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K_a-convergence and Korovkin type approximation

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Abstract In the present paper, we study a Korovkin type approximation theorem in the setting of K_a-convergence that contains the classical result. We also study the rate of K_a-convergence and afterwards, we give some concluding remarks.

Keywords K_a-convergence · Positive linear operator · Korovkin theorem · Statistical convergence · Almost convergence

Mathematics Subject Classification 40A35 · 41A25 · 41A36

1 Introduction

The notion of statistical convergence was first introduced by Fast [3] and Steinhaus [12], independently and studied by many authors [4,5,14].

Let E be a subset of $\mathbb{N}$, the set of natural numbers, then *the natural density of* E, denoted by $\delta(E)$, is given by:

$$\delta(E) := \lim_n \frac{1}{n} |\{k \leq n \ : k \in E\}|$$

whenever the limit exists, where $|B|$ denotes the cardinality of the set B [11]. Then a sequence $x = (x_k)$ of numbers is *statistically convergent* to l provided that, for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}) = 0.$$

In this case we write $st - \lim_k x_k = l$ [3,12]. We note that every convergent sequence is statistically convergent, but not conversely. Also, as it is well known, statistical convergence and almost convergence [9,10] overlap. These convergence methods provide interesting results in summability theory and play a vital role not only in pure mathematics but also in other

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branches of science involving mathematics. Another interesting convergence method is K_a-convergence [8]. If the sequence $x = (x_k)$ is convergent and $\sum_{k=1}^{\infty} |a_k| < \infty$, then it is K_a-convergent, too. It is now natural to ask whether these three convergence methods contain each other. Our answer is no, i.e. these methods are overlap, neither contains the other. In the present paper, we study a Korovkin type approximation theorem via K_a-convergence and the rate of K_a-convergence and afterwards, we give some concluding remarks.

We now turn our attention to the K_a-convergence:

The idea of K_a-convergence was defined by Lazic and Jovovic in 1993 [8], which is obviously associated to the matrix

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 & \ldots \\ a_2 & a_1 & 0 & 0 & \\ a_3 & a_2 & a_1 & 0 & \\ \cdot & & & & \\ \cdot & & & & \end{pmatrix}.$$

Let $a = (a_k)$ and $x = (x_k)$ be number sequences, set $K_a(x) = y$, where $y = (y_k)$ and $y_k = \sum_{i=1}^{k} a_{k-i+1} x_i$ $(k = 1, 2, 3, \ldots)$, then we say that $y = (y_k)$ is the K_a-*transformation* of the $x = (x_k)$.

Definition 1.1 [8] The sequence $x = (x_k)$ of real numbers is said to be K_a-convergent to the number l if, its K_a-transformation $y = (y_k)$ converges to the number l, i.e. $\lim_k y_k = l$. This limit is denoted by $K_a - \lim_k x_k = l$.

Proposition 1.2 [8] *Let $a = (a_k)$ be a number sequence and the series $\sum a_k$ be absolutely convergent, i.e.*

$$\sum_{k=1}^{\infty} |a_k| < \infty. \tag{1.1}$$

(i) *If $x = (x_k)$ is convergent, $\lim_k x_k = l$ and the condition (1.1) is satisfied then,*

$$K_a - \lim_k x_k = l \sum_{k=1}^{\infty} a_k.$$

(ii) *A convergence method K_a is regular if and only if the condition (1.1) and*

$$\sum_{k=1}^{\infty} a_k = 1 \tag{1.2}$$

are valid (for more properties and details, see also [8]).

Now, we will give an example that K_a-convergence but, neither statistical convergence nor almost convergence.

Example 1.3 Let $a = (a_k) = (1, 1, -1, 0, 0, \ldots)$ and let $x = (x_i) = (1, 0, 1, -1, 2, -3, 5, -8, \ldots)$ $\left[x_i = x_{i-2} - x_{i-1} \text{ for } i \geq 3\right]$. Then,

$$(y_k) = \left(\sum_{i=1}^{k} a_{k-i+1} x_i\right) = (1, 1, 0, 0, \ldots).$$

This means that $K_a - \lim_k x_k = 0$, but it is shown just easily that $x = (x_i)$ is not statistical convergence and is not almost convergence, either.

2 A Korovkin-type approximation

In this section, we aim to study a Korovkin type approximation theorem via the concept of K_a-convergence. The fundamental theorem of Korovkin [7] exhibits a variety of test functions which ensure that the approximation property holds on the whole space if it holds for them. Over the years Korovkin type approximation theorem and its generalizations have been worked by several mathematicians. We refer to the papers [1,2,13].

Let I be a compact subset of the real numbers, $C(I)$ be the space of all continuous real valued functions on I and $\|f\|$ denote the usual supremum norm of f in $C(I)$. Let L be a linear operator from $C(I)$ into itself. Then, we say that L is positive linear operator on condition that $f \geq 0$ implies $L(f) \geq 0$. Also, we mean the value of $L(f)$ at a point $x \in I$ by $L(f(t); x)$ or $L(f; x)$. Throughout the paper, we also use the following test functions

$$e_0(x) = 1, \quad e_1(x) = x, \quad e_2(x) = x^2.$$

Now, we begin with the following well known Korovkin type theorems.

Theorem 2.1 [7] *Suppose that (L_i) is a sequence of positive linear operators acting from $C(I)$ into itself, satisfying the following conditions:*

$$\lim_i \|L_i(e_r) - e_r\| = 0, \quad (r = 0, 1, 2).$$

Then, for all $f \in C(I)$,

$$\lim_i \|L_i(f) - f\| = 0.$$

Theorem 2.2 [4] *Assume that (L_i) is a sequence of positive linear operators acting from $C(I)$ into itself, satisfying the following conditions:*

$$st - \lim_i \|L_i(e_r) - e_r\| = 0, \quad (r = 0, 1, 2).$$

Then, for all $f \in C(I)$,

$$st - \lim_i \|L_i(f) - f\| = 0.$$

Theorem 2.3 [6] *Suppose that (L_i) is a sequence of positive linear operators acting from $C(I)$ into itself, satisfying the following conditions:*

$$\lim_k \left\| \frac{1}{k} \sum_{i=n}^{n+k-1} L_i(e_r) - e_r \right\| = 0, \text{ uniformly in } n \ \ (r = 0, 1, 2).$$

Then, for all $f \in C(I)$,

$$\lim_k \left\| \frac{1}{k} \sum_{i=n}^{n+k-1} L_i(f) - f \right\| = 0, \text{ uniformly in } n.$$

Now we have the following Korovkin type approximation theorem for method K_a that is our main result.

Theorem 2.4 *Let $a = (a_k)$ be a number sequence and $\sum_{k=1}^{\infty} |a_k| < \infty$. Suppose that (L_i) is a sequence of positive linear operators acting from $C(I)$ into itself, satisfying the following conditions:*

$$\lim_k \left\| \sum_{i=1}^{k} |a_{k-i+1}| L_i(e_r) - e_r \right\| = 0, \quad (r = 0, 1, 2). \tag{2.1}$$

Then, for all $f \in C(I)$, *we have*

$$K_a - \lim_k \|L_k(f) - f\| = 0, \ \textit{i.e.}$$

$$\lim_k \left\| \sum_{i=1}^{k} a_{k-i+1} L_i(f) - f \right\| = 0.$$

Proof Let $f \in C(I)$ and $x \in I$ be fixed. By the continuity of f on I, we can write

$$|f(x)| \leq K. \tag{2.2}$$

Therefore

$$|f(t) - f(x)| \leq 2K.$$

Also, since f is continuous on I, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ holds for all $t \in I$ satisfying $|t - x| < \delta$. Hence, we get

$$|f(t) - f(x)| < \varepsilon + \frac{2K}{\delta^2}(t - x)^2. \tag{2.3}$$

This means

$$-\varepsilon - \frac{2K}{\delta^2}(t - x)^2 < f(t) - f(x) < \varepsilon + \frac{2K}{\delta^2}(t - x)^2.$$

Using the linearity and the positivity of the operators (L_i) and (2.3), we get,

$$\begin{aligned}
&\left| \sum_{i=1}^{k} a_{k-i+1} L_i(f; x) - f(x) \right| \\
&\quad = \left| \sum_{i=1}^{k} a_{k-i+1} \left(L_i(f(t); x) - L_i(f(x); x) + L_i(f(x); x) \right) - f(x) \right| \\
&\quad \leq \sum_{i=1}^{k} |a_{k-i+1}| \, L_i(|f(t) - f(x)|; x) + |f(x)| \left| \sum_{i=1}^{k} |a_{k-i+1}| \, L_i(e_0; x) - e_0(x) \right| \\
&\quad \leq \sum_{i=1}^{k} |a_{k-i+1}| \, L_i\left(\varepsilon + \frac{2K}{\delta^2}(t - x)^2; x \right) + K \left| \sum_{i=1}^{k} |a_{k-i+1}| \, L_i(e_0; x) - e_0(x) \right| \\
&\quad \leq \varepsilon + \left(\varepsilon + K + \frac{2Kc^2}{\delta^2} \right) \left| \sum_{i=1}^{k} |a_{k-i+1}| \, L_i(e_0; x) - e_0(x) \right| \\
&\qquad + \frac{4Kc}{\delta^2} \left| \sum_{i=1}^{k} |a_{k-i+1}| \, L_i(e_1; x) - e_1(x) \right| + \frac{2K}{\delta^2} \left| \sum_{i=1}^{k} |a_{k-i+1}| \, L_i(e_2; x) - e_2(x) \right|
\end{aligned}$$

where $c := \max_{x \in I} |x|$. Then taking supremum over $x \in I$, we have

$$\left\| \sum_{i=1}^{k} a_{k-i+1} L_i(f) - f \right\| \leq \varepsilon + M \left\{ \sum_{r=0}^{2} \left\| \sum_{i=1}^{k} |a_{k-i+1}| \, L_i(e_r) - e_r \right\| \right\} \tag{2.4}$$

where $M := \max \left\{ \varepsilon + K + \frac{2Kc^2}{\delta^2}, \frac{4Kc}{\delta^2}, \frac{2K}{\delta^2} \right\}$. Then using the hypothesis (2.1), we get

$$\lim_k \left\| \sum_{i=1}^{k} a_{k-i+1} L_i(f) - f \right\| = 0.$$

The proof is complete. □

We now present an example of a sequence of positive linear operators that satisfies the conditions of Theorem 2.4 but does not satisfy the conditions of Theorems 2.1, 2.2 and 2.3.

Example 2.5 Let $a = (a_k) = (1, 0, -1, 0, 0, \ldots)$ and let $x = (x_i) = \left(1, 1, \frac{1}{2^3}, \frac{1}{2^4}, 1, 1, \frac{1}{2^7}, \frac{1}{2^8}, 1, 1, \ldots\right)$. Observe now that, $\sum_{k=1}^{\infty} |a_k| = 2$ and $\sum_{k=1}^{\infty} a_k = 0$. Then, consider the following classical Bernstein operators:

$$B_i(f; x) = \sum_{n=0}^{i} f\left(\frac{n}{i}\right)\binom{i}{n} x^n (1-x)^{i-n}$$

where $x \in [0, 1]$, $f \in C[0, 1]$ and $i \in \mathbb{N}$. *Using these polynomials, we introduce the following positive linear operators on* $C[0, 1]$:

$$T_i(f; x) = x_i B_i(f; x), \quad x \in [0, 1], \quad f \in C[0, 1]. \tag{2.5}$$

We now claim that

$$\lim_k \left\| \sum_{i=1}^{k} |a_{k-i+1}| T_i(e_r) - e_r \right\| = 0 \text{ on } [0, 1] \text{ for each } r = 0, 1, 2. \tag{2.6}$$

Indeed, first observe that

$$\begin{aligned} T_i(e_0; x) &= x_i e_0(x), \\ T_i(e_1; x) &= x_i e_1(x), \\ T_i(e_2; x) &= x_i \left[e_2(x) + \frac{e_1(x) - e_2(x)}{i} \right]. \end{aligned}$$

So, $\left| \sum_{i=1}^{k} |a_{k-i+1}| T_i(e_0; x) - e_0(x) \right| = \left| \sum_{i=1}^{k} |a_{k-i+1}| x_i - 1 \right|$ then

$$\left(\sum_{i=1}^{k} |a_{k-i+1}| x_i \right) = \left(1, 1, 1 + \frac{1}{2^3}, 1 + \frac{1}{2^4}, \frac{1}{2^3} + 1, \frac{1}{2^4} + 1, \ldots\right)$$

and

$$\begin{aligned} &\left(\left| \sum_{i=1}^{k} |a_{k-i+1}| x_i - 1 \right| \right) \\ &= \left(0, 0, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^7}, \frac{1}{2^8}, \ldots\right) \to 0, (k \to \infty). \end{aligned} \tag{2.7}$$

We get

$$\lim_k \left\| \sum_{i=1}^{k} |a_{k-i+1}| T_i(e_0) - e_0 \right\| = 0$$

which guarantees that (2.6) holds true for $r = 0$. Also, since

$$\begin{aligned}\left|\sum_{i=1}^{k} |a_{k-i+1}| T_i\left(e_1; x\right) - e_1\left(x\right)\right| &= \left|\sum_{i=1}^{k} |a_{k-i+1}| x x_i - x\right| \\ &= |x| \left|\sum_{i=1}^{k} |a_{k-i+1}| x_i - 1\right|\end{aligned}$$

then

$$\left\|\sum_{i=1}^{k} |a_{k-i+1}| T_i\left(e_1\right) - e_1\right\| = \left|\sum_{i=1}^{k} |a_{k-i+1}| x_i - 1\right|,$$

using (2.7), we have

$$\lim_k \left\|\sum_{i=1}^{k} |a_{k-i+1}| T_i\left(e_1\right) - e_1\right\| = 0.$$

Hence (2.6) is valid for $r = 1$. Finally, we get

$$\left\|\sum_{i=1}^{k} |a_{k-i+1}| T_i\left(e_2\right) - e_2\right\| \leq \left|\sum_{i=1}^{k} |a_{k-i+1}| x_i - 1\right| + \frac{1}{4}\sum_{i=1}^{k} |a_{k-i+1}| \frac{x_i}{i}, \tag{2.8}$$

$\left(\frac{x_i}{i}\right) = \left(1, \frac{1}{2}, \frac{1}{2^3 3}, \frac{1}{2^4 4}, \frac{1}{5}, \frac{1}{6}, \ldots\right) \to 0$ $(i \to \infty)$ and from Proposition 1.2, we obtain

$$\lim_k \sum_{i=1}^{k} |a_{k-i+1}| \frac{x_i}{i} = 0. \tag{2.9}$$

Supposing that $k \to \infty$ in the inequality (2.8) and using (2.7)–(2.9), we have

$$\lim_k \left\|\sum_{i=1}^{k} |a_{k-i+1}| T_i\left(e_2\right) - e_2\right\| = 0.$$

So, our claim (2.6) holds true for each $r = 0, 1, 2$. Now, from (2.6), we can say that our sequence (T_i) defined by (2.5) satisfies all assumptions of Theorem 2.4. Using these facts, we conclude that

$$\lim_k \left\|\sum_{i=1}^{k} a_{k-i+1} T_i\left(f\right) - f\right\| = 0$$

holds for any $f \in C[0, 1]$.

However, since $\|T_i\left(e_0\right) - e_0\| = |x_i - 1|$ and a sequence $\left(\|T_i\left(e_0\right) - e_0\|\right)$ does not converge, Theorem 2.1 (the classical Korovkin theorem) does not work for the sequence (T_i). The sequence $x = (x_i) = \left(1, 1, \frac{1}{2^3}, \frac{1}{2^4}, 1, 1, \frac{1}{2^7}, \frac{1}{2^8}, 1, 1, \ldots\right)$ can be also given as follows:

$$x_i = \begin{cases} 1, & \text{if } i \text{ is an odd square} \\ \frac{1}{2^i}, & \text{if } i \text{ is an even square} \\ 1, & \text{if } (i = 4k-3 \text{ or } i = 4k-2) \text{ and } i \text{ is nonsquare} \\ \frac{1}{2^i}, & \text{otherwise} \end{cases},$$

$k = 1, 2, 3, \ldots$. Set $K_1 = \{i : i \text{ is an odd square}\}$ and $K_2 = \{i : i \text{ is an even square}\}$, then $\delta\left(K_1\right) = \delta\left(K_2\right) = \delta\left(K_1 \cup K_2\right) = 0$. Since $\|T_i\left(e_0\right) - e_0\| = |x_i - 1|$ and $\left(\|T_i\left(e_0\right) - e_0\|\right)$

is not convergent for $i \notin K_1 \cup K_2$, we get that the sequence $(\|T_i(e_0) - e_0\|)$ is not statistically convergent. Hence, Theorem 2.2 (the statistical Korovkin theorem) does not work for the sequence (T_i). Also, since

$\lim_k \left\| \frac{1}{k} \sum_{i=n}^{n+k-1} T_i(e_0) - e_0 \right\| \neq 0,\ (n \in \mathbb{N})$, Theorem 2.3 does not work for the sequence (T_i), too.

3 Rate of convergence

The main result in this section is studying the rate of K_a-convergence with the aid of the modulus of continuity that is defined by

$$\omega(f,\delta) = \sup_{|t-x|\le\delta,\ x,t\in I} |f(t) - f(x)| \quad (\delta > 0), \quad f \in C(I).$$

It is readily seen that, for any $\lambda > 0$ and for all $f \in C(I)$

$$\omega(f, \lambda\delta) \le (1 + [\lambda])\,\omega(f,\delta)$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ. Then the result is stated as follows.

Theorem 3.1 *Let $a = (a_k)$ be a number sequence and $\sum_{k=1}^{\infty} |a_k| < \infty$. Assume that (L_i) be a sequence of positive linear operators acting from $C(I)$ into itself, satisfying the following conditions:*

(i) $\lim_k \left\| \sum_{i=1}^{k} |a_{k-i+1}| L_i(e_0) - e_0 \right\| = 0$

(ii) $\lim_k \omega(f;\alpha_k) = 0$ where $\alpha_k := \sqrt{\left\| \sum_{i=1}^{k} |a_{k-i+1}| L_i\left((t-.)^2\right) \right\|}$

Then for all $f \in C(I)$,

$$K_a - \lim_k \|L_k(f) - f\| = 0.$$

Proof Let $f \in C(I)$ and $x \in I$ be fixed. Using (2.2), the properties of ω, and the positivity and monotonicity of L_i, we get that

$$\begin{aligned}
&\left| \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) - f(x) \right| \\
&\le \sum_{i=1}^{k} |a_{k-i+1}| L_i(|f(t) - f(x)|;x) + K \left| \sum_{i=1}^{k} |a_{k-i+1}| L_i(e_0;x) - e_0(x) \right| \\
&\le \sum_{i=1}^{k} |a_{k-i+1}| L_i\left(\omega\left(f;\delta\frac{|t-x|}{\delta}\right);x\right) + K \left| \sum_{i=1}^{k} |a_{k-i+1}| L_i(e_0;x) - e_0(x) \right| \\
&\le \omega(f;\delta) \sum_{i=1}^{k} |a_{k-i+1}| L_i\left(1 + \frac{|t-x|}{\delta};x\right) + K \left| \sum_{i=1}^{k} |a_{k-i+1}| L_i(e_0;x) - e_0(x) \right| \\
&\le \omega(f;\delta) \sum_{i=1}^{k} |a_{k-i+1}| L_i\left(1 + \frac{(t-x)^2}{\delta^2};x\right) + K \left| \sum_{i=1}^{k} |a_{k-i+1}| L_i(e_0;x) - e_0(x) \right|
\end{aligned}$$

$$\leq \omega(f;\delta)\left[\sum_{i=1}^{k}|a_{k-i+1}|\,L_i(1;x)+\frac{1}{\delta^2}\sum_{i=1}^{k}|a_{k-i+1}|\,L_i\left((t-x)^2;x\right)\right]$$
$$+K\left|\sum_{i=1}^{k}|a_{k-i+1}|\,L_i(e_0;x)-e_0(x)\right|$$
$$\leq \omega(f;\delta)\left|\sum_{i=1}^{k}|a_{k-i+1}|\,L_i(e_0;x)-e_0(x)\right|+\omega(f;\delta)+K\left|\sum_{i=1}^{k}|a_{k-i+1}|\,L_i(e_0;x)-e_0(x)\right|$$
$$+\frac{\omega(f;\delta)}{\delta^2}\sum_{i=1}^{k}|a_{k-i+1}|\,L_i\left((t-x)^2;x\right)$$

Then taking supremum over $x\in I$, we have

$$\left\|\sum_{i=1}^{k}a_{k-i+1}L_i(f)-f\right\|\leq \omega(f;\delta)\left\|\sum_{i=1}^{k}|a_{k-i+1}|\,L_i(e_0)-e_0\right\|$$
$$+2\omega(f;\delta)+K\left\|\sum_{i=1}^{k}|a_{k-i+1}|\,L_i(e_0)-e_0\right\|$$

where $\delta := \alpha_k = \sqrt{\left\|\sum_{i=1}^{k}|a_{k-i+1}|\,L_i\left((t-.)^2\right)\right\|}$. Then, from the hypotheses we conclude that

$$\lim_k\left\|\sum_{i=1}^{k}a_{k-i+1}L_i(f)-f\right\|=0,$$

we obtain the assertion. □

4 Concluding remarks

Finally, we conclude this paper with the following theorems that establish a relationship between the convergence methods.

Theorem 4.1 *Let $a=(a_k)$ be a number sequence and assume that a method K_a is regular. Suppose that (L_k) be a sequence of positive linear operators acting from $C(I)$ into itself such that*

$$\lim_k\sum_{i=1}^{k}|a_{k-i+1}|\,\|L_k(f)-L_i(f)\|=0.$$

If

$$\lim_k\left\|\sum_{i=1}^{k}|a_{k-i+1}|\,L_i(e_r)-e_r\right\|=0,\ \ (r=0,1,2)\,. \tag{4.1}$$

Then, for all $f\in C(I)$, we have

$$\lim_k\|L_k(f)-f\|=0.$$

Proof From Theorem 2.4, we get that if (4.1) holds then

$$\lim_k \left\| \sum_{i=1}^{k} a_{k-i+1} L_k(f) - f \right\| = 0 \tag{4.2}$$

Also,

$$\begin{aligned} L_k(f;x) - \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) &= \sum_{i=1}^{\infty} a_i L_k(f;x) - \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) \\ &= \sum_{i=1}^{k} a_{k-i+1} \left(L_k(f;x) - L_i(f;x) \right) + \sum_{i=k+1}^{\infty} a_i L_k(f;x), \end{aligned}$$

so that

$$\begin{aligned} |L_k(f;x) - f(x)| &= \left| L_k(f;x) + \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) - \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) - f(x) \right| \\ &\leq \sum_{i=1}^{k} |a_{k-i+1}| \, |L_k(f;x) - L_i(f;x)| + |L_k(f;x)| \sum_{i=k+1}^{\infty} |a_i| \\ &\quad + \left| \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) - f(x) \right|. \end{aligned}$$

Then taking supremum over $x \in I$, we have

$$\begin{aligned} \|L_k(f) - f\| \leq \sum_{i=1}^{k} |a_{k-i+1}| \, \|L_k(f;x) - L_i(f;x)\| + \|L_k(f)\| \sum_{i=k+1}^{\infty} |a_i| \\ + \left\| \sum_{i=1}^{k} a_{k-i+1} L_i(f) - f \right\|. \end{aligned}$$

Since the method K_a is regular, then we can write that $\sum_{i=1}^{\infty} |a_i| < \infty$. So,

$$\lim_k \sum_{i=k+1}^{\infty} |a_i| = 0. \tag{4.3}$$

Hence using (4.1), (4.2) and (4.3), for all $f \in C(I)$, we get

$$\lim_k \|L_k(f) - f\| = 0.$$

□

Theorem 4.2 *Let $a = (a_k)$ be a number sequence such that*

$$\lim_k \sum_{\substack{1 \leq i \leq k \\ i \in E}} |a_{k-i+1}| = 0, \tag{4.4}$$

where $E \subseteq \mathbb{N}$ with $\delta(E) = 0$ and assume that a method K_a is regular. Suppose that (L_k) be a sequence of positive linear operators acting from $C(I)$ into itself. If

$$st - \lim_k \|L_k(e_r) - e_r\| = 0, \quad (r = 0, 1, 2), \tag{4.5}$$

then, for all $f \in C(I)$, we have

$$\lim_k \left\| \sum_{i=1}^{k} a_{k-i+1} L_k(f) - f \right\| = 0.$$

Proof From Theorem 2.2, we get that if (4.5) holds then

$$st - \lim_k \|L_k(f) - f\| = 0.$$

Now, define $E := \{k \in \mathbb{N} : \|L_k(f) - f\| \geq \varepsilon\}$, hence $\delta(E) = 0$. Also,

$$\begin{aligned}
\left| \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) - f(x) \right| &= \left| \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) - \sum_{i=1}^{\infty} a_i f(x) \right| \\
&\leq \sum_{i=1}^{k} |a_{k-i+1}| \, |L_i(f;x) - f(x)| + |f(x)| \sum_{i=k+1}^{\infty} |a_i| .
\end{aligned}$$

Then taking supremum over $x \in I$, we have

$$\begin{aligned}
\left\| \sum_{i=1}^{k} a_{k-i+1} L_i(f;x) - f(x) \right\| &\leq \sum_{i=1}^{k} |a_{k-i+1}| \, \|L_i(f;x) - f(x)\| + \|f\| \sum_{i=k+1}^{\infty} |a_i| \\
&= \sum_{\substack{1 \leq i \leq k \\ i \in E}} |a_{k-i+1}| \, \|L_i(f;x) - f(x)\| \\
&\quad + \sum_{\substack{1 \leq i \leq k \\ i \notin E}} |a_{k-i+1}| \, \|L_i(f;x) - f(x)\| + \|f\| \sum_{i=k+1}^{\infty} |a_i| \\
&\leq \left(\sup_i \|L_i(f;x) - f(x)\| \right) \sum_{\substack{1 \leq i \leq k \\ i \in E}} |a_{k-i+1}| \\
&\quad + \varepsilon \sum_{\substack{1 \leq i \leq k \\ i \notin E}} |a_{k-i+1}| + \|f\| \sum_{i=k+1}^{\infty} |a_i|
\end{aligned}$$

Then, since the method K_a is regular and using (4.4), for all $f \in C(I)$, we get

$$\lim_k \left\| \sum_{i=1}^{k} a_{k-i+1} L_k(f) - f \right\| = 0.$$

$\square$

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