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Approximation via Power Series Method in Two-Dimensional Weighted Spaces

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Abstract

In this work, we obtain a Korovkin-type approximation theorem for double sequences of real-valued functions by using the power series method in two-dimensional weighted spaces. We also study the rate of convergence by using the weighted modulus of continuity, and in the last section, we present an application that satisfies our new Korovkin-type approximation theorem but does not satisfy classical one.

Keywords Korovkin theorem · Weighted space · Double sequence · Power series method

Mathematics Subject Classification 40B05 · 40C15 · 41A36

1 Introduction

The Korovkin theory has a big impact on the development of approximation theory [10]. The first study that considers summability theory in Korovkin-type approximation theory is Gadjiev and Orhan's paper [9]. Within this scope, many mathematicians investigate and improve this theory via different spaces and convergence methods [1,4,7,18]. The concept of power series method is more effective than ordinary convergence, which includes Abel and Borel methods which are many well-known summability methods. Both methods' definitions are based on power series, and they are not matrix methods. Power series methods are considered in the Korovkin-type approximation theory first time with the Abel summability method in [15],

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and many authors show how this concept can be applied to approximation theory [11,12,14,16,17]. The subject of this paper is to give Korovkin theorem for double sequences of real-valued functions by using the power series method in two-dimensional weighted spaces. We study the rate of convergence by using the weighted modulus of continuity and present an application that satisfies our new Korovkin-type approximation theorem but does not satisfy classical one. Let (p_{mn}) be a double sequence of nonnegative numbers with $p_{00} > 0$ and such that the following power series

$$p(t,s) := \sum_{m,n=0}^{\infty} p_{mn} t^m s^n$$

has radius of convergence R with $R \in (0, \infty]$ and $t, s \in (0, R)$. If, for all $t, s \in (0, R)$,

$$\lim_{t,s \to R^-} \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n x_{mn} = A,$$

then we say that the double sequence $x = (x_{mn})$ is convergent to A in the sense of power series method [5]. The power series method for double sequences is regular if and only if

$$\lim_{t,s \to R^-} \frac{\sum_{m=0}^{\infty} p_{m\nu} t^m}{p(t,s)} = 0 \text{ and } \lim_{t,s \to R^-} \frac{\sum_{n=0}^{\infty} p_{\mu n} s^n}{p(t,s)} = 0, \text{ for any } \mu, \upsilon, \tag{1}$$

hold [5]. Throughout the paper, we suppose that power series method is regular.

Remark 1 Note that in the case of $R = 1$, if $p_{mn} = 1$ and $p_{mn} = \frac{1}{(m+1)(n+1)}$, the power series methods coincide with Abel summability method and logarithmic summability method, respectively. In the case of $R = \infty$ and $p_{mn} = \frac{1}{m!n!}$, the power series method coincides with Borel summability method.

The convergence of a sequence of positive linear operators defined on weighted space was first studied by Gadjiev [8], and recently, Cao and Liu [6] study Korovkin-type theorems for two-variable functions by means of single sequence on weighted spaces. More recently, Akdag [2] studied this theorem for double sequences.

We now remind some notations and properties of weighted spaces in two variables.

A function ρ is called a weight function if it is continuous on $\mathbb{R}^2$, the real two-dimensional Euclidean space, and for all $(x, y) \in \mathbb{R}^2$,

$$\rho(x,y) \geq 1 \text{ and } \lim_{\sqrt{x^2+y^2} \to \infty} \rho(x,y) = \infty. \tag{2}$$

Let B_ρ denote the weighted space which is the space of real-valued functions f defined on $\mathbb{R}^2$ and satisfying $|f(x,y)| \leq M_f \rho(x,y)$ (for all $(x,y) \in \mathbb{R}^2$), where M_f is a constant depending on the function f. The weighted subspace C_ρ of B_ρ is given by

$$C_\rho := \left\{ f \in B_\rho : f \text{ is continuous over } \mathbb{R}^2 \right\}.$$

The spaces B_ρ and C_ρ are Banach spaces with the norm

$$\|f\|_\rho := \sup_{(x,y)\in\mathbb{R}^2} \frac{|f(x,y)|}{\rho(x,y)}.$$

Let ρ_1 and ρ_2 be two weight functions satisfying (2). Assume also that the condition

$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{\rho_1(x,y)}{\rho_2(x,y)} = 0 \tag{3}$$

holds. If L is a positive linear operator from C_{ρ_1} into B_{ρ_2}, then the operator norm is given by

$$\|L\|_{C_{\rho_1}\to B_{\rho_2}} := \sup_{\|f\|_{\rho_1}=1} \|Lf\|_{\rho_2} = \|L\rho_1\|_{\rho_2}.$$

Now, let us remind of the concept of Pringsheim convergence for double sequences and Korovkin-type approximation theorem via this convergence method.

Definition 1 [13] A double sequence $x = (x_{mn})$ is said to be convergent in Pringsheim's sense or P-convergent if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, the set of all natural numbers such that $|x_{mn} - l| < \varepsilon$ whenever $m, n > N$ and l is called the Pringsheim limit of x (denoted by $P - \lim_{m,n} x_{mn} = l$).

A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \le M$ for all $(m,n) \in \mathbb{N}^2$. We note that contrary to the case for single sequences, a convergent double sequence need not to be bounded.

Throughout this paper, we use the test functions F_i $(i = 0, 1, 2, 3)$ defined by

$$F_0(x,y) = \frac{\rho_1(x,y)}{1+x^2+y^2},\ F_1(x,y) = \frac{x\rho_1(x,y)}{1+x^2+y^2},$$
$$F_2(x,y) = \frac{y\rho_1(x,y)}{1+x^2+y^2} \text{ and } F_3(x,y) = \frac{\left(x^2+y^2\right)\rho_1(x,y)}{1+x^2+y^2}.$$

Theorem 1 [2]. *Assume that the functions ρ_1 and ρ_2 are weight functions satisfying (3), and let (L_{mn}) be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $f \in C_{\rho_1}$,*

$$P - \lim_{m,n} \|L_{mn}f - f\|_{\rho_2} = 0$$

if

$$P - \lim_{m,n} \|L_{mn}F_i - F_i\|_{\rho_1} = 0, \quad i = 0, 1, 2, 3.$$

2 Approximation via Power Series Method

In this section, we study a Korovkin theorem via power series method for a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} and we obtain the rate of convergence by using the weighted modulus of continuity. We give the proofs in consideration of revised proofs in [3].

Now, we begin the following.

Let (L_{mn}) be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that for every $f \in C_{\rho_1}$

$$\sup_{0<t,s<R} \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n \left\| L_{mn} f \right\|_{\rho_1} < \infty \tag{4}$$

holds. Throughout the paper, the operators fulfill conditions (1) and (4). Also, using (3), (4) and after some simple calculations, we get

$$\sup_{0<t,s<R} \left\| T_{ts} \right\|_{C_{\rho_1} \to B_{\rho_2}} = \sup_{0<t,s<R} \left\| T_{ts} \rho_1 \right\|_{\rho_2} < \infty$$

where

$$T_{ts}\left(f(u,v);x,y\right) = \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n L_{mn}\left(f(u,v);x,y\right),\ t,s \in (0,R).$$

Hence, T_{ts} is also a positive linear operator from C_{ρ_1} into B_{ρ_2}.

We now give with the following lemma before our main theorem.

Lemma 1 *Let (L_{mn}) be a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} where ρ_1 and ρ_2 are weight functions satisfying the condition (3). If*

$$\lim_{t,s\to R^-} \left\| T_{ts} F_i - F_i \right\|_{\rho_1} = 0, \quad i = 0,1,2,3, \tag{5}$$

then, for any $z > 0$ and for all $f \in C_{\rho_1}$, we have

$$\lim_{t,s\to R^-} \left\| T_{ts} f - f \right\|_{\rho_2,[-z,z]} := \lim_{t,s\to R^-} \left(\sup_{\sqrt{x^2+y^2}\le z} \frac{\left| T_{ts}(f;x,y) - f(x,y) \right|}{\rho_2(x,y)} \right) = 0.$$

Proof Let $f \in C_{\rho_1}$ and $\sqrt{x^2+y^2} \le z$. Since f is continuous on $\mathbb{R}^2$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(u,v) - f(x,y)| < \varepsilon$ whenever $|u-x| < \delta$ and $|v-y| < \delta$. When $|u-x| \ge \delta$ or $|v-y| \ge \delta$, we have

$$\begin{aligned} |f(u,v) - f(x,y)| &\le 2M_f \rho_1(x,y) F_0(u,v) \left(1 + u^2 + v^2\right) \\ &\le 4M_f \rho_1(x,y) F_0(u,v) \left((u-x)^2 + (v-y)^2\right) \\ &\quad \times \left(\frac{1+x^2+y^2}{(u-x)^2+(v-y)^2} + 1 \right) \\ &\le K_{\rho_1}(x,y) \left((u-x)^2 + (v-y)^2\right) F_0(u,v) \end{aligned}$$

where $K_{\rho_1}(x,y) := 4M_f\rho_1(x,y)\left(1+\frac{1+x^2+y^2}{\delta^2}\right)$. For all $(u,v) \in \mathbb{R}^2$ and $\sqrt{x^2+y^2} \le z$, we see that

$$|f(u,v)-f(x,y)| < \varepsilon + K_{\rho_1}(x,y)\left((u-x)^2+(v-y)^2\right)F_0(u,v). \tag{6}$$

Then, we can write

$$\begin{aligned}
|T_{ts}(f;x,y)-f(x,y)| &= \left|\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty} p_{mn}t^m s^n L_{mn}(f;x,y)-f(x,y)\right| \\
&\le \frac{1}{p(t,s)}\sum_{m,n=0}^{\infty} p_{mn}t^m s^n L_{mn}(|f(u,v)-f(x,y)|;x,y) \\
&\quad +|f(x,y)|\left|\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty} p_{mn}t^m s^n L_{mn}(1;x,y)-1\right| \\
&\le \varepsilon + K_{\rho_1}(x,y)\Big|T_{ts}\Big(\Big((u-x)^2 \\
&\quad +(v-y)^2\Big)F_0(u,v);x,y\Big)\Big| \\
&\quad +(\varepsilon+|f(x,y)|)\,|T_{ts}(1;x)-1|.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\sup_{\sqrt{x^2+y^2}\le z} |T_{ts}(f;x,y)-f(x,y)| \\
&\le \varepsilon + H_1 \sup_{\sqrt{x^2+y^2}\le z} T_{ts}\left(\left((u-x)^2+(v-y)^2\right)F_0(u,v);x,y\right) \\
&\quad +H_2 \sup_{\sqrt{x^2+y^2}\le z} |T_{ts}(1;x,y)-1|
\end{aligned} \tag{7}$$

where $H_1 := H_1(s) := \sup_{\sqrt{x^2+y^2}\le z} K_{\rho_1}(x,y)$ and $H_2 := H_2(s) := \varepsilon + \sup_{\sqrt{x^2+y^2}\le z} |f(x,y)|$. For any $z \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\sup_{\sqrt{x^2+y^2}\le z} T_{ts}\left(\left((u-x)^2+(v-y)^2\right)F_0(u,v);x,y\right) \\
&= \sup_{\sqrt{x^2+y^2}\le z} \Big\{T_{ts}\left(\left(u^2+v^2\right)F_0(u,v);x,y\right) - 2xL_i(uF_0(u,v);x,y) \\
&\quad -2yL_i(vF_0(u,v);x,y)+\left(x^2+y^2\right)T_{ts}(F_0(u,v);x,y)\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\sqrt{x^2+y^2}\leq z} \{\left|T_{ts}\left(F_3\left(u,v\right);x,y\right)-F_3\left(x,y\right)\right| \\
&\quad +2\left|x\right|\left|T_{ts}\left(F_1\left(u,v\right);x,y\right)-F_1\left(x,y\right)\right| \\
&\quad +2\left|y\right|\left|T_{ts}\left(F_2\left(u,v\right);x,y\right)-F_2\left(x,y\right)\right| \\
&\quad +\left(x^2+y^2\right)\left|T_{ts}\left(F_0\left(u,v\right);x,y\right)-F_0\left(x,y\right)\right|\Big\}, \\
&\leq H_3\left\{\left\|T_{ts}F_0-F_0\right\|_{\rho_1}+\left\|T_{ts}F_1-F_1\right\|_{\rho_1}\right. \\
&\quad \left.+\left\|T_{ts}F_2-F_2\right\|_{\rho_1}+\left\|T_{ts}F_3-F_3\right\|_{\rho_1}\right\}
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
H_3 := H_3\left(s\right) := \max\Bigg\{ &\sup_{\sqrt{x^2+y^2}\leq z}\rho_1\left(x,y\right), 2\sup_{\sqrt{x^2+y^2}\leq z}\left|x\right|\rho_1\left(x,y\right), \\
&2\sup_{\sqrt{x^2+y^2}\leq z}\left|y\right|\rho_1\left(x,y\right), \sup_{\sqrt{x^2+y^2}\leq z}\left(x^2+y^2\right)\rho_1\left(x,y\right)\Bigg\}.
\end{aligned}$$

Since $F_0 \in C_{\rho_1}$ and

$$\begin{aligned}
F_0\left(x,y\right)\left|T_{ts}\left(1;x,y\right)-1\right| \leq &\left|T_{ts}\left(F_0\left(u,v\right);x,y\right)-F_0\left(x,y\right)\right| \\
&+T_{ts}\left(\left|F_0\left(u,v\right)-F_0\left(x,y\right)\right|;x,y\right),
\end{aligned}$$

it follows from (6) that

$$\begin{aligned}
\left|T_{ts}\left(1;x,y\right)-1\right| \leq &\frac{1}{F_0\left(x,y\right)}\{\left|T_{ts}\left(F_0\left(u,v\right);x,y\right)-F_0\left(x,y\right)\right|+\varepsilon T_{ts}\left(1;x,y\right) \\
&+K_{\rho_1}\left(x,y\right)T_{ts}\left(\left(\left(u-x\right)^2+\left(v-y\right)^2\right)F_0\left(u,v\right);x,y\right)\Big\}.
\end{aligned}$$

Hence, we have for any $z \in \mathbb{R}$ that

$$\begin{aligned}
&\sup_{\sqrt{x^2+y^2}\leq z}\left|T_{ts}\left(1;x,y\right)-1\right| \\
&\quad \leq H_4\left\{\left\|T_{ts}F_0-F_0\right\|_{\rho_1}+\varepsilon\sup_{\sqrt{x^2+y^2}\leq z}\frac{T_{ts}\left(1;x,y\right)}{\rho_1\left(x,y\right)}\right\} \\
&\qquad +H_5\sup_{\sqrt{x^2+y^2}\leq z}T_{ts}\left(\left(\left(u-x\right)^2+\left(v-y\right)^2\right)F_0\left(u,v\right);x,y\right)
\end{aligned} \tag{9}$$

where $H_4 := H_4(s) := \sup\limits_{\sqrt{x^2+y^2}\leq z} \frac{\rho_1(x,y)}{F_0(x,y)}$ and $H_5 := H_5(s) := \sup\limits_{\sqrt{x^2+y^2}\leq z} \frac{K_{\rho_1}(x,y)}{F_0(x,y)}$. Considering (7)–(9), we have

$$\sup_{\sqrt{x^2+y^2}\leq z} |T_{ts}(f;x,y) - f(x,y)| \leq \varepsilon + H\left\{\varepsilon \|T_{ts}1\|_{\rho_1} + \|T_{ts}F_0 - F_0\|_{\rho_1}\right.$$
$$\left. + \|T_{ts}F_1 - F_1\|_{\rho_1} + \|T_{ts}F_2 - F_2\|_{\rho_1} + \|T_{ts}F_3 - F_3\|_{\rho_1}\right\},$$

where $H := H_3(H_1 + H_2H_5) + H_2H_4$. By using (4) and taking

$$M = \max\left\{1 + H\|T_{ts}1\|_{\rho_1}, H\right\},$$

we get

$$\|T_{ts}f - f\|_{\rho_2,[-z,z]} \leq M\left\{\varepsilon + \sum_{i=0}^{3}\|T_{ts}F_i - F_i\|_{\rho_1}\right\},$$

for all $t, s \in (0, R)$. Taking limit as $t, s \to R^-$ and using (5), we get the desired result. □

Now, we can give our main Korovkin-type approximation theorem.

Theorem 2 *Let ρ_1 and ρ_2 be as in Lemma 1. Suppose that (L_{mn}) be a double sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then for all $f \in C_{\rho_1}$,*

$$\lim_{t,s\to R^-} \|T_{ts}f - f\|_{\rho_2} = 0 \tag{10}$$

provided that

$$\lim_{t,s\to R^-} \|T_{ts}F_i - F_i\|_{\rho_1} = 0, \quad i = 0, 1, 2, 3. \tag{11}$$

Proof Observe that the hypothesis (11) implies $T_{ts}(F_i; x, y) - F_i(x, y) \in B_{\rho_1}$ and hence $T_{ts}(F_i; x, y) \in B_{\rho_1}$ for $i = 0, 1, 2, 3$. Since $\rho_1 = F_0 + F_3$, we also get $T_{ts}\rho_1 \in B_{\rho_1}$. Hence if $f \in C_{\rho_1}$, then we obtain $T_{ts}f \in B_{\rho_1}$. Hence, we get

$$\begin{aligned}\|T_{ts}\|_{C_{\rho_1}\to B_{\rho_1}} &= \|T_{ts}\rho_1\|_{\rho_1} \\ &= \sup_{(x,y)\in\mathbb{R}^2} \frac{|T_{ts}(\rho_1; x, y)|}{\rho_1(x, y)} \leq M_1 < \infty.\end{aligned}$$

Therefore, we may write for a given $f \in C_{\rho_1}$ that

$$\|T_{ts}f\|_{\rho_1} \leq \|T_{ts}\|_{C_{\rho_1}\to B_{\rho_1}} \|f\|_{\rho_1} \leq M_1 \|f\|_{\rho_1}. \tag{12}$$

Now for given $\varepsilon > 0$, pick an $z_0 > 0$ such that $\frac{\rho_1(x,y)}{\rho_2(x,y)} \leq \varepsilon$ for every $\sqrt{x^2 + y^2} \geq z_0$. This is possible by (3). Using the fact, we may write for $f \in C_{\rho_1}$ that

$$\|T_{ts}f - f\|_{\rho_2} = \sup_{(x,y)\in\mathbb{R}^2} \frac{|T_{ts}(f; x, y) - f(x, y)|}{\rho_2(x, y)}$$

$$
\begin{aligned}
&\le \sup_{\sqrt{x^2+y^2}\le z_0.} \frac{|T_{ts}(f;x,y)-f(x,y)|}{\rho_2(x,y)} \\
&\quad + \sup_{\sqrt{x^2+y^2}\ge z_0.} \frac{|T_{ts}(f;x,y)-f(x,y)|}{\rho_2(x,y)} \frac{\rho_1(x,y)}{\rho_1(x,y)} \\
&\le \|T_{ts}f-f\|_{\rho_2,[-z_0,z_0]} + \varepsilon \|T_{ts}f-f\|_{\rho_1} \\
&\le \|T_{ts}f-f\|_{\rho_2,[-z_0,z_0]} + \varepsilon \left(\|T_{ts}f\|_{\rho_1} + \|f\|_{\rho_1}\right),
\end{aligned}
$$

and by (12), we immediately get

$$
\|T_{ts}f-f\|_{\rho_2} \le \|T_{ts}f-f\|_{\rho_2,[-z_0,z_0]} + \varepsilon \|f\|_{\rho_1}(M_1+1).
$$

Hence, using Lemma 1, the proof is completed. □

Now, defining the weight function ρ_1 in Theorem 2 by $\rho_1(x,y)=1+x^2+y^2$ on $\mathbb{R}^2$, we obtain the rate of convergence by using the following weighted modulus of continuity:

$$
w_{\rho_1}(f,\delta) = \sup_{\sqrt{(u-x)^2+(v-y)^2}\le\delta} \frac{|f(u,v)-f(x,y)|}{\rho_1(x,y)},
$$

where δ is a positive constant and $f \in C_{\rho_1}$. It can be easily seen that, for any $c>0$ and all $f \in C_{\rho_1}$,

$$
w_{\rho_1}(f,c\delta) \le (1+[c])\, w_{\rho_1}(f,\delta)
$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

Following [2], and if we use the same operators L_{mn} in Theorem 2, we can write, for any $\delta>0$, that

$$
\begin{aligned}
|T_{ts}(f;x,y)-f(x,y)| &\le T_{ts}(|f(u,v) \\
&\quad -f(x,y)|;x,y) + |f(x,y)|\,|T_{ts}(F_0;x,y)-F_0(x,y)| \\
&\le T_{ts}\left(\rho_1(x,y)\, w_{\rho_1}\left(f, \delta\frac{\sqrt{(u-x)^2+(v-y)^2}}{\delta}\right);x,y\right) \\
&\quad + |f(x,y)|\,|T_{ts}(F_0;x,y)-F_0(x,y)| \\
&\le \rho_1(x,y)\, w_{\rho_1}(f,\delta)\, T_{ts}\left(1+\left(\frac{(u-x)^2+(v-y)^2}{\delta^2}\right);x,y\right) \\
&\quad + |f(x,y)|\,|T_{ts}(F_0;x,y)-F_0(x,y)|
\end{aligned}
$$

$$\leq \rho_1 (x, y)\, w_{\rho_1} (f, \delta) \left\{ T_{ts} (\rho_1; x, y) + \frac{1}{\delta^2} T_{ts} \left(\varphi_{(x,y)}; x, y \right) \right\}$$
$$+ |f (x, y)| \, |T_{ts} (F_0; x, y) - F_0 (x, y)|$$

where $\varphi_{(x,y)} (u, v) := (u - x)^2 + (v - y)^2$, and we obtain that

$$\|T_{ts} f - f\|_{\rho_2^2} \leq \|\rho_1\|_{\rho_2} w_{\rho_1} (f, \delta) \left\{ \|T_{ts}\rho_1 - \rho_1\|_{\rho_2} + \|\rho_1\|_{\rho_2} + \frac{1}{\delta^2} \left\| T_{ts}\varphi_{(x,y)} \right\|_{\rho_2} \right\} + \|f\|_{\rho_2} \|\rho_1\|_{\rho_2} \|T_{ts} F_0 - F_0\|_{\rho_1}$$

provided that $T_{ts}\varphi_{(x,y)} \in B_{\rho_2}$.

3 Concluding Remarks

We now present an example of a double sequence of positive linear operators that satisfies the conditions of Theorem 2 but does not satisfy the conditions of Theorem 1, and we analyze that the results were given in the previous sections by graphs.

Example 1 Let $\rho_1 (x, y) = 1 + x^2 + y^2$ and $\rho_2 (x, y) = 1 + x^4 + y^4$. In this case, the test functions F_i become $F_0 (x, y) = 1$, $F_1 (x, y) = x$, $F_2 (x, y) = y$ and $F_3 (x, y) = x^2 + y^2$. Then, consider the following bivariate Bernstein–Kantorovich operator (U_{mn}) which is defined by:

$$U_{mn}(f; x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n} p_{j,k}^{(m,n)} (x, y) (m + 1) (n + 1) \int_{j/(m+1)}^{(j+1)/(m+1)} \int_{k/(n+1)}^{(k+1)/(n+1)} f (u, v) \,\mathrm{d}v\mathrm{d}u \tag{13}$$

where $p_{j,k}^{(m,n)} (x, y)$ is defined by

$$p_{j,k}^{(m,n)} (x, y) = \binom{m}{j}\binom{n}{k} x^j y^k (1 - x)^{m-j} (1 - y)^{n-k}.$$

Using the operators (U_{mn}), we define the sequence of positive linear operators $\mathbb{L} = (\mathrm{L_{mn}})$ as follows:

$$L_{mn}(f; x, y) = (1 + s_{mn}) U_{mn}(f; x, y), \text{ for } f \in C_{\rho_1},$$

where $s_{mn} = 1$, m, n are squares and 0 otherwise. Also let $R = 1$, $p (t, s) = \frac{1}{(1-t)(1-s)}$ and for $m, n \geq 0$, $p_{mn} = 1$. As it is well known, in this case the power series method coincides with Abel method. Then, (s_{mn}) is convergent to 0 in the sense of power series method. Now, observe that

$$\begin{aligned}
L_{mn}(F_0;x,y) &= 1+s_{mn},\\
L_{mn}(F_1;x,y) &= (1+s_{mn})\left(\frac{mx}{m+1}+\frac{1}{2(m+1)}\right),\\
L_{mn}(F_2;x,y) &= (1+s_{mn})\left(\frac{ny}{n+1}+\frac{1}{2(n+1)}\right),\\
L_{mn}(F_3;x,y) &= (1+s_{mn})\left(\frac{m(m-1)x^2}{(m+1)^2}+\frac{2mx}{(m+1)^2}+\frac{1}{3(m+1)^2}\right.\\
&\qquad\left.+\frac{n(n-1)y^2}{(n+1)^2}+\frac{2ny}{(n+1)^2}+\frac{1}{3(n+1)^2}\right).
\end{aligned}$$

Also, we get

$$\begin{aligned}
L_{mn}(\rho_1;x,y) &= 1+s_{mn}+(1+s_{mn})\left(\frac{m(m-1)x^2}{(m+1)^2}+\frac{2mx}{(m+1)^2}+\frac{1}{3(m+1)^2}\right.\\
&\qquad\left.+\frac{n(n-1)y^2}{(n+1)^2}+\frac{2ny}{(n+1)^2}+\frac{1}{3(n+1)^2}\right)\\
&< 12\rho_2(x,y)
\end{aligned}$$

which yields $\sup_{0<t,s<R}\|T_{ts}\|_{C_{\rho_1}\to B_{\rho_2}}<\infty$ where

$$T_{ts}(f;x,y)=\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^m s^n L_{mn}(f;x,y).$$

Now observe that

$$\|T_{ts}F_0-F_0\|_{\rho_1}=\left\|\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^m s^n s_{mn}\right\|_{\rho_1}.$$

Since (s_{mn}) is convergent to 0 in the sense of power series method, we get

$$\lim_{t,s\to R^-}\|T_{ts}F_0-F_0\|_{\rho_1}=0.$$

Also, we have

$$\|T_{ts}F_1-F_1\|_{\rho_1}<\left\|\frac{1}{p(t,s)}\sum_{m,n=0}^{\infty}p_{mn}t^m s^n\left(s_{mn}+\frac{1}{m+1}\right)\right\|_{\rho_1},$$

using the method's regularity, and since (s_{mn}) is convergent to 0 in the sense of power series method, we have

$$\lim_{t,s\to R^-} \|T_{ts}F_1 - F_1\|_{\rho_1} = 0.$$

Similarly, we get

$$\lim_{t,s\to R^-} \|T_{ts}F_2 - F_2\|_{\rho_1} = 0.$$

Finally, since

$$\|T_{ts}F_3 - F_3\|_{\rho_1} < \left\| \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n \left(s_{mn} + \frac{4m+2}{(m+1)^2} + \frac{4n+2}{(n+1)^2} \right) \right\|_{\rho_1}$$

we have

$$\lim_{t,s\to R^-} \|T_{ts}F_3 - F_3\|_{\rho_1} = 0.$$

Therefore, we can say that our sequence (L_{mn}) satisfies all assumptions of Theorem 2 and we conclude that for all $f \in C_{\rho_1}$, $\lim_{t,s\to R^-} \|T_{ts}(f) - f\|_{\rho_2} = 0$. However, since (s_{mn}) is not convergent to zero in Pringsheim sense, (L_{mn}) does not satisfy Theorem 1.

Let us consider the function $f \in C_{\rho_1}$ given by $f(x,y) = \frac{1+x^2+y^2}{mn}$. Figures 1 and 2 show that the convergence in the sense of power series method holds. Also, Fig. 2 illustrates the operator does not satisfy Theorem 1. Finally, Fig. 3 shows and compares the error of approximation.

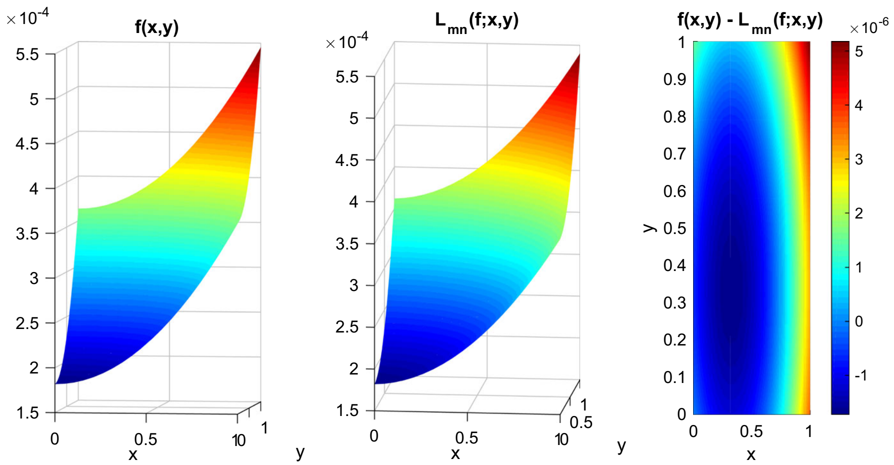

Fig. 1 The function and the operator for $m = 50$, $n = 110$

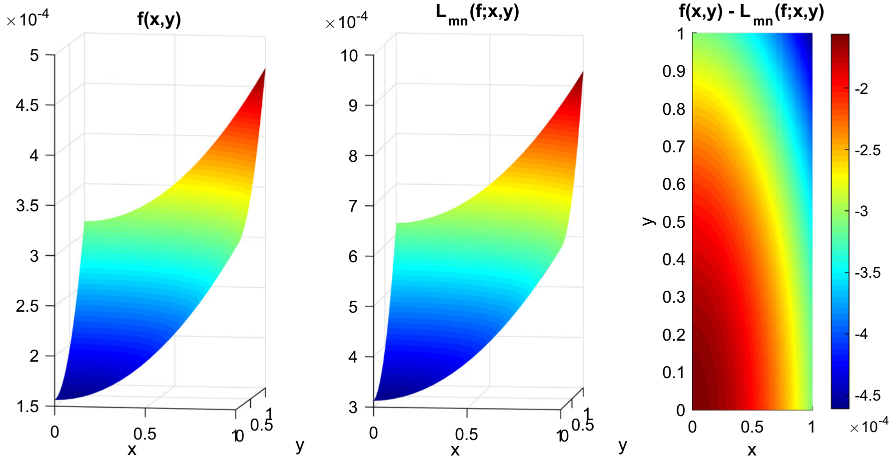

Fig. 2 The function and the operator for $m = 64$, $n = 100$

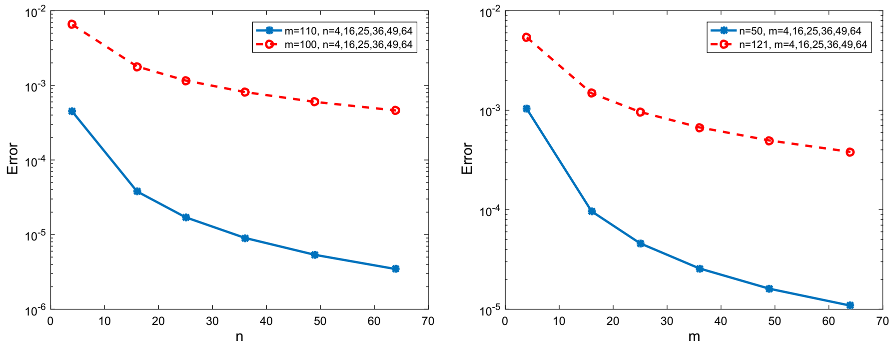

Fig. 3 Error of approximation

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