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A-statistical relative modular convergence of positive linear operators

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Abstract In this paper, we investigate the problem of statistical approximation to a function f by means of positive linear operators defined on a modular space. Particularly, in order to get stronger results than the classical aspects we mainly use the concept of statistical convergence. Also, a non-trivial application is presented.

Keywords Positive linear operators · Modular space · Statistical convergence · Statistical relative modular convergence · Korovkin theorem

Mathematics Subject Classification 41A35 · 41A36 · 46E30

1 Introduction

The concept of uniform convergence of a sequence of functions relative to a scale function was introduced first by Moore [24] and Chittenden [7] developed this concept as follows:

A sequence $\{f_n\}$ of functions, defined on an interval $I \equiv (a \leq x \leq b)$, converges *relatively uniformly to a limit function* f if there exists a function $\sigma(x)$, called a scale function, such that for every $\varepsilon > 0$ there is an integer n_ε such that for every $n > n_\varepsilon$ the inequality

$$|f_n(x) - f(x)| < \varepsilon |\sigma(x)|$$

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holds uniformly in x on the interval I. The sequence $\{f_n\}$ is said to converge *uniformly relative to the scale function* σ or more simply, *relatively uniformly.*

It will be observed that uniform convergence is the special case of relatively uniform convergence in which the scale function is a non-zero constant (for more properties and details, see also [7–9]).

Example 1 For each $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

This sequence does not converge uniformly, but converges to $f = 0$ uniformly relative to a scale function $\sigma(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x}, & 0 < x \leq 1 \end{cases}$.

Demirci and Orhan [12] defined the statistically relatively uniform convergence by using the notions of the natural density and the relatively uniform convergence and studied its use in the Korovkin-type approximation theory. In another direction, a new type of modular convergence, called relative modular convergence, was introduced in [29] inspired by the investigations in [4] in modular spaces. These spaces were extensively investigated by Musielak (see [26,27] and later on in [23]). Later Bardaro, Musielak and Vinti improved an approximation theory in modular spaces for the general case of nonlinear integral operators (see [6]). Also, some versions of abstract Korovkin-type theorems in modular function spaces, with respect to filter convergence for linear positive operators have been studied in [2]. This class of spaces is very wide and contains as particular case the L_p-spaces, Orlicz spaces, the Musielak- Orlicz spaces and others. The concepts of the strong and modular convergence were introduced by Bardaro and Mantellini and they presented some approximation theorems by means of positive linear operators defined on modular spaces in [4]. Then, Karakuş, Demirci and Duman [21] replaced the convergence that Bardaro and Mantellini used in [4], by the more general A-statistical convergence with respect to a fixed summability matrix A and introduced a non-trivial generalization of the classical Korovkin-type results in the modular spaces. In this paper, we improve the notion of relative modular convergence [29] with the help of statistical approach. So we obtain various Korovkin-type approximation theorems in modular spaces via the concept of A-statistical relative modular convergence. Also, we display an example satisfying all conditions of our approximation result in a modular space but not the classic one in [29] and also the other in [21].

We now recall some basic definitions and notations used in the paper.

Let

$$A := [a_{jn}] \ (j, n \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

be an infinite summability matrix. For a given sequence $x := \{x_n\}$, the *A-transform of* x, denoted by

$$Ax := \{(Ax)_j\},$$

is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n,$$

provided the series converges for each $j \in \mathbb{N}$. We say that A is *regular* (see [19]) if

$$\lim Ax = L \quad \text{whenever} \quad \lim x = L.$$

Assume that A is a non-negative regular summability matrix. Then the sequence $x = \{x_n\}$ is called *A-statistically convergent* to L provided that, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:\ |x_n - L| \geq \varepsilon} a_{jn} = 0. \tag{1}$$

We denote this limit as follows (*cf.* [18]; see also [10,11,20])

$$st_A - \lim x = L.$$

Actually, this convergence method is based on the concept of A-density. Recall the *A-density* of a subset $K \subset \mathbb{N}$, denoted by

$$\delta_A (K),$$

is given by

$$\delta_A (K) = \lim_j \sum_{n=1}^{\infty} a_{jn} \chi_K (n),$$

provided the limit exists, where χ_K is the characteristic function of K; or equivalently

$$\delta_A (K) = \lim_j \sum_{n \in K} a_{jn}.$$

So, by (1), we easily see that

$$st_A - \lim x = L \text{iff } \delta_A (\{n : |x_n - L| \geq \varepsilon\}) = 0 \text{ for every } \varepsilon > 0.$$

We should note that if we take $A = C_1 := [c_{jn}]$, the *Cesáro matrix* defined by

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

then the concept of A-density is called natural density and A-statistical convergence reduces to the concept of *statistical convergence* (*cf.* [14]; see also [15–17]). In this case, we write

$$st - \lim x = L \text{ instead of } st_{C_1} - \lim x = L.$$

Further, taking $A = I$, the identity matrix, A-statistical convergence coincides with the ordinary convergence, i.e.,

$$st_I - \lim x = \lim x = L.$$

Observe that every convergent sequence (in the usual sense) is A-statistically convergent to the same value for any non-negative regular matrix A, but its converse is not always true. Actually, in [20], Kolk proved that A-statistical convergence is stronger than convergence when $A = [a_{jn}]$ is a non-negative regular summability matrix such that

$$\lim_j \max_n \{a_{jn}\} = 0.$$

The concepts of *statistical limit superior* and *limit inferior* have been introduced by Fridy and Orhan [17]. A-statistical analogs of these concepts have been examined by Connor and Kline [10], and Demirci [11] as follows. The *A-statistical limit superior* of a number sequence $x = \{x_n\}$, denoted by

$$st_A - \limsup x,$$

is defined by

$$st_A - \limsup x = \begin{cases} \sup B_x, \text{ if } B_x \neq \phi, \\ -\infty, \quad \text{ if } B_x = \phi, \end{cases}$$

where $B_x := \{b \in \mathbb{R} : \delta_A \{n : x_n > b\} \neq 0\}$ and ϕ denotes the empty set. We note that by $\delta_A(K) \neq 0$ we mean either $\delta_A(K) > 0$ or K fails to have A-density. Similarly, the *A-statistical limit inferior* of $\{x_n\}$, denoted by

$$st_A - \liminf x,$$

is defined by

$$st_A - \liminf x = \begin{cases} \inf C_x, \text{ if } C_x \neq \phi, \\ +\infty \ , \text{ if } C_x = \phi, \end{cases}$$

where $C_x := \{c \in \mathbb{R} : \delta_A (\{n : x_n < c\}) \neq 0\}$. Of course, if we take $A = C_1$, then the above definitions reduce to the concepts of $st - \limsup x$ and $st - \liminf x$ given in [17], respectively. As in the ordinary limit superior or inferior, it was proved that

$$st_A - \liminf x \leq st_A - \limsup x$$

and also that, for any sequence $x = \{x_n\}$ satisfying $\delta_A(\{n : |x_n| > M\}) = 0$ for some $M > 0$,

$$st_A - \lim x = L \quad \text{iff} \quad st_A - \liminf x = st_A - \limsup x = L.$$

We now focus on modular spaces.

Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X(I)$ we denote the space of all real-valued measurable functions on I provided with equality *a.e.* As usual, let $C(I)$ denote the space of all continuous real-valued functions, and $C^{\infty}(I)$ denote the space of all infinitely differentiable functions on I . In this case, we say that a functional $\rho : X(I) \to [0, +\infty]$ is a *modular* on $X(I)$ provided that the following conditions hold:

(i) $\rho(f) = 0$ if and only if $f = 0$ *a.e.* in I,
(ii) $\rho(-f) = \rho(f)$ for every $f \in X(I)$,
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(I)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Recall that a modular ρ is called *N-quasi convex* if the following is satisfied:

- there exists a constant $N \geq 1$ such that

$$\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng),$$

holds for every $f, g \in X(I)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*. Furthermore, a modular ρ is called *N-quasi semiconvex* if it holds:

- there exists a constant $N \geq 1$ such that

$$\rho(af) \leq Na\rho(Nf),$$

holds for every $f \in X(I)$ and $a \in (0, 1]$.

It is clear that every N-quasi convex modular is N-quasi semiconvex. We should recall that the above two concepts were introduced and discussed in details by Bardaro et al. [5,6].

We now consider some appropriate vector subspaces of $X(I)$ by means of a modular ρ as follows:

$$L^{\rho}(I) := \left\{ f \in X(I) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\}$$

and

$$E^{\rho}(I) := \left\{ f \in L^{\rho}(I) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0 \right\}.$$

Here, $L^{\rho}(I)$ is called the *modular space* generated by ρ; and also $E^{\rho}(I)$ is called the space of the finite elements of $L^{\rho}(I)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\{ f \in X(I) : \rho(\lambda f) < +\infty \quad \text{for some} \quad \lambda > 0 \}$$

coincides with $L^{\rho}(I)$. The notions about modulars are introduced in [27] and widely discussed in [6] (see also [23,28]).

Now we recall the convergence methods in modular spaces [4].

Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}(I)$. Then, $\{f_n\}$ is *modularly convergent* to a function $f \in L^{\rho}(I)$ iff

$$\lim_n \rho\left(\lambda_0 \left(f_n - f\right)\right) = 0 \quad \text{for some} \quad \lambda_0 > 0. \tag{2}$$

Also, $\{f_n\}$ is *F-norm convergent* (or, *strongly convergent*) to f iff

$$\lim_n \rho\left(\lambda \left(f_n - f\right)\right) = 0 \quad \text{for every} \quad \lambda > 0. \tag{3}$$

It is known from [27] that (2) and (3) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in X(I)$.

Also we recall the definition of relative modular convergence given in [29].

Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}(I)$. Then, $\{f_n\}$ is said to be *relatively modularly convergent* to a function $f \in L^{\rho}(I)$ if there exists a function $\sigma(x)$, called a scale function $\sigma \in X(I)$, $|\sigma(x)| \neq 0$ such that

$$\lim_n \rho\left(\lambda_0 \left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for some} \quad \lambda_0 > 0. \tag{4}$$

Also, if (4) is provided for all $\lambda > 0$, then it is called relatively strong convergence, i.e. if

$$\lim_n \rho\left(\lambda \left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for every} \quad \lambda > 0. \tag{5}$$

Now unifying the approaches (4) and (5) with A-statistical convergence we introduce the notion of the *A-statistical relative modular (or strong) convergence* as follows:

Definition 1 Let $\{f_n\}$ be a function sequence whose terms belong to $L^{\rho}(I)$ and let $A = \left[a_{jn}\right]$ be a non-negative regular summability matrix. Then, $\{f_n\}$ is said to be A-statistical *relatively modularly convergent* to a function $f \in L^{\rho}(I)$ if there exists a function $\sigma(x)$, called a scale function $\sigma \in X(I)$, $|\sigma(x)| \neq 0$ such that

$$st_A - \lim_n \rho\left(\lambda_0 \left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for some} \quad \lambda_0 > 0.$$

Also, if the previous formula is provided for all $\lambda > 0$, then it is called A-statistical relatively strong convergence, i.e. if

$$st_A - \lim_n \rho\left(\lambda \left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for every} \quad \lambda > 0.$$

Now we reveal the special cases of A-*statistical relative modular convergence.* If we take $A = I$, the *identity matrix*, the concept of A-*statistical relative modular convergence* reduces to the concept of *relative modular convergence,* if $A = I$ and the scale function is a non-zero constant, then A-*statistical relative modular convergence* coincides with the *modular convergence* and if we take $A = C_1$, the concept of A-*statistical relative modular convergence* reduces to the concept of *statistical relative modular convergence.* Also, it will be observed that A-*statistical modular convergence* is the special case of A-*statistical relative modular convergence* in which the scale function is a non-zero constant (cf. [21]).Moreover, if $\sigma(x)$ is bounded, A-*statistical relative modular convergence* implies A-*statistical modular convergence*. However, A-*statistical relative modular convergence* does not imply A-*statistical modular convergence*, when $\sigma(x)$ is unbounded. This is illustrated by the following example.

Example 2 Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- *$\varphi$is convex,*
- $\varphi(0) = 0, \varphi(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to\infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^{φ} on $X(I)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \varphi(|f(x)|)\, dx \textit{ for } \quad f \in X(I).$$

In this case, ρ^{φ} is a convex modular on $X(I)$, which satisfies all assumptions listed above (see [4]*). Consider the Orlicz space generated by φ as follows:*

$$L^{\rho}_{\varphi}(I) := \left\{ f \in X(I) : \rho^{\varphi}(\lambda f) < +\infty \text{ for some } \lambda > 0 \right\}.$$

For each $n \in N$, define $g_n : [0, 1] \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} 2, & n = k^2 \\ n^3 x, & 0 < x < \frac{1}{n}, n \neq k^2, k = 1, 2, \dots \\ 0, & x = 0 or \frac{1}{n} \leq x \leq 1, n \neq k^2 \end{cases} \tag{6}$$

If $\varphi(x) = x^p$ for $1 \leq p < \infty$, $x \geq 0$, then $L^{\rho}_{\varphi}(I) = L_p(I)$ ([6])*. Moreover we have for any function $f \in L^{\rho}_{\varphi}(I)$*

$$\rho^{\varphi}(f) = \|f\|^p_{L_p}.$$

Now let $A = C_1$, the *Cesáro matrix of order one. It is clear that $\{g_n\}$ does not converge statistical modularly but converges to $g = 0$ statistical modularly relative to a scale function$\sigma(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x^2}, & 0 < x \leq 1 \end{cases}$ on $L_1([0, 1])$ .Indeed, for some $\lambda_0 > 0$, with the choice of $p = 1$ we have $\rho^{\varphi}(g) = \|g\|_{L_1}$,*

$$\rho\left(\lambda_0\left(g_n-g\right)\right)=\left\|\lambda_0\left(g_n-g\right)\right\|_{L_1}=\begin{cases}2\lambda_0, & n=k^2\\ \frac{\lambda_0 n}{2}, & n\neq k^2\end{cases}, k=1,2,\ldots, \tag{7}$$

since $\lim_n \frac{\lambda_0 n}{2} \neq 0$, *we have*

$$st-\lim_n \left\|\lambda_0\left(g_n-g\right)\right\|_{L_1} \neq 0.$$

Then using the scale function σ

$$\rho\left(\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right)=\left\|\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right\|_{L_1}=\begin{cases}\frac{2\lambda_0}{3}, & n=k^2\\ \frac{\lambda_0}{4n}, & n\neq k^2\end{cases}, k=1,2,\ldots,$$

since $\lim_n \frac{\lambda_0}{4n}=0$,*we get*

$$st-\lim_n \left\|\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right\|_{L_1}=0.$$

On the other hand, we easily see that $\{g_n\}$ *does not converge to* $g=0$ *modularly relative to a scale function* $\sigma(x)=\begin{cases}1, & x=0\\ \frac{1}{x^2}, & 0<x\leq 1\end{cases}$ *on* $L_1([0,1])$ *. Indeed from* (7), *we observed that the sequence* $\left\{\rho\left(\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right)\right\}$ *has two subsequences with different limit points so* $\{g_n\}$ *is not relatively modularly convergent to* $g=0$.

In this paper, we will need the following assumptions on a modular ρ:

- ρ is *monotone* if $\rho(f)\leq\rho(g)$ for $|f|\leq|g|$,
- ρ is*finite* if $\chi_A\in L^\rho(I)$ whenever A is measurable subset of I, with finite measure i.e. $\mu(A)<\infty$,
- ρ is *absolutely finite*,if ρ is finite and, for every $\varepsilon>0$, $\lambda>0$, there exists a $\delta>0$ such that $\rho(\lambda\chi_B)<\varepsilon$ for any measurable subset $B\subset I$ with $\mu(B)<\delta$,
- ρ is *strongly finite* if $\chi_I\in E^\rho(I)$,
- ρ is called *absolutely continuous* provided that there exists an $\alpha>0$ such that, for every $f\in X(I)$ with $\rho(f)<+\infty$, the following condition holds: for every $\varepsilon>0$ there is $\delta>0$ such that $\rho(\alpha f\chi_B)<\varepsilon$ whenever B is any measurable subset of I with $\mu(B)<\delta$.

Observe now that (see [4,5]) if a modular ρ is monotone and finite, then we have $C(I)\subset L^\rho(I)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I)\subset \overline{E^\rho(I)}$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty(I)}=L^\rho(I)$ [25]. Some important relations between the above properties may be found in [3,5,6,25,28].

2 *A*-statistical relative Korovkin theorems in modular spaces

Let ρ be a monotone and finite modular on $X(I)$ and let $A=\left[a_{jn}\right]$ be a non-negative regular summability matrix. Assume that D is a set satisfying $C^\infty(I)\subset D\subset L^\rho(I)$.

We can construct such a subset D since ρ is monotone and finite (see [4]). Here, D is domain of the operator $\mathbb{T}$. Assume further that $\mathbb{T} := \{T_n\}$ is a sequence of positive linear operators from D into $X(I)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ containing $C^{\infty}(I)$ and $\sigma \in X(I)$ is an unbounded function satisfying $\sigma(x) \neq 0$ such that

$$st_A - \limsup_n \rho\left(\lambda\left(\frac{T_n h}{\sigma}\right)\right) \le P\rho(\lambda h) \tag{8}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. We also use the test functions e_i defined by

$$e_i(x) = x^i \quad (i = 0, 1, 2).$$

Theorem 1 *Let $A = [a_{jn}]$ be a non-negative regular summability matrix and let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T} := \{T_n\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (8). Moreover suppose that $\sigma_i(x)$ is an unbounded function satisfying $|\sigma_i(x)| \ge a_i > 0$ $(i = 0, 1, 2)$. Assume that*

$$st_A - \lim_n \rho\left(\lambda\left(\frac{T_n e_i - e_i}{\sigma_i}\right)\right) = 0 \quad \text{for every} \quad \lambda > 0 \text{ and } i = 0, 1, 2. \tag{9}$$

Now let f be any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$. Then, we have

$$st_A - \lim_n \rho\left(\lambda_0\left(\frac{T_n f - f}{\sigma}\right)\right) = 0 \quad \text{for some} \quad \lambda_0 > 0, \tag{10}$$

where $\sigma(x) = \max\{|\sigma_i(x)|; i = 0, 1, 2\}$.

Proof We first claim that

$$st_A - \lim_n \rho\left(\mu\left(\frac{T_n g - g}{\sigma}\right)\right) = 0 \text{ for every } g \in C(I) \cap D \text{ and every } \mu > 0. \tag{11}$$

To see this assume that g belongs to $C(I) \cap D$ and μ is any positive number. By the continuity of g on I and by the linearity and positivity of the operators T_n, we easily see that (see, for instance, [1,22]), for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} |T_n(g;x) - g(x)| \le \varepsilon T_n(e_0;x) + M\,|T_n(e_0;x) - e_0(x)| \\ + \frac{2M}{\delta^2} T_n\left((t-x)^2; x\right) \end{aligned}$$

holds for every $x \in I$ and $n \in \mathbb{N}$, where $M := \sup_{x \in I} |g(x)|$. The above inequality implies that

$$|T_n(g;x) - g(x)| \le \varepsilon + (\varepsilon + M + \frac{2Mc^2}{\delta^2})\,|T_n(e_0;x) - e_0(x)|$$

$$+\frac{4Mc}{\delta^2}\left|T_n\left(e_1;x\right)-e_1(x)\right|$$
$$+\frac{2M}{\delta^2}\left|T_n\left(e_2;x\right)-e_2(x)\right|,$$

where $c := \max|x|$.

Hence, denoting by $K := \max\left\{\varepsilon+M+\frac{2Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$,

$$|T_n(g;x)-g(x)| \le \varepsilon + K\left\{|T_n(e_0)-e_0|+|T_n(e_1)-e_1|+|T_n(e_2)-e_2|\right\}.$$

Now we multiply the both-sides of the above inequality with $\frac{1}{|\sigma(x)|}$, we have

$$\left|\frac{T_n(g;x)-g(x)}{\sigma(x)}\right| \le \frac{\varepsilon}{|\sigma(x)|}+\frac{K}{|\sigma(x)|}\left\{|T_n(e_0;x)-e_0\,(x)|\right.$$
$$\left.+|T_n(e_1;x)-e_1\,(x)|+|T_n(e_2;x)-e_2\,(x)|\right\}.$$

So the last inequality gives, for many $\mu > 0$, that

$$\mu\left\{\left|\frac{T_n(g;x)-g(x)}{\sigma(x)}\right|\right\} \le \frac{\mu\varepsilon}{|\sigma(x)|}+\mu K\left\{\left|\frac{T_n(e_0;x)-e_0\,(x)}{\sigma_0(x)}\right|\right\}$$
$$+\mu K\left\{\left|\frac{T_n(e_1;x)-e_1\,(x)}{\sigma_1(x)}\right|\right\}$$
$$+\mu K\left\{\left|\frac{T_n(e_2;x)-e_2\,(x)}{\sigma_2(x)}\right|\right\}.$$

$$\rho\left(\mu\left(\frac{T_ng-g}{\sigma}\right)\right) \le \rho\left(\frac{\mu\varepsilon}{\sigma}+\mu K\left(\frac{T_ne_0-e_0}{\sigma_0}\right)\right.$$
$$\left.+\mu K\left(\frac{T_ne_1-e_1}{\sigma_1}\right)+\mu K\left(\frac{T_ne_2-e_2}{\sigma_2}\right)\right).$$

Hence, we may write that

$$\rho\left(\mu\left(\frac{T_ng-g}{\sigma}\right)\right) \le \rho\left(\frac{4\mu\varepsilon}{\sigma}\right)+\rho\left(4\mu K\left(\frac{T_ne_0-e_0}{\sigma_0}\right)\right)$$
$$\rho\left(4\mu K\left(\frac{T_ne_1-e_1}{\sigma_1}\right)\right)+\rho\left(4\mu K\left(\frac{T_ne_2-e_2}{\sigma_2}\right)\right).$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0 < \varepsilon \le 1$

$$\rho\left(\mu\left(\frac{T_ng-g}{\sigma}\right)\right) \le N\varepsilon\rho\left(\frac{4\mu N}{\sigma}\right)+\rho\left(4\mu K\left(\frac{T_ne_0-e_0}{\sigma_0}\right)\right)$$

$$\rho\left(4\mu K\left(\frac{T_n e_1 - e_1}{\sigma_1}\right)\right) + \rho\left(4\mu K\left(\frac{T_n e_2 - e_2}{\sigma_2}\right)\right).$$

For a given $r > 0$, choose an $\varepsilon \in (0, 1]$ such that $N\varepsilon\rho\left(\frac{4\mu N}{\sigma}\right) < r$. Now we define the following sets:

$$G_n : = \left\{n : \rho\left(\mu\left(\frac{T_n g - g}{\sigma}\right)\right) \geq r\right\}$$
$$G_{n,i} : = \left\{n : \rho\left(4\mu K\left(\frac{T_n e_i - e_i}{\sigma_i}\right)\right) \geq \frac{r - N\varepsilon\rho\left(\frac{4\mu N}{\sigma}\right)}{3}\right\}, i = 0, 1, 2.$$

Then, it is easy to see that $G_n \subseteq \cup_{i=0}^{2} G_{n,i}$. So, we can write, for all $j \in \mathbb{N}$, that

$$\sum_{n\in G_n} a_{jn} \leq \sum_{n\in G_{n,0}} a_{jn} + \sum_{n\in G_{n,1}} a_{jn} + \sum_{n\in G_{n,2}} a_{jn}. \tag{12}$$

Taking limit as $j \to \infty$ in (12) and using the hypothesis (9), we get

$$\lim_j \sum_{n\in G_n} a_{jn} = 0$$

which proves our claim (11). Observe that (11) also holds for every $g \in C^{\infty}(I)$ because of $C^{\infty}(I) \subset C(I)$. Now let $f \in L^{\rho}(I)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$. Since $\mu(I) < \infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I)$(see [3]). Using these properties of the modular ρ, it is known from [6,25] that the space $C^{\infty}(I)$ is modularly dense in $L^{\rho}(I)$, i.e., there exists a sequence $\{g_k\} \subset C^{\infty}(I)$ such that

$$\lim_k \rho\left(3\lambda_0^*(g_k - f)\right) = 0 \quad \text{for some} \quad \lambda_0^* > 0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*(g_k - f)\right) < \varepsilon \quad \text{for every} \quad k \geq k_0. \tag{13}$$

On the other hand, by the linearity and positivity of the operators T_n, we may write that

$$\lambda_0^*|T_n(f;x) - f(x)| \leq \lambda_0^*\left|T_n(f - g_{k_0};x)\right| + \lambda_0^*\left|T_n(g_{k_0};x) - g_{k_0}(x)\right|$$
$$+\lambda_0^*\left|g_{k_0}(x) - f(x)\right|,$$

which holds for every $x \in I$ and $n \in \mathbb{N}$. Multiplying the last inequality with $\frac{1}{|\sigma(x)|}$ and applying the modular ρ in it and moreover considering the monotonicity of ρ, we have

$$\rho\left(\lambda_0^*\left(\frac{T_nf-f}{\sigma}\right)\right)\leq\rho\left(3\lambda_0^*\left(\frac{T_n\left(g_{k_o}-f\right)}{\sigma}\right)\right)+\rho\left(3\lambda_0^*\left(\frac{T_ng_{k_o}-g_{k_o}}{\sigma}\right)\right)$$
$$+\rho\left(3\lambda_0^*\left(\frac{g_{k_o}-f}{\sigma}\right)\right).$$

Hence, observing $|\sigma|\geq a>0$ ($a=\max\{a_i:i=0,1,2\}$) and taking into consideration that ρ is monotone, we may write that

$$\rho\left(\lambda_0^*\left(\frac{T_nf-f}{\sigma}\right)\right)\leq\rho\left(3\lambda_0^*\left(\frac{T_n\left(g_{k_o}-f\right)}{\sigma}\right)\right)$$
$$+\rho\left(3\lambda_0^*\left(\frac{T_ng_{k_o}-g_{k_o}}{\sigma}\right)\right)+\rho\left(\frac{3\lambda_0^*}{a}\left(g_{k_o}-f\right)\right). \quad (14)$$

Then it follows from (13) and (14) that

$$\rho\left(\lambda_0^*\left(\frac{T_nf-f}{\sigma}\right)\right)\leq\rho\left(3\lambda_0^*\left(\frac{T_n\left(g_{k_o}-f\right)}{\sigma}\right)\right)+\rho\left(3\lambda_0^*\left(\frac{T_ng_{k_o}-g_{k_o}}{\sigma}\right)\right)+\varepsilon.$$

Taking A-statistical limit superior in both-sides of last inequality and also using the facts that $g_{k_o}\in C^{\infty}(I)$ and $f-g_{k_o}\in X_{\mathbb{T}}$, we obtain from (8) that

$$st_A-\limsup_n\rho\left(\lambda_0^*\left(\frac{T_nf-f}{\sigma}\right)\right)\leq\varepsilon+P\rho\left(3\lambda_0^*\left(f-g_{k_o}\right)\right)$$
$$+st_A-\limsup_n\rho\left(3\lambda_0^*\left(\frac{T_ng_{k_o}-g_{k_o}}{\sigma}\right)\right)$$

which gives

$$st_A-\limsup_n\rho\left(\lambda_0^*\left(\frac{T_nf-f}{\sigma}\right)\right)$$
$$\leq\varepsilon(P+1)+st_A-\limsup_n\rho\left(3\lambda_0^*\left(\frac{T_ng_{k_o}-g_{k_o}}{\sigma}\right)\right). \quad (15)$$

By (11), since

$$st_A-\lim_n\rho\left(3\lambda_0^*\left(\frac{T_ng_{k_0}-g_{k_0}}{\sigma}\right)\right)=0,$$

we get

$$st_A-\limsup_n\rho\left(3\lambda_0^*\left(\frac{T_ng_{k_0}-g_{k_0}}{\sigma}\right)\right)=0. \quad (16)$$

Combining (15) with (16), we conclude that

$$st_A - \limsup_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) \leq \varepsilon(P+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$st_A - \limsup_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) = 0.$$

Furthermore, since $\rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right)$ is non-negative for all $n \in \mathbb{N}$, we can easily show that

$$st_A - \lim_n \rho\left(\lambda_0^*\left(\frac{T_n f - f}{\sigma}\right)\right) = 0,$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition then the space $C^\infty(I)$ is strongly dense in $L^\rho(I)$ (see [4]) hence, we get the following result from Theorem 1 at once.

Theorem 2 *Let $A = \left[a_{jn}\right]$, $\mathbb{T} := \{T_n\}$, ρ and σ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $st_A - \lim_n \rho\left(\lambda\left(\frac{T_n e_i - e_i}{\sigma_i}\right)\right) = 0$ *for every $\lambda > 0$ and $i = 0, 1, 2$,*

(b) $st_A - \lim_n \rho\left(\lambda\left(\frac{T_n f - f}{\sigma}\right)\right) = 0$ *for every $\lambda > 0$ provided that f is any function belonging to $L^\rho(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty(I)$.*

If one replaces the scale function by a nonzero constant, then the condition (8) reduces to

$$st_A - \limsup_n \rho\left(\lambda\left(T_n h\right)\right) \leq P\rho\left(\lambda h\right)$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. In this case, the results obtained in [21] follows from our Theorems 1 and 2. In addition to this replacing if we take $A = I$, the identity matrix, then the condition (8) reduces to

$$\limsup_n \rho\left(\lambda\left(T_n h\right)\right) \leq P\rho\left(\lambda h\right)$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P, so the results obtained by Bardaro and Mantellini in [4] immediately follows from our Theorems 1 and 2. Finally if we replace the matrix A by the identity matrix, then the condition (8) reduces to

$$\limsup_n \rho\left(\lambda\left(\frac{T_n h}{\sigma}\right)\right) \leq P\rho\left(\lambda h\right) \tag{17}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant P. In this case, the next results which were obtained by Yilmaz et al. in [29] immediately follows from our Theorems 1 and 2.

Corollary 1 ([29]) *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T} := \{T_n\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (17). Moreover suppose that $\sigma_i(x)$ is an unbounded function satisfying $|\sigma_i(x)| \geq a_i > 0$ $(i = 0, 1, 2)$. If $\{T_n e_i\}$ is relatively strongly convergent to e_i for each $i = 0, 1, 2$, then $\{T_n f\}$ is relatively modularly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$.*

Corollary 2 ([29]) *Let $\mathbb{T} := \{T_n\}$ and ρ be the same as in Corollary 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $\{T_n e_i\}$ is relatively strongly convergent to e_i for each $i = 0, 1, 2$,
(b) $\{T_n f\}$ is relatively strongly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}(I)$.

3 Concluding remarks

In this section, we display an example such that our Korovkin-type approximation results in modular spaces are stronger than the classical ones in [4].

Example 3 Take $I = [0, 1]$ and let A *be* the *Cesáro matrix of order one and* φ, ρ^{φ} and $L^{\rho}_{\varphi}(I)$ be the same as in Example 2. *Then consider the following classical Bernstein-Kantorovich operator $U := \{U_n\}$on the space $L^{\rho}_{\varphi}(I)$ (see [4]) which is defined by:*

$$U_n(f; x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \quad for\ x \in I. \tag{18}$$

Observe that the operators U_n map the Orlicz space $L^{\rho}_{\varphi}(I)$ into itself. Moreover, the property (17) is satisfied with the choice of $X_{\mathbf{U}} := L^{\rho}_{\varphi}(I)$. Then, by Corollary 1, we know that, for any function $f \in L^{\rho}_{\varphi}(I)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}(I)$, $\{U_n f\}$is relatively modularly convergent to f.

If $\varphi(x) = x^p$ for $1 \leq p < \infty$, $x \geq 0$,then $L^{\rho}_{\varphi}(I) = L_p(I)$ ([6]). *Moreover we have*

$$\rho^{\varphi}(f) = \|f\|^p_{L_p}.$$

For $p = 1$, we have $\rho^{\varphi}(f) = \|f\|_{L_1}$. Then, using the operators U_n, we define the sequence of positive linear operators $V := \{V_n\}$ on $L_1(I)$ as follows:

$$V_n(f; x) = (1 + g_n(x))U_n(f; x) \;\; for\ f \in L_1(I), x \in I\ and\ n \in \mathbb{N}, \tag{19}$$

where $\{g_n\}$ *is the same as in (6) and we choose* $\sigma_i(x) = \sigma(x)$ $(i = 0, 1, 2)$, *where* $\sigma(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x^2}, & 0 < x \le 1 \end{cases}$. *Since, for positive constant* C, $\left\| U_j(f; x) \right\|_{L_p} \le C \left\| f \right\|_{L_p}$ ([13]) , *we can easily see that, for every* $\lambda > 0$

$$st - \limsup_n \left\| \lambda \left(\frac{V_n f}{\sigma} \right) \right\|_{L_1} \le C \left\| \lambda f \right\|_{L_1}.$$

We now claim that

$$st - \lim_n \left\| \lambda \left(\frac{V_n e_i - e_i}{\sigma} \right) \right\|_{L_1} = 0, i = 0, 1, 2. \tag{20}$$

Observe that $U_n(e_0; x) = e_0$, $U_n(e_1; x) = \frac{nx}{n+1} + \frac{1}{2(n+1)}$ *and* $U_n(e_2; x) = \frac{n(n-1)x^2}{(n+1)^2} + \frac{2nx}{(n+1)^2} + \frac{1}{3(n+1)^2}$. *So, we can see, for any* $\lambda > 0$, *that*

$$\begin{aligned} \left\| \lambda \left(\frac{V_n(e_0; x) - e_0(x)}{\sigma(x)} \right) \right\|_{L_1} &= \left\| \lambda \left(\frac{1 + g_n(x) - 1}{\sigma(x)} \right) \right\|_{L_1} \\ &= \begin{cases} \frac{2\lambda}{3} & n = k^2 \\ \frac{\lambda}{4n} & n \ne k^2 \end{cases}, k = 1, 2, \ldots, \end{aligned} \tag{21}$$

since $\lim_n \frac{\lambda}{4n} = 0$, *we get*

$$st - \lim_n \left\| \lambda \left(\frac{V_n e_0 - e_0}{\sigma} \right) \right\|_{L_1} = 0.$$

which guarantees that (20) *holds true for* $i = 0$. *Also, we have*

$$\begin{aligned} \left\| \lambda \left(\frac{V_n(e_1; x) - e_1(x)}{\sigma(x)} \right) \right\|_{L_1} &= \left\| \lambda \left(\frac{(1 + g_n(x)) U_n(e_1; x)}{\sigma(x)} - \frac{x}{\sigma(x)} \right) \right\|_{L_1} \\ &\le \left\| \lambda \frac{g_n(x)}{\sigma(x)} \left(\frac{nx}{n+1} + \frac{1}{2(n+1)} \right) \right\|_{L_1} + \left\| \lambda \frac{x - 2x^2}{2(n+1)} \right\|_{L_1} \\ &< \left\| \lambda \frac{g_n(x)}{\sigma(x)} \right\|_{L_1} + \frac{\lambda}{2(n+1)}, \end{aligned}$$

because of $st - \lim_n \left\| \lambda \frac{g_n(x)}{\sigma(x)} \right\|_{L_1} = 0$ *and* $\lim_n \frac{\lambda}{2(n+1)} = 0$, *we get*

$$st - \lim_n \left\| \lambda \left(\frac{V_n(e_1; x) - e_1(x)}{\sigma(x)} \right) \right\|_{L_1} = 0.$$

Finally, since

$$\begin{aligned}
&\left\|\lambda\left(\frac{V_n\left(e_2;x\right)-e_2(x)}{\sigma\left(x\right)}\right)\right\|_{L_1}\\
&\quad=\left\|\lambda\frac{\left(1+g_n\left(x\right)\right)U_n\left(e_2;x\right)}{\sigma\left(x\right)}-\frac{\lambda x^2}{\sigma\left(x\right)}\right\|_{L_1}\\
&\quad\leq\left\|\lambda\frac{g_n\left(x\right)}{\sigma\left(x\right)}\left(\frac{n\left(n-1\right)x^2}{\left(n+1\right)^2}+\frac{2nx}{\left(n+1\right)^2}+\frac{1}{3\left(n+1\right)^2}\right)\right\|_{L_1}\\
&\qquad+\left\|\lambda\left(\frac{\left(3n+1\right)x^3}{\left(n+1\right)^2}+\frac{2nx^2}{\left(n+1\right)^2}+\frac{x}{3\left(n+1\right)^2}\right)\right\|_{L_1}\\
&\quad<3\left\|\lambda\frac{g_n\left(x\right)}{\sigma\left(x\right)}\right\|_{L_1}+\frac{3\lambda}{n+1}+\frac{2\lambda n}{\left(n+1\right)^2}+\frac{\lambda}{3\left(n+1\right)^2},
\end{aligned}$$

we have,

$$st-\lim_n\left\|\lambda\left(\frac{V_n\left(e_2;x\right)-e_2(x)}{\sigma\left(x\right)}\right)\right\|_{L_1}=0.$$

So, our claim (20) holds true for each $i=0,1,2$. $\left\{V_j\right\}$ *satisfies all hypothesis of Theorem 1 and we immediately see that,*

$$st-\lim_n\left\|\lambda_0\left(\frac{V_nf-f}{\sigma}\right)\right\|_{L_1}=0,\quad for\, some\quad\lambda_0>0,\ on\ I\quad forall\quad f\in L_1(I).$$

However, from (21) it can be seen that the sequence $\left\|\lambda\left(\frac{V_n(e_0;x)-e_0(x)}{\sigma(x)}\right)\right\|_{L_1}$ *has two subsequences with different limit points. Since* $\lim_n\left\|\lambda\left(\frac{V_ne_0-e_0}{\sigma(x)}\right)\right\|_{L_1}\neq0$, *Corollary 1 does not work for the sequence* $V:=\{V_n\}$.

With this example, we can immediately say that our Theorem 1 is a non-trivial generalization of the classical Korovkin-type results in the modular spaces.

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