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Results in Mathematics

Statistically Relatively Uniform Convergence of Positive Linear Operators

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Abstract. In this paper we define a new type of statistical convergence by using the notions of the natural density and the relatively uniform convergence. We study its use in the Korovkin-type approximation theory. Then, we construct an example such that our new approximation result works but its classical and statistical cases do not work. We also compute the rates of statistically relatively uniform convergence of sequences of positive linear operators.

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1. Introduction

Moore [15] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Then, Chittenden [5] gave the following definition of relatively uniform convergence is equivalent to the definition given by Moore:

A sequence (f_n) of functions, defined on an interval $I \equiv (a \leq x \leq b)$, converges *relatively uniformly to a limit function* f if there exists a function $\sigma(x)$, called a scale function $\sigma(x)$ such that for every $\varepsilon > 0$ there is an integer n_ε such that for every $n > n_\varepsilon$ the inequality

$$|f_n(x) - f(x)| < \varepsilon |\sigma(x)|$$

holds uniformly in x on the interval I. The sequence (f_n) is said to converge *uniformly relative to the scale function* σ or more simply, *relatively uniformly.*

It will be observed that uniform convergence is the special case of relatively uniform convergence in which the scale function is a non-zero constant (for more properties and details, see also [4–6]).

Example 1. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{nx}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

This sequence does not converge uniformly, but converges to $f = 0$ uniformly, relative to a scale function $\sigma(x) = \begin{cases} \frac{1}{x}, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases}$.

In the present paper, we introduce the notion of statistically uniform convergence of a sequence of functions relative to a scale function. We further apply our new type of convergence to prove a Korovkin type approximation theorem.

Let K be a subset of $\mathbb{N}$, the set of natural numbers, then *the natural density of* K, denoted by $\delta(K)$, is given by:

$$\delta(K) := \lim_n \frac{1}{n} \left| \{ k \leq n \ : k \in K \} \right|$$

whenever the limit exists, where $|B|$ denotes the cardinality of the set B [16].

Let f and f_n belong to $C(X)$, which is the space of all continuous real valued functions on a compact subset X of the real numbers and $\|f\|_{C(X)}$ denotes the usual supremum norm of f in $C(X)$.

Definition 1 [10]. (f_n) is said to be statistically pointwise convergent to f on X if $st - \lim_n f_n(x) = f(x)$ for each $x \in X$, i.e., for every $\varepsilon > 0$ and for each $x \in X$, $\delta(\{n \ : |f_n(x) - f(x)| \geq \ \varepsilon\}) = 0$. Then, it is denoted by $f_n \to f$ $(stat)$ on X.

Definition 2 [10]. (f_n) is said to be statistically uniform convergent to f on X if $st - \lim_n \|f_n - f\|_{C(X)} = 0$, or $\delta\left(\left\{n \ : \|f_n - f\|_{C(X)} \geq \ \varepsilon\right\}\right) = 0$ for every $\varepsilon > 0$. This limit is denoted by $f_n \rightrightarrows f$ $(stat)$ on X.

Definition 3. (f_n) is said to be *statistically relatively uniform convergent* to f on X if there exists a function $\sigma(x)$, $|\sigma(x)| > 0$, called a scale function $\sigma(x)$ such that for every $\varepsilon > 0$, $\delta\big(\big\{n \ : \sup\limits_{x \in X} \big|\frac{f_n(x) - f(x)}{\sigma(x)}\big| \geq \ \varepsilon\big\}\big) = 0$.

This limit is denoted by $(st) - f_n \rightrightarrows f$ $(X; \sigma)$.

Using the above definitions, the next result follows immediately.

Lemma 1. *$f_n \rightrightarrows f$ on X (in the ordinary sense) implies $f_n \rightrightarrows f$ (stat) on X, which also implies $(st) - f_n \rightrightarrows f$ $(X; \sigma)$.*

However, one can construct an example which guarantees that the converses of Lemma 1 are not always true. Such an example is presented as follows.

Example 2. For each $n \in \mathbb{N}$, define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \frac{2nx}{1+n^2x^2} \tag{1}$$

Then observe that $(st) - g_n \rightrightarrows g = 0\ ([0,1]\,;\sigma)$, $\sigma(x) = \begin{cases} \frac{1}{x}, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases}$, however (g_n) is not statistically (or ordinary) uniform convergent to the function $g = 0$ on the interval $[0,1]$.

2. A Korovkin-Type Approximation Theorem

For a sequence (T_n) of positive linear operators on $C(X)$, Korovkin [14] first introduced the necessary and sufficient conditions for the uniform convergence of $T_n(f)$ to a function f by using the test function e_i defined by $e_i(x) = x^i$, $(i = 0,1,2)$ (see, for instance, [2]). More recently, general versions of the Korovkin theorem were studied, in which a more general notion of convergence is used. Some Korovkin-type theorems in the setting of a statistical convergence were given by [1,8,9]. The concept of statistical convergence for sequences of real numbers was introduced by Fast [11] and Steinhaus [17] independently in the same year 1951 and since then several generalizations and applications of this notion have been investigated by various authors [3,13].

In this section we apply the notion of statistically uniform convergence of a sequence of functions relative to a scale function to prove a Korovkin type approximation theorem. Let T be a linear operator from $C(X)$ into itself. Then, as usual, we say that T is positive linear operator provided that $f \geq 0$ implies $T(f) \geq 0$. Also, we denote the value of $T(f)$ at a point $x \in X$ by $T(f(u);x)$ or, briefly, $T(f;x)$.

First we recall the statistical case of the Korovkin-type result introduced in [12],

Theorem 1. *Let (T_n) be a sequence of positive linear operators acting from $C(X)$ into itself. Then, for all $f \in C(X)$,*

$$st - \lim_n \|T_n(f) - f\|_{C(X)} = 0$$

if and only if

$$st - \lim_n \|T_n(e_i) - e_i\|_{C(X)} = 0.$$

Now we have the following main result.

Theorem 2. *Let (T_n) be a sequence of positive linear operators acting from $C(X)$ into itself. Then, for all $f \in C(X)$,*

$$(st) - T_n(f) \rightrightarrows f\ (X;\sigma) \tag{2}$$

if and only if

$$(st) - T_n(e_i) \rightrightarrows e_i\ (X;\sigma_i), \quad i = 0,1,2 \tag{3}$$

where $\sigma(x) = \max\{|\sigma_i(x)|; i = 0, 1, 2\}$, $|\sigma_i(x)| > 0$ *and* $\sigma_i(x)$ *is unbounded,* $i = 0, 1, 2$.

Proof. Condition (3) follows immediately from condition (2), since each of the functions $1, x, x^2$ belongs to $C(X)$. We prove the converse part. By the continuity of f on X, we can write

$$|f(x)| \leq K.$$

Therefore

$$|f(u) - f(x)| \leq 2K.$$

Also, since f is continuous on X, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(u) - f(x)| < \varepsilon$ for all $x \in X$ satisfying $|u - x| < \delta$. Hence, putting $\varphi(u) = (u - x)^2$, we get

$$|f(u) - f(x)| < \varepsilon + \frac{2K}{\delta^2}\varphi.$$

This means

$$-\varepsilon - \frac{2K}{\delta^2}\varphi < f(u) - f(x) < \varepsilon + \frac{2K}{\delta^2}\varphi.$$

Now operating $T_n(e_0, x)$ to this inequality since $T_n(f, x)$ is monotone and linear. Hence,

$$T_n(e_0; x)\left(-\varepsilon - \frac{2K}{\delta^2}\varphi\right) < T_n(e_0; x)\left(f(u) - f(x)\right) < T_n(e_0; x)\left(\varepsilon + \frac{2K}{\delta^2}\varphi\right).$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$-\varepsilon T_n(e_0; x) - \frac{2K}{\delta^2}T_n(\varphi; x) < T_n(f; x) - f(x)\, T_n(e_0; x) < \varepsilon T_n(e_0; x) + \frac{2K}{\delta^2}T_n(\varphi; x). \tag{4}$$

Also

$$\begin{aligned} T_n(f; x) - f(x) &= T_n(f; x) - f(x)\, T_n(e_0; x) + f(x)\, T_n(e_0; x) - f(x) \\ &= [T_n(f; x) - f(x)\, T_n(e_0; x)] + f(x)\, [T_n(e_0; x) - e_0(x)], \end{aligned} \tag{5}$$

using (4) and (5), we get

$$\begin{aligned} T_n(f; x) - f(x) &< \varepsilon T_n(e_0; x) + \frac{2K}{\delta^2}\left\{[T_n(e_2; x) - e_2(x)] - 2x\,[T_n(e_1; x) - e_1(x)]\right. \\ &\qquad \left. + x^2\,[T_n(e_0; x) - e_0(x)]\right\} + f(x)\,[T_n(e_0; x) - e_0(x)] \\ &= \varepsilon\,[T_n(e_0; x) - e_0(x)] + \varepsilon + \frac{2K}{\delta^2}\left\{[T_n(e_2; x) - e_2(x)]\right. \\ &\qquad - 2x\,[T_n(e_1; x) - e_1(x)] + x^2\,[T_n(e_0; x) - e_0(x)]\left.\right\} \\ &\qquad + f(x)\,[T_n(e_0; x) - e_0(x)] \end{aligned}$$

then,

$$
\begin{aligned}
|T_n(f)-f| &\leq \varepsilon+\left(\varepsilon+K+\frac{2K\|e_2\|_{C(X)}}{\delta^2}\right)|T_n(e_0)-e_0| \\
&\quad+\frac{4K\|e_1\|_{C(X)}}{\delta^2}|T_n(e_1)-e_1| \\
&\quad+\frac{2K}{\delta^2}|T_n(e_2)-e_2| \\
&\leq \varepsilon+M\{|T_n(e_0)-e_0|+|T_n(e_1)-e_1|+|T_n(e_2)-e_2|\}
\end{aligned}
$$

where $M=\varepsilon+K+\frac{2K}{\delta^2}\left(\|e_2\|_{C(X)}+2\|e_1\|_{C(X)}+1\right)$. We get

$$
\begin{aligned}
\sup_{x\in X}\left|\frac{T_n(f;x)-f(x)}{\sigma(x)}\right| &\leq \sup_{x\in X}\frac{\varepsilon}{|\sigma(x)|}+M\left\{\sup_{x\in X}\left|\frac{T_n(e_0;x)-e_0(x)}{\sigma_0(x)}\right|\right. \\
&\quad\left.+\sup_{x\in X}\left|\frac{T_n(e_1;x)-e_1(x)}{\sigma_1(x)}\right|+\sup_{x\in X}\left|\frac{T_n(e_2;x)-e_2(x)}{\sigma_2(x)}\right|\right\}
\end{aligned}
\tag{6}
$$

where $\sigma(x)=\max\{|\sigma_i(x)|;i=0,1,2\}$. Now, for a given $r>0$, choose $\varepsilon>0$ such that $\sup_{x\in X}\frac{\varepsilon}{|\sigma(x)|}<r$. Then,

$$
R_n(x,r):=\left\{k\leq n:\sup_{x\in X}\left|\frac{T_k(f;x)-f(x)}{\sigma(x)}\right|\geq r\right\}
$$

and

$$
R_{i,n}(x,r):=\left\{k\leq n:\sup_{x\in X}\left|\frac{T_k(e_i;x)-e_i(x)}{\sigma_i(x)}\right|\geq\frac{r-\sup_{x\in X}\frac{\varepsilon}{|\sigma(x)|}}{3M}\right\},\qquad i=0,1,2
$$

It follows from (6) that $R_n(x,r)\subset\bigcup_{i=0}^{2}R_{i,n}(x,r)$ and so $\delta(R_n(x,r))\leq\sum_{i=0}^{2}\delta(R_{i,n}(x,r))$. Then using the hypothesis (3), we get

$$
(st)-T_n(f)\rightrightarrows f\ (X;\sigma)
$$

where $\sigma(x)=\max\{|\sigma_i(x)|;i=0,1,2\}$.

This completes the proof of the theorem.

Now let $X=[0,1]$ and consider the classical Bernstein polynomials $B_n(f;x)$ on $C[0,1]$. Using these polynomials, we introduce the following positive linear operators on $C[0,1]$:

$$
P_n(f;x)=(1+g_n(x))B_n(f;x),\quad x\in[0,1]\ \text{and}\ f\in C[0,1],\tag{7}
$$

where $g_n(x)$ is given by (1). Then, observe that

$$
\begin{aligned}
P_n(e_0;x) &= (1+g_n(x))e_0(x),\\
P_n(e_1;x) &= (1+g_n(x))e_1(x),\\
P_n(e_2;x) &= (1+g_n(x))\left[e_2(x)+\frac{x(1-x)}{n}\right].
\end{aligned}
$$

Since $(st)-g_n \rightrightarrows g=0\ ([0,1]\,;\sigma)$, $\sigma(x)=\begin{cases}\frac{1}{x}, & 0<x\le 1\\ 1, & x=0\end{cases}$, we conclude that

$$(st)-P_n(e_i)\rightrightarrows e_i\ ([0,1];\sigma)\quad \text{for each } i=0,1,2.$$

So, by Theorem 2, we immediately see that

$$(st)-P_n(f)\rightrightarrows f\ \ ([0,1];\sigma)\quad \text{for all } f\in C[0,1].$$

However, since (g_n) is not statistically uniform convergent to the function $g=0$ on the interval $[0,1]$, we can say that Theorem 1 does not work for our operators defined by (7). Furthermore, since (g_n) is not uniformly convergent (in the ordinary sense) to the function $g=0$ on $[0,1]$, the classical Korovkin theorem does not work either. Therefore, this application clearly shows that our Theorem 2 is a non-trivial generalization of the classical and the statistical cases of the Korovkin results introduced in [14] and [12], respectively.

3. Rates of Statistically Relatively Uniform Convergence in Theorem 2

In this section we study the rates of statistically relatively uniform convergence of a sequence of positive linear operators defined on $C(X)$ with the help of modulus of continuity.

We now present the following definition.

Definition 4. A sequence (f_n) is *said to converge statistically uniformly relative to the scale function* $\sigma(x)$, $|\sigma(x)|>0$, to f on X with the rate of $\beta\in(0,1)$ if for every $\varepsilon>0$,

$$\lim_n \frac{\left|\left\{k\le n:\sup\limits_{x\in X}\left|\frac{f_k(x)-f(x)}{\sigma(x)}\right|\ge\ \varepsilon\right\}\right|}{n^{1-\beta}}=0.$$

In this case, it is denoted by

$$(st)-(f_n-f)=o(n^{-\beta})\ (X;\sigma).$$

Then we first need the following lemma to get the rates of convergence in Theorem 2 by using Definition 4.

Lemma 2. *Let* (f_n) *and* (g_n) *be function sequences belonging to* $C(X)$*. Assume that* $(st)-(f_n-f)=o(n^{-\beta_0})\ (X;\sigma_0)$ *and* $(st)-(g_n-g)=o(n^{-\beta_1})\ (X;\sigma_1)$, $|\sigma_i(x)|>0$, $i=0,1$*. Let* $\beta=\min\{\beta_0,\beta_1\}$*. Then the following statements hold:*

(i) $(st)-(f_n+g_n)-(f+g)=o(n^{-\beta})\ (X;\max\{|\sigma_i(x)|\,;i=0,1\})$

(ii) $(st)-(f_n-f)(g_n-g)=o(n^{-\beta})\ (X;\sigma_0(x)\,\sigma_1(x))$,

(iii) $(st)-(\lambda(f_n-f))=o(n^{-\beta_0})\ (X;\sigma_0(x))$ *for any real number* λ,
(iv) $(st)-\sqrt{|f_n-f|}=o(n^{-\beta_0})\ (X;\sqrt{|\sigma_0(x)|})$.

Proof. (i) Assume that $(st)-(f_n-f)=o(n^{-\beta_0})\ (X;\sigma_0)$ and that $(st)-(g_n-g)=o(n^{-\beta_1})\ (X;\sigma_1)$. Also, for every $\varepsilon>0$ define

$$\Psi_n(x,\varepsilon):=\left|\left\{k\le n:\sup_{x\in X}\left|\frac{(f_k+g_k)(x)-(f+g)(x)}{\sigma(x)}\right|\ge\varepsilon\right\}\right|,$$
$$\Psi_{0,n}(x,\varepsilon):=\left|\left\{k\le n:\sup_{x\in X}\left|\frac{f_k(x)-f(x)}{\sigma_0(x)}\right|\ge\frac{\varepsilon}{2}\right\}\right|,$$
$$\Psi_{1,n}(x,\varepsilon):=\left|\left\{k\le n:\sup_{x\in X}\left|\frac{g_k(x)-g(x)}{\sigma_1(x)}\right|\ge\frac{\varepsilon}{2}\right\}\right|,$$

where $\sigma(x)=\max\{|\sigma_i(x)|;\quad i=0,1\}$. Then observe that

$$\frac{\Psi_n(x,\varepsilon)}{n^{1-\beta}}\le\frac{\Psi_{0,n}(x,\varepsilon)+\Psi_{1,n}(x,\varepsilon)}{n^{1-\beta}}. \tag{8}$$

Since $\beta=\min\{\beta_0,\beta_1\}$, by (8), we get

$$\frac{\Psi_n(x,\varepsilon)}{n^{1-\beta}}\le\frac{\Psi_{0,n}(x,\varepsilon)}{n^{1-\beta_0}}+\frac{\Psi_{1,n}(x,\varepsilon)}{n^{1-\beta_1}}. \tag{9}$$

Now by taking limit as $n\to\infty$ in (9) and using the hypotheses, we conclude that

$$\lim_n\frac{\Psi_n(x,\varepsilon)}{n^{1-\beta}}=0,$$

which completes the proof of (i). Since the proofs of (ii), (iii) and (iv) are similar, we omit them.

On the other hand, we recall that the modulus of continuity of a function $f\in C(X)$ is defined by

$$w(f,\delta)=\sup_{|y-x|\le\delta,x,y\in X}|f(y)-f(x)|\quad(\delta>0).$$

Then we have the following result.

Theorem 3. *Let X be a compact subset of the real numbers, and let (T_n) be a sequence of positive linear operators acting from $C(X)$ into itself. Assume that the following conditions hold:*

(a) $(st)-(T_n(e_0)-e_0)=o(n^{-\beta_0})\ (X;\sigma_0)$,
(b) $(st)-w(f,\delta_n)=o(n^{-\beta_1})\ (X;\sigma_1)$, *where* $\delta_n(x)=\sqrt{T_n(\varphi^2;x)}$ *with* $\varphi(y)=(y-x)$.

Then we have, for all $f\in C(X)$,

$$(st)-(T_n(f)-f)=o(n^{-\beta})\ (X;\sigma),$$

where $\beta=\min\{\beta_0,\beta_1\}$ *and* $\sigma(x)=\max\{|\sigma_0(x)|,\ |\sigma_1(x)|,\ |\sigma_0(x)\sigma_1(x)|\}$, $|\sigma_i(x)|>0$ *and* $\sigma_i(x)$ *is unbounded*, $i=0,1$.

Proof. Let $f \in C(X)$ and $x \in X$. Then it is well-known that,

$$\begin{aligned}|T_n(f;x) - f(x)| &\leq M\,|T_n(e_0;x) - e_0(x)| \\ &\quad + \left(T_n(e_0;x) + \sqrt{T_n(e_0;x)}\right) w(f,\delta_n)\end{aligned}$$

where $M = \|f\|_{C(X)}$ (see, for instance, [2,7]). This yields that

$$\begin{aligned}\sup_{x\in X}\left|\frac{T_n(f;x)-f(x)}{\sigma(x)}\right| &\leq M\sup_{x\in X}\left|\frac{T_n(e_0;x)-e_0(x)}{\sigma_0(x)}\right| + 2\sup_{x\in X}\frac{w(f,\delta_n)}{|\sigma_1(x)|} \\ &\quad + \sup_{x\in X}\frac{w(f,\delta_n)}{|\sigma_1(x)|}\sup_{x\in X}\left|\frac{T_n(e_0;x)-e_0(x)}{\sigma_0(x)}\right| \\ &\quad + \sup_{x\in X}\frac{w(f,\delta_n)}{|\sigma_1(x)|}\sup_{x\in X}\sqrt{\left|\frac{T_n(e_0;x)-e_0(x)}{\sigma_0(x)}\right|}.\end{aligned}$$

Now considering the above inequality, the hypotheses (a), (b), and Lemma 2, the proof is completed at once.

References

[1] Anastassiou, G.A., Duman, O.: A Baskakov type generalization of statistical Korovkin theory. J. Math. Anal. Appl. **340**, 476–486 (2008)

[2] Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and Its Applications, de Gruyter Stud. Math. 17, Walter de Gruyter, Berlin (1994)

[3] Balcerzak, M., Dems, K., Komisarski, A.: Statistical convergence and ideal convergence for sequences of functions. J. Math. Anal. Appl. **328**, 715–729 (2007)

[4] Chittenden, E.W.: Relatively uniform convergence of sequences of functions. Trans. AMS **15**, 197–201 (1914)

[5] Chittenden, E.W.: On the limit functions of sequences of continuous functions converging relatively uniformly. Trans. AMS **20**, 179–184 (1919)

[6] Chittenden, E.W.: Relatively uniform convergence and classification of functions. Trans. AMS **23**, 1–15 (1922)

[7] Devore, R.A.: The Approximation of Continuous Functions by Positive Linear Operators, Lecture Notes in Math., vol. 293, Springer, Berlin (1972)

[8] Duman, O., Khan, M.K., Orhan, C.: A-statistical convergence of approximating operators. Math. Inequal. Appl. **4**, 689–699 (2003)

[9] Duman, O., Erkuş, E., Gupta, V.: Statistical rates on the multivariate approximation theory. Math. Comput. Modell. **44**, 763–770 (2006)

[10] Duman, O., Orhan, C.: μ-Statistically convergent function sequences. Czechoslov. Math. J. **54**, 413–422 (2004)

[11] Fast, H.: Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)

[12] Gadjiev, A.D., Orhan, C.: Some approximation theorems via statistical convergence. Rocky Mount. J. Math. **32**, 129–138 (2002)

[13] Karakuş, S., Demirci, K., Duman, O.: Equi-statistical convergence of positive linear operators. J. Math. Anal. Appl. **339**, 1065–1072 (2008)

[14] Korovkin, P.P.: Linear Operators and Approximation Theory. Hindustan Publ. Co., Delhi (1960)

[15] Moore, E.H.: An Introduction to a Form of General Analysis, The New Haven Mathematical Colloquium. Yale University Press, New Haven (1910)

[16] Niven, I., Zuckerman, H.S.: An Introduction to the Theory of Numbers. Fourt Ed. Wiley, New York (1980)

[17] Steinhaus, H.: Sur la convergence ordinaire et la convergence asymtotique. Colloq. Math. **2**, 73–74 (1951)

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