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Relative modular convergence of positive linear operators

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Abstract In this paper we define a new type of modular convergence by using the notion of the relatively uniform convergence. We prove a Korovkin-type approximation theorem via this type of convergence in modular spaces. Then, we construct an example such that our new approximation result works but its classical cases do not work.

Keywords Positive linear operators · Modular space · Relative modular convergence · Korovkin theorem

Mathematics Subject Classification 41A35 · 41A36 · 46E30

1 Introduction and preliminaries

The concept of uniform convergence of a sequence of functions relative to a scale function was first introduced by Moore [12]. Then, Chittenden aimed to study this type of convergence in the field of functions of a real variable [6] and in [7] he gave the following definition of relatively uniform convergence, equivalent to the definition given by Moore:

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A sequence $\{f_n\}$ of functions, defined on an interval $I \equiv (a \leq x \leq b)$, converges *relatively uniformly to a limit function* f if there exists a function $\sigma(x)$, called a scale function $\sigma(x)$ such that for every $\varepsilon > 0$ there is an integer n_ε such that for every $n > n_\varepsilon$ the inequality

$$|f_n(x) - f(x)| < \varepsilon |\sigma(x)|$$

holds uniformly in x on the interval I. The sequence $\{f_n\}$ is said to converge *uniformly relative to the scale function* σ or more simply, *relatively uniformly*.

It will be observed that uniform convergence is a special case of relatively uniform convergence in which the scale function is a non-zero constant (for more properties and details, see also [6–8]).

Example 1 For each $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \frac{n^2 x}{1 + n^3 x^2}.$$

This sequence does not converge uniformly, but converges to $f = 0$ uniformly relative to a scale function $\sigma(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x}, & 0 < x \leq 1 \end{cases}$.

In this paper, we introduce a new definition of modular (or strong) convergence, called relative modular convergence and investigate the problem of modular approximation to a function f by means of positive linear operators defined on a modular space. So we obtain various Korovkin-type approximation theorems in modular spaces via the concept of relative modular convergence. Also, we display an example satisfying all conditions of our approximation result in a modular space but not the classic one in [3]. Modular spaces were extensively investigated by Musielak (see [14,15] and later on in [11]). Later Bardaro, Musielak and Vinti improved an abstract approximation theory in modular spaces for the general case of nonlinear integral operators (see [5]). This class of spaces is very wide and contains as particular case the L_p-spaces, Orlicz spaces, the Musielak–Orlicz spaces and others. Bardaro and Mantellini [3] presented some approximation theorems by means of positive linear operators defined on modular spaces and after they studied a modular version of the classical Korovkin theorem in multivariate modular function spaces in [4].

We now recall some basic definitions and notations in modular spaces.

Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X(I)$ we denote the space of all real-valued measurable functions on I provided with equality a.e. As usual, let $C(I)$ denote the space of all continuous real-valued functions, and $C^\infty(I)$ denote the space of all infinitely differentiable functions on I. In this case, we say that a functional $\rho : X(I) \to [0, +\infty]$ is a *modular* on $X(I)$ provided that the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ a.e. in I,
(ii) $\rho(-f) = \rho(f)$ for every $f \in X(I)$,
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(I)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Recall that a modular ρ is called *N-quasi convex* if the following is satisfied:

- there exists a constant $N \geq 1$ such that

$$\rho\left(\alpha f+\beta g\right) \leq N\alpha\rho\left(Nf\right)+N\beta\rho\left(Ng\right),$$

holds for every $f, g \in X\left(I\right)$, $\alpha, \beta \geq 0$ with $\alpha+\beta=1$. In particular, if $N=1$, then ρ is called *convex*.

Furthermore, a modular ρ is called *N-quasi semiconvex* if:

- there exists a constant $N \geq 1$ such that

$$\rho(af) \leq Na\rho(Nf),$$

holds for every $f \in X\left(I\right)$ and $a \in (0,1]$.

It is clear that every N-quasi semiconvex modular is N-quasi convex. We should recall that the above two concepts were introduced and discussed in details by Bardaro et. al. [4,5].

We now consider some appropriate vector subspaces of $X(I)$ by means of a modular ρ as follows:

$$L^{\rho}\left(I\right):=\left\{f \in X\left(I\right): \lim_{\lambda \to 0^{+}} \rho\left(\lambda f\right)=0\right\}$$

and

$$E^{\rho}\left(I\right):=\left\{f \in L^{\rho}\left(I\right): \rho\left(\lambda f\right)<+\infty \quad \text{for all} \quad \lambda>0\right\}.$$

Here, $L^{\rho}\left(I\right)$ is called the *modular space* generated by ρ; and also $E^{\rho}\left(I\right)$ is called the space of the finite elements of $L^{\rho}\left(I\right)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\left\{f \in X\left(I\right): \rho\left(\lambda f\right)<+\infty \quad \text{for some} \quad \lambda>0\right\}$$

coincides with $L^{\rho}\left(I\right)$. The notions about modulars are introduced in [15] and widely discussed in [5] (see also [11,16]).

Now we recall the convergence notions in modular spaces.

Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}\left(I\right)$. Then, $\{f_n\}$ is *modularly convergent* to a function $f \in L^{\rho}\left(I\right)$ iff

$$\lim_{n} \rho\left(\lambda_0\left(f_n-f\right)\right)=0 \quad \text{for some } \lambda_0>0. \tag{1}$$

Also, $\{f_n\}$ is *F-norm convergent* (or, *strongly convergent*) to f iff

$$\lim_{n} \rho\left(\lambda\left(f_n-f\right)\right)=0 \quad \text{for every } \lambda>0. \tag{2}$$

It is known from [15] that (1) and (2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e. there exists a constant $M>0$ such that $\rho\left(2f\right) \leq M\rho\left(f\right)$ for every $f \in X\left(I\right)$.

In this paper, we will need the following assumptions on a modular ρ:

- ρ is *monotone* if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$,
- ρ is *finite* if $\chi_A \in L^{\rho}(I)$ whenever A is measurable subset of I such that $\mu(A) < \infty$,
- ρ is *absolutely finite*, if ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda \chi_B) < \varepsilon$ for any measurable subset $B \subset I$ with $\mu(B) < \delta$,
- ρ is *strongly finite* if $\chi_I \in E^{\rho}(I)$,
- ρ is called *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X(I)$ with $\rho(f) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ whenever B is any measurable subset of I with $\mu(B) < \delta$.

Observe now that (see [3,4]) if a modular ρ is monotone and finite, then we have $C(I) \subset L^{\rho}(I)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I) \subset E^{\rho}(I)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^{\infty}(I)} = L^{\rho}(I)$ [13]. Some important relations between the above properties may be found in [2,4,5,13,16].

We give now a definition by using the notion of the modular (or strong) convergence and relatively uniform convergence.

Definition 1 Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}(I)$. Then, $\{f_n\}$ is said to be *relatively modularly convergent* to a function $f \in L^{\rho}(I)$ if there exists a function $\sigma(x)$, called a scale function $\sigma \in X(I)$, $|\sigma(x)| \neq 0$ such that

$$\lim_n \rho\left(\lambda_0\left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for some } \lambda_0 > 0. \tag{3}$$

Also, it is called relatively strongly convergent if

$$\lim_n \rho\left(\lambda\left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for every } \lambda > 0.$$

It will be observed that modular convergence is the special case of relative modular convergence in which the scale function is a non-zero constant. Also, if $\sigma(x)$ is bounded, relative modular convergence implies modular convergence. However, relative modular convergence does not imply modular convergence, when $\sigma(x)$ is unbounded. This is illustrated by the following example.

Example 2 Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u \to \infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^{φ} on $X(I)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \varphi(|f(x)|)\, dx \quad \textit{for } f \in X(I). \tag{4}$$

In this case, ρ^{φ} is a convex modular on $X(I)$, which satisfies all assumptions listed in Sect. 1 (see [3]). Consider the Orlicz space generated by φ as follows:

$$L_{\varphi}^{\rho}(I) := \left\{ f \in X(I) : \rho^{\varphi}(\lambda f) < +\infty \ \textit{for some} \lambda > 0 \right\}.$$

For each $n \in N$, define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} n(1-nx), & 0 < x < \frac{1}{n} \\ 0, & x = 0 \, or \, \frac{1}{n} \leq x \leq 1. \end{cases} \tag{5}$$

If $\varphi(x) = x^p$ for $1 \leq p < \infty$, $x \geq 0$, then $L_{\varphi}^{\rho}(I) = L_p(I)$ ([5]). Moreover we have for any function $f \in L_{\varphi}^{\rho}(I)$

$$\rho^{\varphi}(f) = \|f\|_{L_p}^{p}.$$

Then observe that $\{g_n\}$ does not converge modularly but converges to $g = 0$ modularly relative to a scale function $\sigma(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x}, & 0 < x \leq 1 \end{cases}$ on $L_1([0,1])$. Indeed, for some $\lambda_0 > 0$, with the choise of $p = 1$ we have $\rho^{\varphi}(g) = \|g\|_{L_1}$,

$$\rho(\lambda_0(g_n - g)) = \|\lambda_0(g_n - g)\|_{L_1} = \lambda_0 \int_0^{\frac{1}{n}} |n(1-nx)| \, dx = \frac{\lambda_0}{2},$$

we have

$$\lim_n \|\lambda_0(g_n - g)\|_{L_1} \neq 0.$$

Then using the scale function σ

$$\rho\left(\lambda_0\left(\frac{g_n - g}{\sigma}\right)\right) = \left\|\lambda_0\left(\frac{g_n - g}{\sigma}\right)\right\|_{L_1} = \lambda_0 \int_0^{\frac{1}{n}} \left|\frac{n(1-nx)}{\frac{1}{x}}\right| dx = \lambda_0\left(\frac{1}{2n} - \frac{1}{3n}\right),$$

since $\lim_n \lambda_0\left(\frac{1}{2n} - \frac{1}{3n}\right) = 0$, we get

$$\lim_n \left\|\lambda_0\left(\frac{g_n - g}{\sigma}\right)\right\|_{L_1} = 0.$$

2 Relatively modular Korovkin theorem

Let ρ be a monotone and finite modular on $X(I)$. Assume that D is a set satisfying $C^{\infty}(I) \subset D \subset L^{\rho}(I)$. We can construct such a subset D since ρ is monotone and finite (see [3]). Assume that $\mathbb{T} := \{T_n\}$ is a sequence of positive linear operators from D into $X(I)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ containing $C^{\infty}(I)$ and $\sigma \in X(I)$ is an unbounded function satisfying $\sigma(x) \neq 0$ such that

$$\limsup_{n} \rho\left(\lambda\left(\frac{T_n h}{\sigma}\right)\right) \leq P\rho\left(\lambda h\right) \tag{6}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. We also use the test functions e_i defined by

$$e_i(x) = x^i (i = 0, 1, 2).$$

Theorem 1 *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T} := \{T_n\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (6). Moreover suppose that $\sigma_i(x)$ is an unbounded function satisfying $|\sigma_i(x)| \geq a_i > 0$ $(i = 0, 1, 2)$. Assume that*

$$\lim_{n} \rho\left(\lambda\left(\frac{T_n e_i - e_i}{\sigma_i}\right)\right) = 0 \quad \textit{for every } \lambda > 0 \quad \textit{and} \quad i = 0, 1, 2. \tag{7}$$

Now let f be any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$. Then, we have

$$\lim_{n} \rho\left(\lambda_0\left(\frac{T_n f - f}{\sigma}\right)\right) = 0 \quad \textit{for some } \lambda_0 > 0, \tag{8}$$

where $\sigma(x) = \max\{|\sigma_i(x)|; i = 0, 1, 2\}$.

Proof We first claim that

$$\lim_{n} \rho\left(\mu\left(\frac{T_n g - g}{\sigma}\right)\right) = 0 \quad \text{for every } g \in C(I) \cap D \quad \text{and every } \mu > 0. \tag{9}$$

To see this assume that g belongs to $C(I) \cap D$ and μ is any positive number. By the continuity of g on I and by the linearity and positivity of the operators T_n, we easily see that (see, for instance, [1,10]), for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} |T_n(g; x) - g(x)| &\leq \varepsilon T_n(e_0; x) + M\left|T_n(e_0; x) - e_0(x)\right| \\ &\quad + \frac{2M}{\delta^2} T_n\left((t - x)^2; x\right) \end{aligned}$$

holds for every $x \in I$ and $n \in \mathbb{N}$, where $M := \sup_{x \in I} |g(x)|$. The above inequality implies that

$$\begin{aligned} |T_n(g; x) - g(x)| &\leq \varepsilon + \left(\varepsilon + M + \frac{2Mc^2}{\delta^2}\right)\left|T_n(e_0; x) - e_0(x)\right| \\ &\quad + \frac{4Mc}{\delta^2}\left|T_n(e_1; x) - e_1(x)\right| \\ &\quad + \frac{2M}{\delta^2}\left|T_n(e_2; x) - e_2(x)\right|, \end{aligned}$$

where $c := \max |x|$.

Hence, denoting by $K := \max\left\{\varepsilon + M + \frac{2Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$,

$$|T_n(g;x) - g(x)| \leq \varepsilon + K\left\{|T_n(e_0) - e_0| + |T_n(e_1) - e_1| + |T_n(e_2) - e_2|\right\}.$$

Now we multiply the both-sides of the above inequality with $\frac{1}{|\sigma(x)|}$, we have

$$\begin{aligned}\left|\frac{T_n(g;x) - g(x)}{\sigma(x)}\right| \leq \frac{\varepsilon}{|\sigma(x)|} + \frac{K}{|\sigma(x)|}&\left\{|T_n(e_0;x) - e_0(x)|\right.\\ &\left.+|T_n(e_1;x) - e_1(x)| + |T_n(e_2;x) - e_2(x)|\right\}.\end{aligned}$$

So the last inequality gives, for $\mu > 0$, that

$$\begin{aligned}\mu\left\{\left|\frac{T_n(g;x) - g(x)}{\sigma(x)}\right|\right\} \leq \frac{\mu\varepsilon}{|\sigma(x)|} &+ \mu K\left\{\left|\frac{T_n(e_0;x) - e_0(x)}{\sigma_0(x)}\right|\right\}\\ &+ \mu K\left\{\left|\frac{T_n(e_1;x) - e_1(x)}{\sigma_1(x)}\right|\right\}\\ &+ \mu K\left\{\left|\frac{T_n(e_2;x) - e_2(x)}{\sigma_2(x)}\right|\right\}.\end{aligned}$$

Now, applying the modular ρ in the both-sides of the above inequality, since ρ is monotone, we get

$$\begin{aligned}\rho\left(\mu\left(\frac{T_n g - g}{\sigma}\right)\right) \leq \rho&\left(\frac{\mu\varepsilon}{\sigma} + \mu K\left(\frac{T_n e_0 - e_0}{\sigma_0}\right)\right.\\ &\left.+ \mu K\left(\frac{T_n e_1 - e_1}{\sigma_1}\right) + \mu K\left(\frac{T_n e_2 - e_2}{\sigma_2}\right)\right).\end{aligned}$$

Hence, we may write that

$$\begin{aligned}\rho\left(\mu\left(\frac{T_n g - g}{\sigma}\right)\right) \leq \rho\left(\frac{4\mu\varepsilon}{\sigma}\right) &+ \rho\left(4\mu K\left(\frac{T_n e_0 - e_0}{\sigma_0}\right)\right)\\ &\times\rho\left(4\mu K\left(\frac{T_n e_1 - e_1}{\sigma_1}\right)\right) + \rho\left(4\mu K\left(\frac{T_n e_2 - e_2}{\sigma_2}\right)\right).\end{aligned}$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0 < \varepsilon \leq 1$

$$\begin{aligned}\rho\left(\mu\left(\frac{T_n g - g}{\sigma}\right)\right) \leq N\varepsilon\rho\left(\frac{4\mu N}{\sigma}\right) &+ \rho\left(4\mu K\left(\frac{T_n e_0 - e_0}{\sigma_0}\right)\right)\\ &\times\rho\left(4\mu K\left(\frac{T_n e_1 - e_1}{\sigma_1}\right)\right) + \rho\left(4\mu K\left(\frac{T_n e_2 - e_2}{\sigma_2}\right)\right).\end{aligned}$$

By using the hypothesis (7), we get for every $\lambda > 0$ and so far every constant $\mu > 0$

$$\lim_n \rho\left(\mu\left(\frac{T_n g - g}{\sigma}\right)\right) \leq N\varepsilon\rho\left(\frac{4\mu N}{\sigma}\right),$$

so we deduce easily the strong convergence to a scale function σ for the continuous function g. Obviously (9) also holds for every $g \in C^{\infty}(I)$. Now let $f \in L^{\rho}(I)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$. Since $\mu(I) < \infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I)$. Using these properties of the modular ρ, it is known from [5,13] that the space $C^{\infty}(I)$ is modularly dense in $L^{\rho}(I)$, i.e., there exists a sequence $\{g_k\} \subset C^{\infty}(I)$ such that

$$\lim_k \rho\left(3\lambda_0^*(g_k - f)\right) = 0 \text{ for some } \lambda_0^* > 0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*(g_k - f)\right) < \varepsilon \text{ for every } k \geq k_0. \tag{10}$$

On the other hand, by the linearity and positivity of the operators T_n, we may write that

$$\begin{aligned}\lambda_0^*\left|T_n(f;x) - f(x)\right| \leq \lambda_0^*\left|T_n(f - g_{k_0};x)\right| + \lambda_0^*\left|T_n(g_{k_0};x) - g_{k_0}(x)\right| \\ + \lambda_0^*\left|g_{k_0}(x) - f(x)\right|,\end{aligned}$$

which holds for every $x \in I$ and $n \in \mathbb{N}$. Multiplying the last inequality with $\frac{1}{|\sigma(x)|}$ applying the modular ρ and moreover considering the monotonicity of ρ, we have

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) \leq \rho\left(3\lambda_0^*\left(\frac{T_n\left(g_{k_o} - f\right)}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_o} - g_{k_o}}{\sigma}\right)\right) \\ + \rho\left(3\lambda_0^*\left(\frac{g_{k_o} - f}{\sigma}\right)\right).\end{aligned}$$

Hence, observing that $|\sigma| \geq a > 0$ ($a = \max\{a_i : i = 0, 1, 2\}$) and taking into consideration that ρ is monotone, we may write that

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) \leq \rho\left(3\lambda_0^*\left(\frac{T_n\left(g_{k_o} - f\right)}{\sigma}\right)\right) \\ + \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_o} - g_{k_o}}{\sigma}\right)\right) + \rho\left(\frac{3\lambda_0^*}{a}\left(g_{k_o} - f\right)\right).\end{aligned} \tag{11}$$

Then it follows from (10) and (11) that

$$\rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) \leq \rho\left(3\lambda_0^*\left(\frac{T_n\left(g_{k_o} - f\right)}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_o} - g_{k_o}}{\sigma}\right)\right) + \varepsilon.$$

taking limit superior in both-sides of last inequality and also using (6)

$$\begin{aligned}\limsup_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) &\leq \varepsilon + \limsup_n \rho\left(3\lambda_0^*\left(\frac{T_n\left(g_{k_o} - f\right)}{\sigma}\right)\right)\\ &\quad + \limsup_n \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_o} - g_{k_o}}{\sigma}\right)\right)\\ &\leq \varepsilon + P\rho\left(3\lambda_0^*\left(f - g_{k_o}\right)\right)\\ &\quad + \limsup_n \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_o} - g_{k_o}}{\sigma}\right)\right)\end{aligned}$$

which gives

$$\limsup_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) \leq \varepsilon(P+1) + \limsup_n \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_o} - g_{k_o}}{\sigma}\right)\right). \tag{12}$$

By (9), since

$$\lim_n \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_0} - g_{k_0}}{\sigma}\right)\right) = 0,$$

we get

$$\limsup_n \rho\left(3\lambda_0^*\left(\frac{T_n g_{k_0} - g_{k_0}}{\sigma}\right)\right) = 0. \tag{13}$$

Combining (12) with (13), we conclude that

$$\limsup_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) \leq \varepsilon(P+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$\limsup_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) = 0.$$

Furthermore, since $\rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right)$ is non-negative for all $n \in \mathbb{N}$, we can easily show that

$$\lim_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) = 0,$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition then the space $C^{\infty}(I)$ is strongly dense in $L^{\rho}(I)$ [3] and so we say $T_n f$ converges strongly to f hence, $T_n f$ converges relatively strongly to f so we get the following result from Theorem 1 at once.

Theorem 2 *Let $\mathbb{T} := \{T_n\}$, ρ and σ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $\lim_n \rho\left(\lambda\left(\frac{T_n e_i - e_i}{\sigma_i}\right)\right) = 0$ *for every* $\lambda > 0$ *and* $i = 0, 1, 2$,
(b) $\lim_n \rho\left(\lambda\left(\frac{T_n f - f}{\sigma}\right)\right) = 0$ *for every* $\lambda > 0$ *provided that* f *is any function belonging to* $L^{\rho}(I)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}(I)$.

If one replaces the scale function by a non-zero constant, then the condition (6) reduces to

$$\limsup_n \rho\left(\lambda\left(T_n h\right)\right) \le P\rho\left(\lambda h\right) \tag{14}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. In this case, the next results which were obtained by Bardaro and Mantellini [3] immediately follows from our Theorems 1 and 2.

Corollary 1 [3] *Let* ρ *be a monotone, strongly finite, absolutely continuous and* N*-quasi semiconvex modular on* $X(I)$. *Let* $\mathbb{T} := \{T_n\}$ *be a sequence of positive linear operators from* D *into* $X(I)$ *satisfying* (6)*: If* $\{T_n e_i\}$ *is strongly convergent to* e_i *for each* $i = 0, 1, 2$, *then* $\{T_n f\}$ *is modularly convergent to* f *provided that* f *is any function belonging to* $L^{\rho}(I)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}(I)$.

Corollary 2 [3] $\mathbb{T} := \{T_n\}$ *and* ρ *be the same as in Corollary* 1. *If* ρ *satisfies the* Δ_2*-condition, then the following statements are equivalent:*

(a) $\{T_n e_i\}$ *is strongly convergent to* e_i *for each* $i = 0, 1, 2$,
(b) $\{T_n f\}$ *is strongly convergent to* f *provided that* f *is any function belonging to* $L^{\rho}(I)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}(I)$.

3 Concluding remarks

In this section, we display an example such that our Korovkin-type approximation results in modular spaces are stronger than the classical ones in [3].

Example 3 Take $I = [0, 1]$ and let φ, ρ^{φ} and $L_{\varphi}^{\rho}(I)$ be the same as in Example 2. Then consider the following classical Bernstein–Kantorovich operator $U := \{U_n\}$ on the space $L_{\varphi}^{\rho}(I)$ (see [3]) which is defined by:

$$U_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt \ \ \text{for } x \in I.$$

Observe that the operators U_n map the Orlicz space $L_{\varphi}^{\rho}(I)$ into itself. Moreover, the property (6) is satisfied with the choice of $X_{\mathbf{U}} := L_{\varphi}^{\rho}(I)$. Then, by Corollary 1, we know that, for any function $f \in L_{\varphi}^{\rho}(I)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}(I)$, $\{U_n f\}$ is modularly convergent to f.

If $\varphi(x) = x^p$ for $1 \le p < \infty$, $x \ge 0$, then $L_{\varphi}^{\rho}(I) = L_p(I)$ ([5]). For $p = 1$, we have $\rho^{\varphi}(f) = \|f\|_{L_1}$. Then, using the operators U_n, we define the sequence of positive linear operators $V := \{V_n\}$ on $L_1(I)$ as follows:

$$V_n(f;x) = (1+g_n(x))\,U_n(f;x) \text{ for } f \in L_1(I),\, x \in [0,1] \quad \text{and } n \in \mathbb{N},$$

where $\{g_n\}$ is the same as in (5) and we choose $\sigma_i(x) = \sigma(x)$ $(i = 0, 1, 2)$, where $\sigma(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x}, & 0 < x \leq 1 \end{cases}$. Since, for positive constant C, $\left\| U_j(f;x) \right\|_{L_p} \leq C \left\| f \right\|_{L_p}$ ([9]), we can easily see that, for every $\lambda > 0$

$$\limsup_n \left\| \lambda \left(\frac{V_n f}{\sigma} \right) \right\|_{L_1} \leq C \left\| \lambda f \right\|_{L_1}.$$

We now claim that

$$\lim_n \left\| \lambda \left(\frac{V_n e_i - e_i}{\sigma} \right) \right\|_{L_1} = 0, \quad i = 0, 1, 2. \tag{15}$$

Observe that $U_n(e_0;x) = e_0$, $U_n(e_1;x) = \frac{nx}{n+1} + \frac{1}{2(n+1)}$ and $U_n(e_2;x) = \frac{n(n-1)x^2}{(n+1)^2} + \frac{2nx}{(n+1)^2} + \frac{1}{3(n+1)^2}$. So, we can see

$$\begin{aligned} \left\| \lambda \left(\frac{V_n(e_0;x) - e_0(x)}{\sigma(x)} \right) \right\|_{L_1} &= \left\| \lambda \left(\frac{1 + g_n(x) - 1}{\sigma(x)} \right) \right\|_{L_1} = \left\| \lambda \frac{g_n(x)}{\sigma(x)} \right\|_{L_1} \\ &= \lambda \int_0^{\frac{1}{n}} \left| \frac{n(1-nx)}{\frac{1}{x}} \right| dx = \lambda \left(\frac{1}{2n} - \frac{1}{3n} \right), \end{aligned}$$

since $\lim_n \lambda \left(\frac{1}{2n} - \frac{1}{3n} \right) = 0$, we get

$$\lim_n \left\| \lambda \left(\frac{V_n e_0 - e_0}{\sigma} \right) \right\|_{L_1} = 0.$$

which guarantees that (15) holds true for $i = 0$. Also, we have

$$\begin{aligned} \left\| \lambda \left(\frac{V_n(e_1;x) - e_1(x)}{\sigma(x)} \right) \right\|_{L_1} &= \left\| \lambda \left(\frac{(1 + g_n(x))\,U_n(e_1;x)}{\sigma(x)} - \frac{x}{\sigma(x)} \right) \right\|_{L_1} \\ &\leq \left\| \lambda \frac{g_n(x)}{\sigma(x)} \left(\frac{nx}{n+1} + \frac{1}{2(n+1)} \right) \right\|_{L_1} + \left\| \lambda \frac{x - 2x^2}{2(n+1)} \right\|_{L_1} \\ &< \left\| \lambda \frac{g_n(x)}{\sigma(x)} \right\|_{L_1} + \frac{\lambda}{2(n+1)}, \end{aligned}$$

because of $\lim_n \left\| \lambda \frac{g_n(x)}{\sigma(x)} \right\|_{L_1} = 0$ and $\lim_n \frac{\lambda}{2(n+1)} = 0$ we get

$$\lim_n \left\| \lambda \left(\frac{V_n(e_1;x) - e_1(x)}{\sigma(x)} \right) \right\|_{L_1} = 0.$$

Finally, since

$$\begin{aligned}
&\left\|\lambda\left(\frac{V_n\left(e_2;x\right)-e_2(x)}{\sigma\left(x\right)}\right)\right\|_{L_1}\\
&=\left\|\lambda\frac{\left(1+g_n\left(x\right)\right)U_n\left(e_2;x\right)}{\sigma\left(x\right)}-\frac{\lambda x^2}{\sigma\left(x\right)}\right\|_{L_1}\\
&\leq\left\|\lambda\frac{g_n\left(x\right)}{\sigma\left(x\right)}\left(\frac{n\left(n-1\right)x^2}{\left(n+1\right)^2}+\frac{2nx}{\left(n+1\right)^2}+\frac{1}{3\left(n+1\right)^2}\right)\right\|_{L_1}\\
&\quad+\left\|\lambda\left(\frac{\left(3n+1\right)x^3}{\left(n+1\right)^2}+\frac{2nx^2}{\left(n+1\right)^2}+\frac{x}{3\left(n+1\right)^2}\right)\right\|_{L_1}\\
&<3\left\|\lambda\frac{g_n\left(x\right)}{\sigma\left(x\right)}\right\|_{L_1}+\frac{3\lambda}{n+1}+\frac{2\lambda n}{\left(n+1\right)^2}+\frac{\lambda}{3\left(n+1\right)^2}.
\end{aligned}$$

We have,

$$\lim_n\left\|\lambda\left(\frac{V_n\left(e_2;x\right)-e_2(x)}{\sigma\left(x\right)}\right)\right\|_{L_1}=0.$$

So, our claim (15) holds true for each $i = 0, 1, 2$. $\left\{V_j\right\}$ satisfies all hypothesis of Theorem 1 and we immediately see that,

$$\lim_n\left\|\lambda_0\left(\frac{V_nf-f}{\sigma}\right)\right\|_{L_1}=0,\ \text{for some } \lambda_0>0,\quad on\ [0,1]\quad for\ all\ f\in L_1(I).$$

Since $\lim_n\|\lambda\left(V_ne_0-e_0\right)\|_{L_1}\neq 0$, Corollary 1 does not work for the sequence $V:=\{V_n\}$.

With this example, we can immediately say that our Theorem 1 is a non-trivial generalization of the classical Korovkin-type results in the modular spaces.

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