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RESEARCH PAPER

Statistical Relative $\mathcal{A}$-Summation Process and Korovkin-Type Approximation Theorem on Modular Spaces

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Abstract In this paper, we present the notion of relative $\mathcal{A}$-summation process. Then we give a statistical Korovkin-type approximation theorem via relative $\mathcal{A}$-summation process in the setting of modular spaces, which includes as particular cases Lp, Orlicz and Musielak–Orlicz spaces. Finally, we give an example showing that our results are proper extensions of the corresponding ones.

Keywords Positive linear operators · Modular space · Matrix summability · Statistical convergence · Korovkin Theorem

1 Introduction

Classical approximation theory has started with the proof of Weierstrass theorem which states that every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated as closely as desired by a polynomial function. After that Korovkin (1960) first established the necessary and sufficient conditions for the uniform convergence of a sequence $\{T_n\}$ of positive linear operators to a function f (see also Altomare 2010; Altomare and Campiti 1994). In classical Korovkin theorem, most of the classical operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. In this case, the matrix summability methods of the Cesaro type are applicable to correct the lack of convergence Bojanic and Khan (1992). The main purpose of using summability theory has always been to make a non-convergent sequence convergent. More generally, in 1981, Nishishiraho introduced the notion of $\mathcal{A}$-summation process on a compact Hausdorff space (Nishishiraho 1981, see also Nishishiraho 1983). Afterwards Korovkin-type approximation theorems are studied via $\mathcal{A}$-summation process in various spaces like weighted spaces (see Atlıhan and Orhan 2007, 2008), modular spaces (Karakus et al. 2010; Orhan and Demirci 2014). In addition, general versions of the Korovkin theorem were studied in which a various kind of convergence methods is used, particularly statistical convergence (Anastassiou and Duman 2008; Duman et al. 2003; Gadjiev and Orhan 2002; Karakus et al. 2008), and it had become one of the most active areas of research in the field of approximation theory. Some Korovkin-type theorems in the setting of a general notion of convergence involving abstract filters were given by Bardaro et al. (2013) (see also Bardaro et al. 2013; Boccuto et al. 2014; Candeloro and Sambucini 2015). In this study, we will investigate a statistical Korovkin-type theorem via an improved $\mathcal{A}$-summation process in the settings of modular function spaces which are the natural generalization of L_p $(p > 0)$, Orlicz, Lorentz, and K öthe spaces.

2 Preliminaries

We begin with some definitions and notations which we will use in the sequel. First, we focus on modular spaces.

Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X(I)$, we denote the space of all real-valued measurable functions on

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I provided with equality *a.e.* As usual, let $C(I)$ denote the space of all continuous real-valued functions, and $C^\infty(I)$ denotes the space of all infinitely differentiable functions on I. We say that a functional $\rho : X(I) \to [0, +\infty]$ is called a *modular* on $X(I)$ if (*i*) $\rho(f) = 0$ if and only if $f = 0$ *a.e.* in I, (*ii*) $\rho(-f) = \rho(f)$ for every $f \in X(I)$ and (*iii*) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ for every $f, g \in X(I)$ and for any $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

A modular ρ is said to be *N-quasi convex* if there exists a constant $N \ge 1$ such that $\rho(\alpha f + \beta g) \le N\alpha\rho(Nf) + N\beta\rho(Ng)$ holds for every $f, g \in X(I)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*. A modular ρ is said to be *N-quasi semiconvex* if there exists a constant $N \ge 1$ such that $\rho(af) \le Na\rho(Nf)$ holds for every $f \in X(I)$ and $a \in (0, 1]$.

It is clear that every N-quasi convex modular is N-quasi semiconvex. We should recall that the above two concepts were introduced and discussed in detail by Bardaro et al. (2003).

We now consider some appropriate vector subspaces of $X(I)$ by means of a modular ρ as follows:

$$L^\rho(I) := \left\{ f \in X(I) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\}$$

and

$$E^\rho(I) := \{ f \in L^\rho(I) : \rho(\lambda f) < +\infty \quad \text{for all} \quad \lambda > 0 \}.$$

Here, $L^\rho(I)$ is called the *modular space* generated by ρ, and $E^\rho(I)$ is called the space of the finite elements of $L^\rho(I)$. Observe that if ρ is N-quasi semiconvex, then the space

$$\{ f \in X(I) : \rho(\lambda f) < +\infty \quad \text{for some} \quad \lambda > 0 \}$$

coincides with $L^\rho(I)$. The notions about modulars are introduced by Musielak (1993) and widely discussed by Bardaro et al. (2003) (see also Kozlowski 1988; Musielak 1983).

With the help of the notions of modular convergence and strong convergence, some approximation theorems were introduced by Bardaro and Mantellini (2007).

Now we recall the convergence notions in modular spaces.

Let $\{f_n\}$ be a function sequence whose terms belong to $L^\rho(I)$. Then, $\{f_n\}$ is *modularly convergent* to a function $f \in L^\rho(I)$ iff

$$\lim_n \rho(\lambda_0(f_n - f)) = 0 \quad \text{for some} \quad \lambda_0 > 0. \tag{1}$$

In addition, $\{f_n\}$ is *F-norm convergent* (or *strongly convergent*) to f iff

$$\lim_n \rho(\lambda(f_n - f)) = 0 \quad \text{for every} \quad \lambda > 0. \tag{2}$$

It is known from the study by Musielak (1983) that (1) and (2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e., there exists a constant $M > 0$ such that $\rho(2f) \le M\rho(f)$ for every $f \in X(I)$.

We also recall the definition of relative modular convergence given by Yilmaz et al. (2016). Let $\{f_n\}$ be a function sequence whose terms belong to $L^\rho(I)$. Then, $\{f_n\}$ is said to be*relatively modularly convergent* to a function $f \in L^\rho(I)$ if there exists a function $\sigma \in X(I)$, called a scale function satisfying $|\sigma(x)| \ne 0$ such that

$$\lim_n \rho\left(\lambda_0\left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for some} \quad \lambda_0 > 0. \tag{3}$$

In addition, if (3) is provided for all $\lambda > 0$, then it is called relative strong convergence, i.e., if

$$\lim_n \rho\left(\lambda\left(\frac{f_n - f}{\sigma}\right)\right) = 0 \quad \text{for every} \quad \lambda > 0.$$

It will be observed that modular convergence is the special case of relative modular convergence in which the scale function is a non-zero constant.

A modular ρ is *monotone* if $\rho(f) \le \rho(g)$ for $|f| \le |g|$, ρ is said to be *finite* if $\chi_A \in L^\rho(I)$ whenever A is measurable subset of I such that $\mu(A) < \infty$. If ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset I$ with $\mu(B) < \delta$, then ρ is *absolutely finite* and if $\chi_I \in E^\rho(I)$, then ρ is *strongly finite.* A modular ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X(I)$ with $\rho(f) < +\infty$, the following condition holds:

- for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f\chi_B) < \varepsilon$ whenever B is any measurable subset of I with $\mu(B) < \delta$.

Observe now that (see Bardaro and Mantellini 2007) if a modular ρ is monotone and finite, then we have $C(I) \subset L^\rho(I)$. In a similar manner, if ρ is monotone and strongly finite, then $C(I) \subset E^\rho(I)$. In addition, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty(I)} = L^\rho(I)$. Some important relations between the above properties may be found in the studies by Bardaro and Mantellini (2006), Bardaro et al. (2003), Mantellini (1998), Musielak (1993).

Now, most of the classical operators converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. There are, however, some exceptions that do not converge at points of simple discontinuity Bojanic and Cheng (1983). As it is mentioned before, in this case the matrix summability methods or more generally $\mathcal{A}$-summation methods are strong enough to correct the lack of convergence.

Let $\mathcal{A} := \{A^{(n)}\} = \left\{\left(a_{kj}^{(n)}\right)\right\}$ be a sequence of infinite non-negative real matrices. For a sequence of real numbers, $x = \{x_j\}$, the double sequence $\mathcal{A}x := \{(Ax)_k^n : k, n \in \mathbb{N}\}$ defined by $(Ax)_k^n := \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j$ is called the $\mathcal{A}-$ transform of x whenever the series converges for all k and n. A sequence x is said to be $\mathcal{A}-$summable to L if

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = L$$

uniformly in n (Bell 1973; Steiglitz 1973). If $A^{(n)} = \mathcal{B}$ for some matrix $\mathcal{B}$, then $\mathcal{A}-$summability is the ordinary matrix summability by $\mathcal{B}$. If, $a_{kj}^{(n)} = \frac{1}{k+1}$, for $n \le j \le k+n$, $(n = 1, 2, \ldots)$, and $a_{kj}^{(n)} = 0$ otherwise, then $\mathcal{A}-$summability reduces to almost convergence Lorentz (1948).

Orhan and Demirci (2014) presented $\mathcal{A}-$summation process on a modular space as follows:

Let D be a set satisfying $C^{\infty}(I) \subset D \subset L^{\rho}(I)$. A sequence $\mathbb{T} := \{T_j\}$ of positive linear operators from D into $X(I)$ is called an $\mathcal{A}-$summation process on D if $\{T_j(f)\}$ is $\mathcal{A}-$summable to f (with respect to modular ρ) for every $f \in D$, i.e.,

$$\lim_k \rho\left[\lambda\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f\right)\right] = 0, \text{ uniformly in } n, \text{ for some } \lambda > 0. \tag{4}$$

In addition, they obtained an extension of the classical Korovkin theorem on a modular space using the statistical $\mathcal{A}-$summation process. Our interest in the present paper is to give a Korovkin theorem for a sequence of positive linear operators using a statistical relative $\mathcal{A}$-summation process on a modular space, which is an extension of Theorem 1 by Orhan and Demirci (2014) and as well Theorem 2 by Orhan and Sakaoglu (2014). Some results concerning summation processes in the space $L_p[a, b]$ of Lebesgue integrable functions on a compact interval may be found in the study by Orhan and Sakaoglu (2014), Sakaoğlu Özgüç and Orhan (2013).

Statistical convergence was first introduced independently by Fast and Steinhaus (see Fast 1951; Steinhaus 1951) and then developed by many authors (see for instance Connor and Kline 1996; Demirci 2006; Fridy 1985; Fridy and Orhan 1997). Now we recall some basic notions and definitions on statistical convergence.

Let $\mathcal{B} := (b_{lk})(l, k \in \mathbb{N} := \{1, 2, 3, \ldots\})$ be an infinite summability matrix. For a given sequence of real numbers $x := \{x_k\}$, the *$\mathcal{B}$-transform of x*, denoted by $\mathcal{B}x := \{(\mathcal{B}x)_l\}$, is given by

$$(\mathcal{B}x)_l = \sum_{k=1}^{\infty} b_{lk} x_k,$$

provided the series converges for each $l \in \mathbb{N}$. We say that $\mathcal{B}$ is *regular* (see Hardy 1949) if $\lim \mathcal{B}x = L$ whenever $\lim x = L$. Assume that $\mathcal{B}$ is a non-negative regular summability matrix. Then the sequence $x = \{x_k\}$ is called *$\mathcal{B}$- statistically convergent* to L provided that, for every $\varepsilon > 0$,

$$\lim_l \sum_{k:|x_k - L| \ge \varepsilon} b_{lk} = 0. \tag{5}$$

We denote this limit by $st_{\mathcal{B}} - \lim x = L$ (*cf.* Freedman and Sember 1981; see also Demirci 2006; Kolk 1993). Actually, this convergence method is based on the concept of $\mathcal{B}$-density. Recall the *$\mathcal{B}$-density* of a subset $K \subset \mathbb{N}$, denoted by $\delta_{\mathcal{B}}\{K\}$, is given by

$$\delta_{\mathcal{B}}\{K\} = \lim_l \sum_{k=1}^{\infty} b_{lk} \chi_K(k),$$

provided the limit exists, where χ_K is the characteristic function of K, or equivalently

$$\delta_{\mathcal{B}}\{K\} = \lim_l \sum_{k \in K} b_{lk}.$$

So, by (5), we easily see that $st_{\mathcal{B}} - \lim x = L$ iff $\delta_{\mathcal{B}}\{k : |x_k - L| \ge \varepsilon\} = 0$ for every $\varepsilon > 0$. We should note that if we take $\mathcal{B} = C_1 := (c_{lk})$, the *Cesáro matrix* is defined by

$$c_{lk} := \begin{cases} \dfrac{1}{l}, & \text{if} \quad 1 \le k \le l, \\ 0, & \text{otherwise}, \end{cases}$$

then $\mathcal{B}$-statistical convergence reduces to the concept of *statistical convergence* (*cf.* Fast 1951; see also Fridy and Orhan 1997). In this case, we write $st - \lim x = L$ instead of $st_{C_1} - \lim x = L$. Further, taking $\mathcal{B} = \mathcal{I}$, the identity matrix, $\mathcal{B}$-statistical convergence coincides with the ordinary convergence, i.e., $st_{\mathcal{I}} - \lim x = \lim x = L$. Observe that every convergent sequence (in the usual sense) is $\mathcal{B}$-statistically convergent to the same value for any non-negative regular matrix $\mathcal{B}$, but its converse is not always true. Actually, in the study by Kolk (1993), Kolk proved that $\mathcal{B}$-statistical convergence is stronger than convergence when $\mathcal{B} = (b_{lk})$ is a non-negative regular summability matrix such that $\lim_l \max_k\{b_{lk}\} = 0$.

The concepts of *statistical limit superior* and *limit inferior* have been introduced by Fridy and Orhan (1997). $\mathcal{B}$-statistical analogs of these concepts have been examined by Connor and Kline (1996), and Demirci (2006) as follows. The *$\mathcal{B}$-statistical limit superior* of a number sequence $x=\{x_k\}$, denoted by $st_{\mathcal{B}}-\limsup x$, is defined by

$$st_{\mathcal{B}}-\limsup x=\begin{cases}\sup B_x, & \text{if} \quad B_x\neq\phi,\\ -\infty, & \text{if} \quad B_x=\phi,\end{cases}$$

where $B_x:=\{b\in\mathbb{R}:\delta_{\mathcal{B}}\{k:x_k>b\}\neq 0\}$ and ϕ denotes the empty set. We note that by $\delta_{\mathcal{B}}\{K\}\neq 0$ we mean either $\delta_{\mathcal{B}}\{K\}>0$ or K fails to have $\mathcal{B}$-density. Similarly, the *$\mathcal{B}$-statistical limit inferior* of $\{x_k\}$, denoted by $st_{\mathcal{B}}-\liminf x$, is defined by

$$st_{\mathcal{B}}-\liminf x=\begin{cases}\inf C_x, & \text{if} \quad C_x\neq\phi,\\ +\infty, & \text{if} \quad C_x=\phi,\end{cases}$$

where $C_x:=\{c\in\mathbb{R}:\delta_{\mathcal{B}}\{k:x_k<c\}\neq 0\}$. Of course, if we take $\mathcal{B}=C_1$, then the above definitions reduce to the concepts of $st-\limsup x$ and $st-\liminf x$ given in Fridy and Orhan (1997), respectively. As in the ordinary limit superior or inferior, it was proved that $st_{\mathcal{B}}-\liminf x\leq st_{\mathcal{B}}-\limsup x$ and also that, for any sequence $x=\{x_k\}$ satisfying $\delta_{\mathcal{B}}\{k:|x_k|>M\}=0$ for some $M>0$, $st_{\mathcal{B}}-\lim x=L$ iff $st_{\mathcal{B}}-\liminf x=st_{\mathcal{B}}-\limsup x=L$.

3 Korovkin-Type Theorems

Let ρ be a monotone and finite modular on $X(I)$, and let $\mathcal{B}=(b_{lk})$ be a non-negative regular summability matrix. Assume that D is a set satisfying $C^{\infty}(I)\subset D\subset L^{\rho}(I)$. We will assume that $\mathbb{T}:=\{T_j\}$ is a sequence of positive linear operators from D to $X(I)$ and for all $k,n\in\mathbb{N}$, $f\in D$ the series

$$A_{k,n}^{\mathbb{T}}(f):=\sum_{j=1}^{\infty}a_{kj}^{(n)}T_jf \tag{6}$$

is absolutely convergent almost everywhere with respect to Lebesgue measure. In addition, assume that there exists a subset $X_{\mathbb{T}}\subset D$ with $C^{\infty}(I)\subset X_{\mathbb{T}}$ and an unbounded function $\sigma\in X(I)$ satisfying $|\sigma(x)|\neq 0$ such that

$$st_{\mathcal{B}}-\limsup_k \rho\left(\lambda\left(\frac{A_{k,n}^{\mathbb{T}}(h)}{\sigma}\right)\right)\leq P\rho(\lambda h),\ \text{uniformly in}\ n \tag{7}$$

holds for every $h\in X_{\mathbb{T}}$, $\lambda>0$ and for an absolute positive constant P.

A sequence $\mathbb{T}:=\{T_j\}$ of positive linear operators of D into $X(I)$ is called a relative $\mathcal{A}-$summation process on D if $\{T_j(f)\}$ is relatively $\mathcal{A}-$summable to f (with respect to modular ρ) for every $f\in D$, i.e.,

$$\lim_k \rho\left[\lambda\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right]=0,\ \text{uniformly in}\ n\ \text{for some}\ \lambda>0. \tag{8}$$

It will be observed that *A-summation process* is the special case of *relative A-summation process* in which the scale function is a non-zero constant.

Throughout the paper, we use the test functions e_i defined by $e_i(x)=x^i (i=0,1,2,\ldots)$.

Theorem 1 *Let $\mathcal{A}=\{A^{(n)}\}$ be a sequence of infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathcal{B}=(b_{lk})$ be a non-negative regular summability matrix and $\mathbb{T}:=\{T_j\}$ be a sequence of positive linear operators from D to $X(I)$ satisfying (7) for each $f\in D$. Moreover, suppose that $\sigma_i(x)$ is an unbounded function satisfying $|\sigma_i(x)|\geq a_i>0$ $(i=0,1,2)$. Suppose that*

$$st_{\mathcal{B}}-\lim_k \rho\left(\lambda\left(\frac{A_{k,n}^{\mathbb{T}}(e_i)-e_i}{\sigma_i}\right)\right)=0,\text{uniformly in}\ n \tag{9}$$

for every $\lambda>0$ and $i=0,1,2$. Now let f be any function belonging to $L^{\rho}(I)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}(I)$. Then we have

$$st_{\mathcal{B}}-\lim_k \rho\left(\lambda_0\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)=0,\text{uniformly in}\ n \tag{10}$$

for some $\lambda_0>0$ where $\sigma(x)=\max\{|\sigma_i(x)|;i=0,1,2\}$.

Proof We first claim that

$$st_{\mathcal{B}}-\lim_k \rho\left(\eta\left(\frac{A_{k,n}^{\mathbb{T}}(g)-g}{\sigma}\right)\right)=0\,\text{uniformly in}\ n \tag{11}$$

for every $g\in C(I)\cap D$ and $\eta>0$ where $A_{k,n}^{\mathbb{T}}(g)=\sum_{j=1}^{\infty}a_{kj}^{(n)}T_jg$. To see this assume that g belongs to $C(I)\cap D$ and η is any positive number. By the continuity of g on I and by the linearity and positivity of the operators T_j, we easily see that (see, for instance Altomare 2010; Kolk 1993), for a given $\varepsilon>0$, there exists a number $\delta>0$ such that $|g(y)-g(x)|<\varepsilon+\frac{2M}{\delta^2}(y-x)^2$ where $M:=\sup_{x\in I}|g(x)|$. Since T_j is a positive linear operator, we get

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g;x) - g(x)\right| \\
&\quad = \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g(.) - g(x);x) + g(x)\right. \\
&\qquad \left. \times \left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)\right)\right| \\
&\quad \le \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(|g(.) - g(x)|;x) + |g(x)| \\
&\qquad \times \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)\right| \\
&\quad \le \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j\left(\varepsilon + \frac{2M}{\delta^2}(. - x)^2;x\right) + |g(x)| \\
&\qquad \times \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)\right| \\
&\quad \le \varepsilon + (\varepsilon + M)\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)\right| \\
&\qquad + \frac{2M}{\delta^2}\left[\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_2(.);x) - e_2(x)\right)\right. \\
&\qquad - 2e_1(x)\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_1(.);x) - e_1(x)\right) \\
&\qquad \left. + e_2(x)\left(\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)\right)\right]
\end{aligned}$$

for every $x \in I$ and $n \in \mathbb{N}$. The last inequality implies that

$$\begin{aligned}
\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g;x) - g(x)\right| \le \varepsilon + \left(\varepsilon + M + \frac{2Mc^2}{\delta^2}\right) & \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)\right| \\
& + \frac{4Mc}{\delta^2}\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_1(.);x) - e_1(x)\right| \\
& + \frac{2M}{\delta^2}\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_2(.);x) - e_2(x)\right|
\end{aligned}$$

where $c := \max|x|$. Hence, denoting by $K := \max\left\{\varepsilon + M + \frac{2Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$,

$$\begin{aligned}
\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g;x) - g(x)\right| \le \varepsilon + K\left\{\left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)\right|\right. & \\
+ \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_1(.);x) - e_1(x)\right| & \\
\left. + \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_2(.);x) - e_2(x)\right|\right\}. &
\end{aligned}$$

Now we multiply both sides of the above inequality by $\frac{1}{|\sigma(x)|}$ and in view of the fact that for any $\eta > 0$, we have

$$\begin{aligned}
\eta\left|\frac{\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(g;x) - g(x)}{\sigma(x)}\right| \le \frac{\eta\varepsilon}{|\sigma(x)|} + \eta K\left\{\left|\frac{\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_0(.);x) - e_0(x)}{\sigma_0(x)}\right|\right. & \\
+ \left|\frac{\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_1(.);x) - e_1(x)}{\sigma_1(x)}\right| & \\
\left. + \left|\frac{\sum_{j=1}^{\infty} a_{kj}^{(n)} T_j(e_2(.);x) - e_2(x)}{\sigma_2(x)}\right|\right\}. &
\end{aligned}$$

Applying the modular ρ on both sides of the above inequality, since ρ is monotone, we get

$$\begin{aligned}
\rho\left(\eta\left(\frac{A_{k,n}^{\mathbb{T}}(g) - g}{\sigma}\right)\right) \le \rho\left(\frac{\eta\varepsilon}{\sigma} + \eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_0) - e_0}{\sigma_0}\right)\right. & \\
+ \eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_1) - e_1}{\sigma_1}\right) & \\
\left. + \eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_2) - e_2}{\sigma_2}\right)\right). &
\end{aligned}$$

So, we may write that

$$\begin{aligned}
\rho\left(\eta\left(\frac{A_{k,n}^{\mathbb{T}}(g) - g}{\sigma}\right)\right) \le \rho\left(\frac{4\eta\varepsilon}{\sigma}\right) + \rho\left(4\eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_0) - e_0}{\sigma_0}\right)\right) & \\
+ \rho\left(4\eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_1) - e_1}{\sigma_1}\right)\right) & \\
+ \rho\left(4\eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_2) - e_2}{\sigma_2}\right)\right). &
\end{aligned}$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0 < \varepsilon \le 1$,

$$\rho\left(\eta\left(\frac{A_{k,n}^{\mathbb{T}}(g)-g}{\sigma}\right)\right)\leq N\varepsilon\rho\left(\frac{4\eta N}{\sigma}\right)+\rho\left(4\eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_0)-e_0}{\sigma_0}\right)\right)$$
$$+\rho\left(4\eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_1)-e_1}{\sigma_1}\right)\right)$$
$$+\rho\left(4\eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_2)-e_2}{\sigma_2}\right)\right).$$

For a given $r>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(\frac{4\eta N}{\sigma}\right)<r$. Now we define the following sets:

$$G_\eta := \left\{k:\rho\left(\eta\left(\frac{A_{k,n}^{\mathbb{T}}(g)-g}{\sigma}\right)\right)\geq r\right\},$$
$$G_{\eta,i} := \left\{k:\rho\left(4\eta K\left(\frac{A_{k,n}^{\mathbb{T}}(e_i)-e_i}{\sigma_i}\right)\right)\geq \frac{r-N\varepsilon\rho\left(\frac{4\eta N}{\sigma}\right)}{3}\right\}, i=0,1,2.$$

Then, it is easy to see that $G_\eta\subseteq\bigcup_{i=0}^{2}G_{\eta,i}$. So we can write, for all $l\in\mathbb{N}$, that

$$\sum_{k\in G_\eta}b_{lk}\leq\sum_{k\in G_{\eta,0}}b_{lk}+\sum_{k\in G_{\eta,1}}b_{lk}+\sum_{k\in G_{\eta,2}}b_{lk}. \tag{12}$$

Taking limit as $l\to\infty$ in (12) and using the hypothesis (9), we get $\lim_l\sum_{k\in G_\eta}b_{lk}=0$, which proves our claim (11). Observe that (11) also holds for every $g\in C^\infty(I)$. Now let $f\in L^\rho(I)$ satisfying $f-g\in X_{\mathbb{T}}$ for every $g\in C^\infty(I)$. Since $\mu(I)<\infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X(I)$ (see Bardaro and Mantellini 2006). Using these properties of the modular ρ, it is known from Bardaro et al. (2003); Mantellini (1998) that the space $C^\infty(I)$ is modularly dense in $L^\rho(I)$, i.e., there exists a sequence $\{g_k\}\subset C^\infty(I)$ such that $\lim_k\rho(3\lambda_0^*(g_k-f))=0$ for some $\lambda_0^*>0$. This means that, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*(g_k-f)\right)<\varepsilon\qquad\text{for every } k\geq k_0. \tag{13}$$

On the other hand, by the linearity and positivity of the operators T_j, we may write that

$$\lambda_0^*\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j(f;x)-f(x)\right|\leq\lambda_0^*\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j(f-g_{k_0};x)\right|$$
$$+\lambda_0^*\left|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j(g_{k_0};x)-g_{k_0}(x)\right|$$
$$+\lambda_0^*|g_{k_0}(x)-f(x)|,$$

holds for every $x\in I$ and $n\in\mathbb{N}$. Multiplying the last inequality by $\frac{1}{|\sigma(x)|}$, applying the modular ρ and, moreover, considering the monotonicity of ρ, we have

$$\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)\leq\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f-g_{k_0})}{\sigma}\right)\right)$$
$$+\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(g_{k_0})-g_{k_0}}{\sigma}\right)\right)$$
$$+\rho\left(3\lambda_0^*\left(\frac{g_{k_0}-f}{\sigma}\right)\right).$$

Hence, observing that $|\sigma|\geq a>0$ ($a=\max\{a_i:i=0,1,2\}$) and taking into consideration that ρ is monotone, we may write that

$$\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)\leq\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f-g_{k_0})}{\sigma}\right)\right)$$
$$+\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(g_{k_0})-g_{k_0}}{\sigma}\right)\right)$$
$$+\rho\left(\frac{3\lambda_0^*}{a}(g_{k_0}-f)\right). \tag{14}$$

Then, it follows from (13) and (14) that

$$\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)\leq\varepsilon+\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f-g_{k_0})}{\sigma}\right)\right)$$
$$+\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(g_{k_0})-g_{k_0}}{\sigma}\right)\right). \tag{15}$$

So taking the $\mathcal{B}$-statistical limit superior as $k\to\infty$ on both sides of (15) and also using the facts that $g_{k_0}\in C^\infty(I)$ and $f-g_{k_0}\in X_{\mathbb{T}}$, we obtain from (7) that

$$st_{\mathcal{B}}-\limsup_k\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)\leq\varepsilon+P\rho\left(3\lambda_0^*(f-g_{k_0})\right)+st_{\mathcal{B}}$$
$$-\limsup_k\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(g_{k_0})-g_{k_0}}{\sigma}\right)\right),$$

which gives

$$st_{\mathcal{B}}-\limsup_k\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)$$
$$\leq\varepsilon(P+1)+st_{\mathcal{B}}-\limsup_k\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(g_{k_0})-g_{k_0}}{\sigma}\right)\right). \tag{16}$$

By (11), since $st_{\mathcal{B}}-\lim_k\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(g_{k_0})-g_{k_0}}{\sigma}\right)\right)=0$, uniformly in n, we get

$$st_{\mathcal{B}}-\limsup_{k}\rho\left(3\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(g_{k_0})-g_{k_0}}{\sigma}\right)\right)=0,\ \text{uniformly in } n. \tag{17}$$

Combining (16) with (17), we conclude that

$$st_{\mathcal{B}}-\limsup_{k}\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)\le\varepsilon(P+1).$$

Since $\varepsilon>0$ is arbitrary, we find $st_{\mathcal{B}}-\limsup_k\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)=0$ uniformly in n. Furthermore, since $\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)$ is non-negative for all $k,n\in\mathbb{N}$, we can easily show that

$$st_{\mathcal{B}}-\lim_{k}\rho\left(\lambda_0^*\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)=0,\ \text{uniformly in } n$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition, then one can get immediately the following result from Theorem 1.

Theorem 2 *Let $\mathcal{A}=\{A^{(n)}\}$ be a sequence of infinite non-negative real matrices, $\mathcal{B}=(b_{lk})$be a non-negative regular summability matrix and $\mathbb{T}:=\{T_j\}$, ρ and σ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

1. $st_{\mathcal{B}}-\lim_k\rho\left(\lambda\left(\frac{A_{k,n}^{\mathbb{T}}(e_i)-e_i}{\sigma_i}\right)\right)=0$ *uniformly in n, for every $\lambda>0$ and $i=0,1,2,$*
2. $st_{\mathcal{B}}-\lim_k\rho\left(\lambda\left(\frac{A_{k,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)=0$ *uniformly in n, for every $\lambda>0$ provided that f is any function belonging to $L^{\rho}(I)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}(I)$.*

If one replaces the scale function by a non-zero constant, then the condition (7) reduces to the condition (4) by Orhan and Demirci (2014). In this case, the results which were obtained by Orhan and Demirci in their study Orhan and Demirci (2014) immediately follow from our Theorems. If one replaces the matrices $\mathcal{B}$ and $A^{(n)}$ $(n\ge1)$ by the identity matrix and take the scale function as a non-zero constant, then the condition (7) reduces to

$$\limsup_{k}\rho\left(\lambda\left(T_jh\right)\right)\le P\rho(\lambda h) \tag{18}$$

for every $h\in X_{\mathbb{T}}$, $\lambda>0$ and for an absolute positive constant P. In this case, the next results which were obtained by Bardaro and Mantellini (2007) immediately follow from our Theorems.

Corollary 1 (Bardaro and Mantellini (2007)) *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X(I)$. Let $\mathbb{T}:=\{T_j\}$ be a sequence of positive linear operators from D into $X(I)$ satisfying (18). If $\{T_je_i\}$is strongly convergent to $e_i$for each $i=0,1,2,$ then $\{T_jf\}$ is modularly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}(I)$.*

Corollary 2 (Bardaro and Mantellini 2007) *Let $\mathbb{T}:=\{T_j\}$and ρ be the same as in Corollary 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

1. *$\{T_je_i\}$ is strongly convergent to e_i for each $i=0,1,2,$*
2. *$\{T_jf\}$ is strongly convergent to f provided that f is any function belonging to $L^{\rho}(I)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}(I)$.*

4 Application

In this section, we give an example of positive linear operators satisfying the conditions of Theorem 1.

Example 1 Take $I=[0,1]$ and let $\varphi:[0,\infty)\to[0,\infty)$ be a continuous function such that φ is a convex function, $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$ and $\lim_{u\to\infty}\varphi(u)=\infty$. Hence, consider the functional ρ^{φ} on $X(I)$ defined by

$$\rho^{\varphi}(f):=\int_0^1\varphi(|f(x)|)dx\text{for} f\in X(I). \tag{19}$$

In this case, ρ^{φ} is a convex modular on $X(I)$, which satisfies all assumptions listed in Section 1 (see Bardaro and Mantellini 2007). Consider the Orlicz space generated by φ as follows:

$$L^{\varphi}(I):=\{f\in X(I):\rho^{\varphi}(\lambda f)<+\infty\quad\text{for some }\lambda>0\}.$$

Then consider the following classical Bernstein–Kantorovich operator $\mathbf{U}:=\{U_j\}$ on the space $L^{\varphi}(I)$ (see Bardaro and Mantellini 2007) which is defined by:

$$U_j(f;x):=\sum_{k=0}^{j}\binom{j}{k}x^k(1-x)^{j-k}(j+1)\int_{k/(j+1)}^{(k+1)/(j+1)}f(t)dt\quad\text{for } x\in I.$$

Observe that the operators U_j map the Orlicz space $L^{\varphi}(I)$ into itself. Moreover, the property (18) is satisfied with the choice of $X_{\mathbf{U}}:=L^{\varphi}(I)$. Then, by Corollary 1, we know that, for any function $f\in L^{\varphi}(I)$ such that $f-g\in X_{\mathbf{U}}$ for every $g\in C^{\infty}(I)$, $\{U_jf\}$ is modularly convergent to f.

If $\varphi(x)=x^p$ for $1\le p<\infty$, $x\ge0$, then $L^{\varphi}(I)=L_p(I)$. Moreover, we have $\rho^{\varphi}(f)=\|f\|_{L_p(I)}^p$. Now take $\mathcal{B}=C_1=(c_{lk})$, the Cesáro matrix of order one. In this case, we know that C_1-statistical convergence coincides with statistical convergence denoted by $st-\lim$. Also for each $n\in\mathbb{N}$, define $g_n:[0,1]\to\mathbb{R}$ by

$$g_n(x)=\begin{cases}3, & n=k^2\\ \dfrac{n^4x^2}{1+n^2x}, & 0<x<\dfrac{1}{n};n\neq k^2\\ 0, & x=0 \quad or \quad \dfrac{1}{n}\leq x\leq 1;n\neq k^2.\end{cases} \tag{20}$$

Now observe that $\{g_n\}$ converges to $g=0$ statistical modular relative to the scale function $\sigma(x)=\begin{cases}1, & x=0\\ \frac{1}{x^2}, & 0<x\leq 1\end{cases}$ on $L_1([0,1])$. Indeed, for some $\lambda_0>0$, with the choice of $p=1$ we have $\rho^{\varphi}(g)=\|g\|_{L_1}$ and using the scale function σ,

$$\rho\left(\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right)=\left\|\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right\|_{L_1} =\begin{cases}\lambda_0, & n=k^2\\ \dfrac{\lambda_0}{n^6}\left(\dfrac{n^4}{4}+\dfrac{n^3}{3}-\dfrac{n^2}{2}+n-\ln(n+1)\right), & n\neq k^2\end{cases}.$$

Since $\lim_n\frac{\lambda_0}{n^6}\left(\frac{n^4}{4}+\frac{n^3}{3}-\frac{n^2}{2}+n-\ln(n+1)\right)=0$, we get

$$st-\lim_n\left\|\lambda_0\left(\frac{g_n-g}{\sigma}\right)\right\|_{L_1}=0. \tag{21}$$

However, since $st-\lim_n\rho(\lambda_0(g_n-g))=st-\lim_n\|\lambda_0(g_n-g)\|_{L_1}=\frac{1}{2}$, $\{g_n\}$ does not converge to $g=0$ statistical modular. Also assume that $\mathcal{A}:=A^{(n)}=\left(a_{kj}^{(n)}\right)$ is a sequence of infinite matrices defined by $a_{kj}^{(n)}=\frac{1}{k+1}$ if $n\leq j\leq n+k$, $(n=1,2,\ldots)$ and $a_{kj}^{(n)}=0$ otherwise. Then, using the operators U_j, we define the sequence of positive linear operators $\mathbb{V}:=\{V_j\}$ on $L_1([0,1])$ as follows:

$$V_j(f;x)=\left(1+g_j(x)\right)U_j(f;x)\quad for\quad f\in L_1(I),x\in[0,1]\quad \text{and}\quad n\in\mathbb{N}, \tag{22}$$

where $\{g_j\}$ is the same as in (20) and we choose $\sigma_i(x)=\sigma(x)$ $(i=0,1,2)$, where $\sigma(x)=\begin{cases}1, & x=0\\ \frac{1}{x^2}, & 0<x\leq 1\end{cases}$. Since, for positive constant C, $\|U_j(f;x)\|_{L_1}\leq C\|f\|_{L_1}$(Ditzian and Totik 1987), we can easily see that

$$st-\lim_k\sup\left\|\lambda\frac{A_{k,n}^{\mathbb{V}}(f)}{\sigma}\right\|_{L_1}\leq P\|\lambda f\|_{L_1},\quad \text{uniformly in } n.$$

where $A_{k,n}^{\mathbb{V}}(f)=\sum_{j=1}^{\infty}a_{kj}^{(n)}V_jf$ as in (6). We now claim that

$$st-\lim_k\left\|\lambda\frac{A_{k,n}^{\mathbb{V}}(e_i)-e_i}{\sigma}\right\|_{L_1}=0, \text{uniformly in } n, i=0,1,2 \tag{23}$$

where $\sigma_i(x)=\sigma(x)$ for $i=0,1,2$. Observe that $U_j(e_0;x)=e_0(x)$, $U_j(e_1;x)=\frac{jx}{j+1}+\frac{1}{2(j+1)}$ and $U_j(e_2;x)=\frac{j(j-1)x^2}{(j+1)^2}+\frac{2jx}{(j+1)^2}+\frac{1}{3(j+1)^2}$. So, we can see,

$$\left\|\lambda\frac{\sum_{j=1}^{\infty}a_{kj}^{(n)}(1+g_j)U_j(e_0)-e_0}{\sigma}\right\|_{L_1}=\left\|\lambda\frac{\sum_{j=n}^{n+k}\frac{1}{k+1}(1+g_j)-1}{\sigma}\right\|_{L_1} \leq\frac{1}{k+1}\sum_{j=n}^{n+k}\left\|\lambda\frac{g_j}{\sigma}\right\|_{L_1}.$$

As is known, if a sequence is convergent then the arithmetic mean of the sequence converges to the same value. Thus, by virtue of statistical convergence and from (21) it can be easily seen that

$$st-\lim_k\left(\sup_n\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\left\|\lambda\frac{g_j}{\sigma}\right\|_{L_1}\right)\right)=0, \tag{24}$$

so we get

$$st-\lim_k\left\|\lambda\frac{A_{k,n}^{\mathbb{V}}(e_0)-e_0}{\sigma}\right\|_{L_1}=0, \text{uniformly in } n,$$

which guarantees that (23) holds true for $i=0$. In addition, we have

$$\begin{aligned}&\left\|\lambda\frac{\sum_{j=1}^{\infty}a_{kj}^{(n)}V_j(e_1)-e_1}{\sigma}\right\|_{L_1}\\ &=\left\|\lambda\left(\sum_{j=n}^{n+k}\frac{1}{k+1}\frac{(1+g_j)}{\sigma}U_j(e_1)-\frac{e_1}{\sigma}\right)\right\|_{L_1}\\ &\leq\left\|\lambda x^3\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j}{j+1}-1\right)\right\|_{L_1}+\left\|\lambda x^2\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{2(j+1)}\right)\right\|_{L_1}\\ &\quad+\frac{1}{k+1}\sum_{j=n}^{n+k}\left(\left\|\lambda\frac{g_j(x)}{\sigma(x)}\frac{jx}{j+1}\right\|_{L_1}+\left\|\lambda\frac{g_j(x)}{\sigma(x)}\frac{1}{2(j+1)}\right\|_{L_1}\right)\\ &<\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j}{j+1}-1\right)\|\lambda x^3\|_{L_1}+\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{2(j+1)}\right)\|\lambda x^2\|_{L_1}\\ &\quad+2\frac{1}{k+1}\sum_{j=n}^{n+k}\left\|\lambda\frac{g_j}{\sigma}\right\|_{L_1}.\end{aligned}$$

Since $st-\lim_k\left(\sup_n\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j}{j+1}-1\right)\right)=0$, $st-\lim_k\left(\sup_n\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{2(j+1)}\right)=0$ and from (24) we have,

$$st-\lim_k\left(\sup_n\left\|\lambda\frac{\sum_{j=1}^{\infty}a_{kj}^{(n)}V_j(e_1)-e_1}{\sigma}\right\|_{L_1}\right)=0$$

which gives $st-\lim_k\left\|\lambda\frac{A_{k,n}^{\mathbb{V}}(e_1)-e_1}{\sigma}\right\|_{L_1}=0$, uniformly in n. So (23) holds true for $i=1$. Finally, since

$$\begin{aligned}
&\left\|\lambda\frac{\sum_{j=1}^{\infty}a_{kj}^{(n)}V_j(e_2)-e_2}{\sigma}\right\|_{L_1}\\
&=\left\|\lambda\left(\sum_{j=n}^{n+k}\frac{1}{k+1}\frac{(1+g_j(x))}{\sigma(x)}U_j(e_2;x)-\frac{e_2(x)}{\sigma(x)}\right)\right\|_{L_1}\\
&\le\left\|\lambda x^4\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j(j-1)}{(j+1)^2}-1\right)\right\|_{L_1}+\left\|\lambda x^3\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{2j}{(j+1)^2}\right)\right\|_{L_1}\\
&\quad+\left\|\lambda x^2\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{3(j+1)^2}\right)\right\|_{L_1}+\frac{1}{k+1}\sum_{j=n}^{n+k}\left(\left\|\lambda\left(\frac{g_j(x)}{\sigma(x)}\frac{j(j-1)x^2}{(j+1)^2}\right)\right\|_{L_1}\right.\\
&\quad\left.+\left\|\lambda\left(\frac{g_j(x)}{\sigma(x)}\frac{2jx}{(j+1)^2}\right)\right\|_{L_1}+\left\|\lambda\left(\frac{g_j(x)}{\sigma(x)}\frac{1}{3(j+1)^2}\right)\right\|_{L_1}\right)\\
&<\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j(j-1)}{(j+1)^2}-1\right)\|\lambda x^4\|_{L_1}+\left(\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{2j}{(j+1)^2}\right)\|\lambda x^3\|_{L_1}\\
&\quad+\left(\frac{1}{k+1}\sum_{j=1}^{n+k}\frac{1}{3(j+1)^2}\right)\|\lambda x^2\|_{L_1}+3\frac{1}{k+1}\sum_{j=n}^{n+k}\left\|\lambda\frac{g_j}{\sigma}\right\|_{L_1}
\end{aligned}$$

and also since $st-\lim_k\left(\sup_n\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{j(j-1)}{(j+1)^2}\right)=0$, $st-\lim_k\left(\sup_n\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{2j}{(j+1)^2}\right)=0$, $st-\lim_k\left(\sup_n\frac{1}{k+1}\sum_{j=n}^{n+k}\frac{1}{3(j+1)^2}\right)=0$ and from (24) we have,

$$st-\lim_k\left(\sup_n\left\|\lambda\frac{\sum_{j=1}^{\infty}a_{kj}^{(n)}V_j(e_2)-e_2}{\sigma}\right\|_{L_1}\right)=0$$

which gives $st-\lim_k\left(\left\|\lambda\frac{A_{k,n}^{\mathbb{V}}(e_2)-e_2}{\sigma}\right\|_{L_1}\right)=0$, uniformly in n. So, our claim (23) holds true for each $i=0,1,2$. $\{V_j\}$ satisfies all hypothesis of Theorem 1 and we immediately see that

$$st-\lim_k\left\|\lambda\frac{A_{k,n}^{\mathbb{V}}(f)-f}{\sigma}\right\|_{L_1}=0,\ \text{uniformly in}\ n,\ on$$
$$I=[0,1]\quad\text{for all}\quad f\in L_1(I).$$

It can be easily seen that $\{V_jf\}$ is not modularly convergent to f. So, Corollary 1 does not work for the sequence $\{V_j\}$. In addition, we can immediately say that our Theorem 1 is a non-trivial generalization of the Korovkin-type results by Orhan and Demirci (2014).

References

Atlıhan ÖG, Orhan C (2007) Matrix summability and positive linear operators. Positivity 11:387–389

Atlıhan ÖG, Orhan C (2008) Summation process of positive linear operators. Comp Math Appl 56:1188–1195

Altomare F (2010) Korovkin-type theorems and approximation by positive linear operators. Surv Approx Theory 5:92–164

Altomare F, Campiti M (1994) Korovkin type approximation theory and its application. Walter de Gruyter Publ, Berlin

Anastassiou GA, Duman O (2008) A Baskakov type generalization of statistical Korovkin theory. J Math Anal Appl 340:476–486

Bardaro C, Boccuto A, Dimitriou X, Mantellini I (2013) Abstract Korovkintype theorems in modular spaces and applications. Central Euro J Math 11(10):1774–1784

Bardaro C, Boccuto A, Dimitriou X, Mantellini I (2013) Modular filter convergence theorems for abstract sampling-type operators. Appl Anal 92(11):2404–2423

Bardaro C, Mantellini I (2006) Approximation properties in abstract modular spaces for a class of general sampling-type operators. Appl Anal 85:383–413

Bardaro C, Musielak J, Vinti G (2003) Nonlinear integral operators and applications. de Gruyter Series Nonlinear Anal Appl 9

Bardaro C, Mantellini I (2007) Korovkin's theorem in modular spaces. Comment Math 47:239–253

Bell HT (1973) Order summability and almost convergence. Proc Am Math Soc 38:548–553

Boccuto A, Candeloro D, Sambucini AR (2014) Vitali-type theorems for filter convergence related to vector lattice-valued modulars and applications to stochastic processes. J Math Anal Appl 419:818–838

Bojanic R, Cheng F (1983) Estimates for the rate of approximation of functions of bounded variation by Hermite-Fejer polynomials. Proc Conf Can Math Soc 3:5–17

Bojanic R, Khan MK (1992) Summability of hermite-fejer interpolation for functions of bounded variation. J Nat Sci Math 32:5–10

Candeloro D, Sambucini AR (2015) Filter convergence and decompositions for vector lattice-valued measures. Mediterr J Math 12(3):621–637

Connor JS, Kline J (1996) On statistical limit points and the consistency of statistical convergence. J Math Anal Appl 197:392–399

Demirci K (2006) A-statistical core of a sequence. Demonst Math 33:343–353

Ditzian Z, Totik V (1987) Moduli of smoothness, Springer series in computational mathematics. Springer-Verlag, New York vol. 9

Duman O, Khan MK, Orhan C (2003) A-statistical convergence of approximation operators. Math Inequal Appl 6(4):689–699
Fast H (1951) Sur la convergence statistique. Colloq Math 2:241–244
Freedman AR, Sember JJ (1981) Densities and summability. Pacific J Math 95:293–305
Fridy JA (1985) On statistical convergence. Analysis 5:301–313
Fridy JA, Orhan C (1997) Statistical limit superior and limit inferior. Proc Am Math Soc 125:3625–3631
Gadjiev AD, Orhan C (2002) Some approximation theorems via statistical convergence. Rocky Mountain J Math 32:129–138
Hardy GH (1949) Divergent series. Oxford University Press, London
Kolk E (1993) Matrix summability of statistically convergent sequences. Analysis 13:77–83
Karakus S, Demirci K (2010) Matrix summability and Korovkin type approximation theorem on modular spaces. Acta Math Univ Comen LXXIX (2):281-292
Karakus S, Demirci K, Duman O (2008) Equi-statistical convergence of positive linear operators. J Math Anal Appl 339:1065–1072
Korovkin PP (1960) Linear operators and approximation theory. Hindustan Publ. Corp, Delhi
Kozlowski WM (1988) Modular function spaces. Pure Appl Math 122
Lorentz GG (1948) A contribution to the theory of divergent sequences. Acta Math 80:167–190
Mantellini I (1998) Generalized sampling operators in modular spaces. Commen Math 38:77–92
Musielak J (1983) Orlicz spaces and modular spaces, vol 1034., Lecture Notes in Mathematics Springer-Verlag, Berlin
Musielak J (1993) Nonlinear approximation in some modular function spaces I. Math Japon 38:83–90
Nishishiraho T (1981) Quantitative theorems on linear approximation processes of convolution operators in Banach spaces. Tôhoku Math J 33:109–126
Nishishiraho T (1983) Convergence of positive linear approximation processes. Tôhoku Math J 35:441–458
Orhan C, Sakaoglu İ (2014) Rate of convergence in Lp approximation. Periodica Math Hungarica 68(2):176–184
Orhan S, Demirci K (2014) Statistical $\mathcal{A}$-Summation Process and Korovkin Type Approximation Theorem on Modular Spaces. Positivity 18:669–686
Özgüç Sakaoğlu (2013) İ., Orhan, C., Strong summation process in L_p spaces, Nonlinear. Analysis 86:89–94
Steiglitz M (1973) Eine verallgemeinerung des begriffs der fastkonvergenz. Math Japon 18:53–70
Steinhaus H (1951) Sur la convergence ordinaire et la convergence asymptotique. Colloq Math 2:73–74
Yilmaz B, Demirci K, Orhan S (2016) Relative modular convergence of positive linear operators. Positivity 20:565–577