{(m+1)^2} \right) + \frac{s_{m,n}}{3\,(m+1)^2} \right. \\
&\quad \left. + \left(1 - \frac{n\,(n-1)\,s_{m,n}}{(n+1)^2} \right) + \left(\frac{2ns_{m,n}}{(n+1)^2} \right) + \frac{s_{m,n}}{3\,(n+1)^2} \right\} \\
&\quad = \lambda \left\{ 2(1 - s_{m,n}) + s_{m,n} \left(\frac{15m+4}{3(m+1)^2} \right) + s_{m,n} \left(\frac{15n+4}{3(n+1)^2} \right) \right\},
\end{aligned}
$$

we get

$$
\begin{aligned}
\rho^{\varphi}\left(\lambda\left(V_{m,n}e_3 - e_3\right)\right) &\leq \rho^{\varphi}\left(6\lambda\left(1 - s_{m,n}\right)\right) + \rho^{\varphi}\left(3\lambda s_{m,n}\left(\frac{15m+4}{3(m+1)^2}\right)\right) \\
&\quad + \rho^{\varphi}\left(3\lambda s_{m,n}\left(\frac{15n+4}{3(n+1)^2}\right)\right) \\
&= \varphi\left(6\lambda(1 - s_{m,n})\right) + \varphi\left(3\lambda s_{m,n}\left(\frac{15m+4}{3(m+1)^2}\right)\right) \\
&\quad + \varphi\left(3\lambda s_{m,n}\left(\frac{15n+4}{3(n+1)^2}\right)\right),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\rho^{\varphi}\left(\lambda\left(V_{m,n}e_3 - e_3\right)\right) &\leq (1 - s_{m,n})\varphi\left(6\lambda\right) + s_{m,n}\varphi\left(3\lambda\left(\frac{15m+4}{3(m+1)^2}\right)\right) \\
&\quad + s_{m,n}\varphi\left(3\lambda\left(\frac{15n+4}{3(n+1)^2}\right)\right)
\end{aligned}
\tag{23}
$$

Considering the continuity of φ, it follows from (23) that

$$
st_2 - \lim_{m,n} \rho^{\varphi}\left(\lambda\left(V_{m,n}e_3 - e_3\right)\right) = 0 \quad \text{for every } \lambda > 0.
$$

So, our claim (21) holds true for each $i = 0, 1, 2, 3$ and for any $\lambda > 0$. Now, from (20) and (21), we can say that our sequence $\mathbb{V} := \{V_{m,n}\}$ defined by (19) satisfy all assumptions of Theorem 1. Therefore, we conclude that

$$
st_2 - \lim_{m,n} \rho^{\varphi}\left(\lambda_0\left(V_{m,n}f - f\right)\right) = 0 \quad \text{for some } \lambda_0 > 0
$$

holds for any $f \in L_{\varphi}^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbb{V}} = L_{\varphi}^{\rho}(I^2)$ for every $g \in C^{\infty}\left(I^2\right)$.

However, for the same f's and for any $\lambda > 0$, since

$$\rho^{\varphi}\left(\lambda\left(V_{m,n}f - f\right)\right) = \rho^{\varphi}\left(\lambda\left(s_{m,n}U_{m,n}f - f\right)\right)$$
$$= \begin{cases} \rho^{\varphi}(\lambda f), & \text{if } m, n \text{ are squares} \\ \rho^{\varphi}\left(\lambda\left(U_{m,n}f - f\right)\right), & \text{otherwise,} \end{cases}$$

the sequence $\left\{\rho^{\varphi}\left(\lambda\left(V_{m,n}f - f\right)\right)\right\}$ has two subsequences with different limit points. In other words, there is not any $\lambda > 0$ such that the sequence $\left\{\rho^{\varphi}\left(\lambda\left(V_{m,n}f - f\right)\right)\right\}$ goes to zero as $m,n \to \infty$. This clearly shows that $\{V_{m,n}f\}$ is not modularly convergent to f. So, Corollary 1 does not work for the sequence $\mathbb{V} := \{V_{m,n}\}$.

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