

Strong and A -statistical comparisons for sequences

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Abstract

Let T and A be two nonnegative regular summability matrices and $W(T, p) \cap l^\infty$ and $c_A(b)$ denote the spaces of all bounded strongly T -summable sequences with index $p > 0$, and bounded summability domain of A , respectively. In this paper we show, among other things, that $\chi_{\mathbb{N}}$ is a multiplier from $W(T, p) \cap l^\infty$ into $c_A(b)$ if and only if any subset K of positive integers that has T -density zero implies that K has A -density zero. These results are used to characterize the A -statistical comparisons for both bounded as well as arbitrary sequences. Using the concept of A -statistical Tauberian rate, we also show that $\chi_{\mathbb{N}}$ is not a multiplier from $W(T, p) \cap l^\infty$ into $c_A(b)$ that leads to a Steinhaus type result.

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1. Introduction

Let $\chi_{\mathbb{N}}$ be the characteristic function of the positive integers. In this paper we investigate if the sequence $\chi_{\mathbb{N}}$ is a multiplier from $W(T, p) \cap l^\infty$, the space of all bounded strongly T -summable sequences with index $p > 0$, into the bounded summability domain $c_A(b)$ when T, A are two nonnegative regular summability matrices. For example we show, among other things, that this is indeed true if and only if any subset K of positive integers

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that has T -density zero implies that it has A -density zero. These results are also used to characterize the A -statistical comparisons for both bounded as well as arbitrary sequences. In the final section, using the concept of A -statistical Tauberian rate, we show that $\chi_{\mathbb{N}}$ is not a multiplier from $W(T, p) \cap l^\infty$ into $c_A(b)$ that leads to a Steinhaus type result.

We will use the common notation for matrix summability: if $x = (x_k)$ is a sequence of real numbers and $A = (a_{nk})$ is an infinite matrix, then Ax is the sequence whose n th term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$, where we assume that the series $\sum_{k=1}^{\infty} a_{nk}x_k$ is convergent for each $n \in \mathbb{N}$. If $\lim_n (Ax)_n = L$, we then say that x is A -summable to L . In this case, we write $A\text{-}\lim x = L$. We say that A is regular if $A\text{-}\lim x = L$ whenever $\lim x = L$ [13]. Let X and Y be two sequence spaces. If Ax exists for each $x \in X$ and $Ax \in Y$ then we say that A maps X into Y , and write $A \in (X, Y)$.

Let A be a nonnegative regular summability matrix. Following Freedman and Sember [6], we say that a set $K \subseteq \mathbb{N}$ has A -density if $\delta_A(K) := \lim_n \sum_{k=1}^{\infty} a_{nk} \chi_K(k)$ exists where χ_K is the characteristic function of the set K . A sequence of real numbers $x = (x_k)$ is A -statistically convergent to L provided that for every $\varepsilon > 0$ the set $K := K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has A -density zero [1,3,6,8,14]. By st_A , st_A^0 , $st_A(b)$, $st_A^0(b)$ we denote the set of all A -statistically convergent sequences, the set of all A -statistically null sequences, the set of all bounded A -statistically convergent sequences, and the set of all bounded A -statistically null sequences, respectively. The case in which $A = C_1$, the Cesàro matrix of order one, reduces to the usual definition of statistical convergence [5,7,19,20].

Statistical convergence, while introduced over nearly fifty years ago, has only recently become an area of active research. It has been discussed in trigonometric series [21], summability theory [11], strong summability [1], strong integral summability [2], measure theory [17,18], Banach space theory [4], and approximation theory [12]. Connections have also been made between statistical convergence, Tauberian theorems and central limit theorem [10].

By c and l^∞ we denote the sets of all convergent and bounded sequences, respectively. From now on the summability field of a matrix A will be denoted by c_A , i.e.,

$$c_A = \left\{ x : \lim_n (Ax)_n \text{ exists} \right\},$$

and $c_A(b) := c_A \cap l^\infty$.

Let p be a positive real number and let $A = (a_{nk})$ be a nonnegative regular infinite matrix. Write

$$W(A, p) := \left\{ x : \lim_n \sum_{k=1}^{\infty} a_{nk} |x_k - L|^p = 0 \text{ for some } L \right\};$$

in this case we say that x is strongly A -summable with index $p > 0$. Let $W_b(A) := W(A, 1) \cap l^\infty$. There is a deep relationship between A -statistical convergence and strong summability [1,6].

We now recall the definition of multipliers. Let w denote the space of all real- (or complex-) valued sequences. Any linear subspace of w is called a sequence space. Suppose that two sequence spaces, E and F , are given. A multiplier from E into F is a sequence u such that $ux = (u_n x_n) \in F$ whenever $x \in E$. The linear space of all such multipliers is denoted by $m(E, F)$. Hence the inclusion $X \subset Y$ may be interpreted as saying that the sequence $\chi_{\mathbb{N}}$ is a multiplier from X to Y .

2. Strong and A -statistical comparisons

In this section our primary objective is to give equivalent forms of $\chi_{\mathbb{N}} \in m(W(T, p) \cap l^\infty, c_A(b))$ that compares bounded strong summability fields of the nonnegative regular summability matrices T and A . We will show that this also characterizes the A -statistical comparisons for both bounded as well as arbitrary sequences. The following is an extension of a result of Maddox [16] so that part of the proof will be omitted, see also [14].

Theorem 1. *Let A and T be nonnegative regular summability matrices. The following statements are equivalent:*

- (i) $\chi_{\mathbb{N}} \in m(W(T, p) \cap l^\infty, c_A(b))$.
- (ii) $W(T, p) \cap l^\infty \subseteq c_A(b)$.
- (iii) $A \in (W(T, p) \cap l^\infty, c)$.
- (iv) For any subset $K \subseteq \mathbb{N}$, $\delta_T(K) = 0$ implies that $\delta_A(K) = 0$.
- (v) $A \in (W(T, p) \cap l^\infty, c)$ and A preserves the strong limits of T .

Proof. The first three parts are equivalent is trivial. To show that (iii) implies (iv), assume that (iii) holds. Suppose the contrary and let K be a subset of nonnegative integers with $\delta_T(K) = 0$ but

$$\limsup_n \sum_{k \in K} a_{nk} > 0. \quad (1)$$

Hence, K must be an infinite set since A is regular and all its columns go to zero. (Since $\delta_T(K) = 0$, and T is regular, it must be that $\mathbb{N} - K$ must also be infinite.) Now take a sequence x which is the indicator of the set K . Note that for any $p > 0$, we have

$$\lim_n \sum_k |t_{nk}| |x_k - 0|^p = \lim_n \sum_k t_{nk} x_k = \lim_n \sum_{k \in K} t_{nk} = \delta_T(K) = 0.$$

Hence, $x \in W(T, p) \cap l^\infty$. Since $A \in (W(T, p) \cap l^\infty, c)$, it must be that $(Ax)_n$ is convergent. Combining this with (1) we see that the density $\delta_A(K)$ exists and so

$$\lim_n (Ax)_n = \delta_A(K) > 0.$$

Now consider the matrix D that keeps all the columns of A whose positions correspond with the set K and fills the rest of the columns with zeros. Since $\lim_n (Dx)_n = \lim_n (Ax)_n > 0$, a straight forward extension of an argument of Maddox [16] provides a contradiction.

Assume now (iv) holds, and let $x \in W(T, p) \cap l^\infty$, so that

$$\sum_k t_{nk} |x_k - L|^p \rightarrow 0 \quad (\text{as } n \rightarrow \infty),$$

for some number L . Hence x is T -statistically convergent. Then for any $\epsilon > 0$, define the set $K = \{k: |x_k - L| > \epsilon\}$. So we have $\delta_T(K) = 0$. By the assumption, it must be that $\delta_A(K) = 0$. Since x is bounded, let $|x_k| \leq C$ for all k . So, for any $q > 0$, we have

$$\begin{aligned}
\sum_k a_{nk} |x_k - L|^q &= \sum_{k \in K} a_{nk} |x_k - L|^q + \sum_{k \in K^c} a_{nk} |x_k - L|^q \\
&\leq (2C)^q \sum_{k \in K} a_{nk} + \epsilon^q \sum_{k \in K^c} a_{nk} \\
&\leq (2C)^q \sum_{k \in K} a_{nk} + \epsilon^q \sum_k a_{nk}.
\end{aligned}$$

Letting n go to infinity, we see that

$$\lim_n \sum_k a_{nk} |x_k - L|^q = 0.$$

Hence, in particular, $(Ax)_n \rightarrow L$ and A preserves the strong limit of T , which gives (v). Observe that (v) trivially implies (iii). $\square$

The following proposition collects various equivalent forms of the last result. For this purpose we introduce the notation

$$W^L(T, p) := \left\{ x : \lim_n \sum_k t_{nk} |x_k - L|^p = 0 \right\}.$$

Proposition 2. *Let A and T be nonnegative regular summability matrices. Then the following statements are equivalent:*

- (i) $\text{st}_T(b) \subseteq \text{st}_A(b)$.
- (ii) $W(T, p) \cap l^\infty \subseteq W(A, q) \cap l^\infty$ for some $p, q > 0$.
- (iii) $A \in (W(T, p) \cap l^\infty, c)$ and A preserves the strong limits of T . That is, $W^L(T, p) \cap l^\infty \subseteq W^L(A, q) \cap l^\infty$ for every L .
- (iv) For any subset $K \subseteq \mathbb{N}$, $\delta_T(K) = 0$ implies $\delta_A(K) = 0$.
- (v) $\text{st}_T^0(b) \subseteq \text{st}_A^0(b)$.
- (vi) $\text{st}_T(b) \subseteq \text{st}_A(b)$ and A preserves the T -statistical limits.
- (vii) $W^L(T, p) \cap l^\infty \subseteq W^L(A, q) \cap l^\infty$ for some $p, q > 0$ and some real number L .
- (viii) $W(T, p) \cap l^\infty \subseteq c_A(b)$ for some $p > 0$.
- (ix) $\text{st}_T \subseteq \text{st}_A$ and A preserves the T -statistical limits.
- (x) $\text{st}_T \subseteq \text{st}_A$.

Proof. Make the notation

$$\text{st}_T^L(b) := \{x \in l^\infty : x \text{ is } T\text{-statistically convergent to } L\}.$$

First note that

$$\text{st}_T^L(b) = W^L(T, p) \cap l^\infty$$

for any $p > 0$. Hence, taking union over all L shows that (i) and (ii) are equivalent.

By Theorem 1, we notice that (iii) and (iv) are equivalent. Taking union over L gives that (iii) implies (ii). To show that (ii) implies (iii), clearly (ii) implies that $W(T, p) \cap l^\infty \subseteq c_A$. Hence, $A \in (W(T, p) \cap l^\infty, c)$. Therefore, by Theorem 1, (iv) holds. Therefore, (iii) holds. Also Theorem 1 implies that (iv) and (viii) are equivalent.

When (vii) holds for some L , if $x \in W^M(T, p) \cap l^\infty$ then define a new sequence $y_k = x_k - M + L$. Since $y \in W^L(T, p) \cap l^\infty$, we have $y \in W^L(A, q) \cap l^\infty$. This implies that

$$\sum_k a_{nk} |x_k - M|^q = \sum_k a_{nk} |y_k - L|^q \rightarrow 0.$$

Hence, $y \in W^M(A, q) \cap l^\infty$. That is,

$$W^M(T, p) \cap l^\infty \subseteq W^M(A, q) \cap l^\infty$$

for every M . Taking supremum over all M , gives (ii). Now (ii) implies (iii) and then trivially (iii) implies (vii). Hence (i) and (iii) together imply (vi). Trivially (vi) implies (i). Furthermore, (vi) implies (v). Conversely (v) implies (vii) with $L = 0$. Hence, (i) through (viii) are all equivalent.

So far all arguments were for bounded sequences. Now (ix) implies (x), and (x) implies (i). To show that (i) implies (ix), let $x \in \text{st}_T$ with T -statistical limit L . For $\epsilon > 0$, define $h_k = 0$ if $|x_k - L| < \epsilon$ and $h_k = 1$ otherwise. Hence, any such $h \in \text{st}_T^0(b) \subseteq \text{st}_A^0(b)$ by (v). This implies that $x \in \text{st}_A$ with L being the A -statistical limit. This finishes the proof. $\square$

3. Steinhaus type results

The well-known theorem of Steinhaus asserts that if T is a regular matrix then $\chi_{\mathbb{N}}$ is not a multiplier from l^∞ into c_T . The result remains true if regularity condition on A is replaced by coregularity. Maddox [16] proved that $\chi_{\mathbb{N}}$ is not a multiplier from $l^\infty \cap W(C_1, p)$ into the summability domain of the Borel matrix. We show that this is a particular case of the following more general result, involving the concept of A -statistical Tauberian rate. Following [9] and [10], we will say that $0 < \Omega(x) \nearrow \infty$ is an A -statistical Tauberian rate if the condition

$$\lim_{\delta \searrow 0} \liminf_n \min_{n \leq m < n + \delta \Omega(n)} \{x_m - x_n\} \geq 0$$

is an A -statistical Tauberian condition for the nonnegative regular matrix A . In order for Theorem 2.2 of [9] be applicable, we will assume that Ω obeys condition GC of [9], which ensures that Ω does not diverge too rapidly. We recall that some results on summability functions may be found in [15].

Theorem 3. *Let A and T be nonnegative regular summability matrices. If Ω is both a summability function of T and a statistical Tauberian rate of A then*

$$\chi_{\mathbb{N}} \notin m(W(T, p) \cap l^\infty, c_A(b)).$$

Proof. We will define an infinite subset $K \subseteq \mathbb{N}$ as follows. Pick a positive integer $w(1)$ and define

$$K(1) = \left\{ k: w(1) \leq k < w(1) + \frac{\Omega(w(1))}{2} \right\}.$$

Having picked $w(1), w(2), \dots, w(q-1)$, and defining

$$K(j) = \left\{ k: w(j) \leq k < w(j) + \frac{\Omega(w(j))}{2} \right\}, \quad j = 1, 2, \dots, q-1,$$

let

$$\kappa = \sum_{k \in \bigcup_{j=1}^{q-1} K(j)} 1. \quad (2)$$

Pick an $w(q)$ large enough such that $w(q) > w(q-1) + \Omega(w(q-1))/2$ and $\Omega(w(q)) > 2\kappa$. This is possible since Ω is a summability function and hence nondecreasing and diverges to infinity. Then take

$$K(q) = \left\{ k: w(q) \leq k < w(q) + \frac{\Omega(w(q))}{2} \right\}.$$

Let $K = \bigcup_q K(q)$ and let $x = \chi_K$. We claim that the counting function $c_x(k)$ of x is bounded above by $\Omega(k)$. To see this note that our assertion trivially holds over the interval $K(1)$ since

$$\Omega(w(1)) \geq \frac{\Omega(w(1))}{2}.$$

Assuming that the assertion holds up till some $q-1$, note that $\Omega(w(q)) \geq 2\kappa$ implies that

$$\Omega(w(q)) \geq \frac{\Omega(w(q))}{2} + \kappa$$

where κ is as defined in (2). This implies that our assertion holds for $k \in K(q)$. Now since Ω is a summability function of T , and the counting function of x is dominated by Ω , it must be that

$$\sum_k t_{nk} |x_k - 0|^p = \sum_k t_{nk} x_k = (Tx)_n \rightarrow 0, \quad p > 0.$$

However, since Ω is a statistical Tauberian rate of A , by [10] it must be

$$\limsup_n (Ax)_n = \limsup_n \sum_q \sum_{k \in K(q)} a_{nk} > 0.$$

The results of last section imply that

$$W(T, p) \cap l^\infty \not\subseteq c_A(b)$$

which completes the proof. $\square$

To show how a result of Maddox [16] becomes a special case of this theorem, we need to recall some terminology from [9] involving convolution methods of summability. Let $p = (p_0, p_1, \dots)$ and $q = (q_0, q_1, \dots)$ be two nonnegative sequences of numbers, each adding up to 1 (which may be considered as discrete probability densities of some nonnegative integer valued random variables). Construct a summability matrix C whose first row is q . After defining the first $n-1$ rows, the n th row of C is the Cauchy product of the $(n-1)$ th row with p . The resulting method, C , is called a convolution method.

It is regular if and only if $p_0 < 1$. Borel, Euler, Meyer–König methods are special cases of the convolution method. For instance, if we take $p = q = (1/en!)$, we get the Borel matrix method. As shown in [9], $\Omega(k) = \sqrt{k}$ is a statistical Tauberian rate for convolution methods having finite variance (in particular the Borel matrix method). It is easy to see that $o(k)$ is a summability function for the Cesàro method. Hence, by taking $\Omega(k) = \sqrt{k}$ in our last theorem we get the following result of Maddox [16]:

Theorem 4. *The Borel matrix does not sum all sequences in $W(C_1, p) \cap l^\infty$.*

This result, in turn, implies that the bounded summability field of the Borel method is strictly contained in that of the C_1 method. Since all convolution methods with finite variance [9] have the statistical Tauberian rate of $\Omega(k) = \sqrt{k}$, similar comparisons with C_1 and other classical summability methods could be deduced.

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