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ORIGINAL PAPER

# Statistical relative $\mathcal{A}$-summation process for double sequences on modular spaces

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**Abstract** In the present paper, we extend the Korovkin type approximation theorem via statistical relative $\mathcal{A}$-summation process onto the double sequences of positive linear operators in a modular space. Then we discuss the reduced results which are obtained by special choice of the scale function and the matrix sequences. We apply our new result to bivariate Bernstein–Kantorovich operators in Orlicz spaces and hence we show that it is stronger than the results obtained previously.

**Keywords** Positive linear operators · Modular spaces · Double sequences · Matrix summability · Statistical convergence

**Mathematics Subject Classification** 41A36 · 47G10 · 47B38 · 46E30

## 1 Introduction

The classical theorem of Korovkin [13] on approximation of continuous functions on a compact interval has the advantage of deducing the sufficient conditions for the uniform convergence of $(L_j)$ to a function $f$, only using the test functions $1, x, x^2$ (see also [1,2]). This theorem has been extended in several directions. One of the most important papers on these extensions is [5] in which the authors obtained Korovkin-type theorem on modular

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spaces. In some Korovkin type theorems, in the case of the lack of convergence, it is effective to use the matrix summability methods or more generally, summation methods. As it is well known, a nonconvergent sequence can be made convergent using these methods. In 1981 Nishishiraho introduced the notion of $\mathcal{A}$-summation process on a compact Hausdorff space [20,21]. Afterwards Korovkin-type approximation theorems are studied via $\mathcal{A}$-summation process in various spaces like the space of all continuous real valued functions on any compact subset of the real two-dimensional space (see [10]), modular spaces [11,22]. Recently, Yılmaz et al. [27] extend the notion of $\mathcal{A}$-summation process on modular spaces to the notion of relative $\mathcal{A}$-summation process. In this study we will investigate the Korovkin theorem for double sequences of positive linear operators via relative $\mathcal{A}$-summation process in the setting of modular function spaces which are natural generalization of $L_p$ $(p > 0)$, Orlicz, Lorentz, and Köthe spaces.

For the purposes of this paper, we begin by reminding of the concept of statistical convergence for double sequences.

A double sequence $x = \left(x_{i,j}\right)$ is convergent to $L$ in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\left|x_{i,j} - L\right| < \varepsilon$ whenever $i, j > N$ and denote it by $P - \lim x = L$ (see [25]). A double sequence is bounded if there exists a positive number $M$ such that $\left|x_{i,j}\right| \leq M$ for all $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. As it is well known in contrast to the case for single sequences, a convergent double sequence does not need to be bounded.

The concept of statistical convergence for double sequences was introduced and studied by Moricz [17] and can be reformulated in terms of natural density. Let $E \subset \mathbb{N}^2$ be a two-dimensional subset of positive integers and let $E_{m,n} = \{(i, j) \in E : i \leq m,\ j \leq n\}$. Then the two-dimensional analogue of natural density can be defined as follows:

$$\delta_2(E) := P - \lim_{m,n} \frac{1}{mn} \left|E_{m,n}\right|$$

if it exists. The number sequence $x = \left(x_{i,j}\right)$ is statistically convergent to $L$ provided that for every $\varepsilon > 0$, the set $E := E_{m,n}(\varepsilon) := \left\{i \leq m,\ j \leq n :\ \left|x_{i,j} - L\right| \geq \varepsilon\right\}$ has natural density zero; in that case we write $st_2 - \lim_{i,j} x_{i,j} = L$.

Clearly, a $P$-convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, the statistical convergence for double sequences was characterized in [17] as given below:

A double sequence $x = \left(x_{i,j}\right)$ is statistically convergent to $L$ if and only if there exists a set $S \subset \mathbb{N}^2$ such that the natural density of $S$ is 1 and

$$P - \lim_{\substack{i,j\to\infty \\ \text{and } (i,j)\in S}} x_{i,j} = L.$$

Now we recall the frequently cited concepts, given by Çakan and Altay [7]. For any real double sequence $x = \left(x_{i,j}\right)$, *the statistical limit superior of* $x$ is

$$st_2 - \limsup_{i,j} x_{i,j} = \begin{cases} \sup G_x, & \text{if } G_x \neq \varnothing, \\ -\infty, & \text{if } G_x = \varnothing, \end{cases}$$

where $G_x := \left\{C \in \mathbb{R} : \delta_2\left(\left\{(i, j) : x_{i,j} > C\right\}\right) \neq 0\right\}$ and $\varnothing$ denotes the empty set. Let $K$ be a subset of $\mathbb{N}^2$. We note that $\delta_2\left(K\right) \neq 0$ means that either $\delta_2\left(K\right) > 0$ or $K$ fails to have the double natural density. Similarly, *the statistical limit inferior of* $x$ is

$$st_2 - \liminf_{i,j} x_{i,j} = \begin{cases} \inf F_x, & \text{if } F_x \neq \varnothing, \\ \infty, & \text{if } F_x = \varnothing, \end{cases}$$

where $F_x := \left\{D \in \mathbb{R} : \delta_2\left(\left\{(i,j) : x_{i,j} < D\right\}\right) \neq 0\right\}$. The ordering relation between the statistical superior and inferior limit is similar as in the ordinary superior or inferior limit, i.e.,

$$st_2 - \liminf_{i,j} x_{i,j} \leq st_2 - \limsup_{i,j} x_{i,j}.$$

Let

$$A = [a_{k,l,i,j}], \quad k,l,i,j \in \mathbb{N},$$

be a four-dimensional infinite matrix. For a given double sequence $x = \left(x_{i,j}\right)$, the $A$-transform of $x$, denoted by $Ax := \left((Ax)_{k,l}\right)$, is given by

$$(Ax)_{k,l} = \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j} x_{i,j}, \quad k,l \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l) \in \mathbb{N}^2$. We say that a sequence $x$ is $A$-summable to $L$ if the $A$-transform of $x$ exists for all $k,l \in \mathbb{N}$ and converges in the Pringsheim's sense i.e.,

$$P - \lim_{p,q} \sum_{i\in\mathbb{N}}^{p} \sum_{j\in\mathbb{N}}^{q} a_{k,l,i,j} x_{i,j} = y_{k,l} \quad \text{and} \quad P - \lim_{k,l} y_{k,l} = L.$$

In summability theory, a two-dimensional matrix transformation is called regular if it maps every convergent sequence into a convergent sequence with the same limit.

Now let $\mathcal{A} := \left(A^{(m,n)}\right) = \left(a^{(m,n)}_{k,l,i,j}\right)$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x = \left(x_{i,j}\right)$ is said to be $\mathcal{A}$-*summable* to $L$ if

$$P - \lim_{k,l} \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} x_{i,j} = L$$

uniformly in $m$ and $n$.

If $A^{(m,n)} = A$, four-dimensional infinite matrix, then $\mathcal{A}$-summability is the $A$-summability for four-dimensional infinite matrix. Some results regarding matrix summability method for double sequences may be found in the papers [24,26].

Let us put our attention to modular spaces.

Let $G = [a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X\left(G^2\right)$ we denote the space of all real-valued measurable functions on $G^2$ provided with equality *a.e.* As usual, let $C\left(G^2\right)$ denote the space of all continuous real-valued functions, and $C^\infty\left(G^2\right)$ denote the space of all infinitely differentiable functions on $G^2$. In this case, we say that a functional $\rho : X\left(G^2\right) \to [0, +\infty]$ is called a *modular* on $X\left(G^2\right)$ if (i) $\rho(f) = 0$ if and only if $f = 0$ *a.e.* in $G^2$, (ii) $\rho(-f) = \rho(f)$ for every $f \in X\left(G^2\right)$ and (iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(G^2)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular $\rho$ is said to be $N$-*quasi convex* if there exists a constant $N \geq 1$ such that $\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng)$ holds for every $f, g \in X\left(G^2\right)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then $\rho$ is called *convex*. A modular $\rho$ is said to be $N$-*quasi semiconvex* if there exists a constant $N \geq 1$ such that $\rho(af) \leq Na\rho(Nf)$ holds for every $f \in X\left(G^2\right)$ and $a \in (0,1]$. It is clear that every $N$-quasi convex modular is $N$-quasi semiconvex. In [4,6], Bardaro et al. introduced and discussed the above two concepts in details.

Some vector subspaces of $X(G^2)$ are considered by means of a modular $\rho$ as follows:

$$L^\rho\left(G^2\right) := \left\{f \in X(G^2) : \lim_{\lambda\to 0^+} \rho(\lambda f) = 0\right\}$$

and

$$E^{\rho}\left(G^{2}\right):=\left\{f\in L^{\rho}\left(G^{2}\right):\rho\left(\lambda f\right)<+\infty \text{ for all } \lambda>0\right\}.$$

are called the *modular space* generated by $\rho$ and the space of the finite elements of $L^{\rho}\left(G^{2}\right)$, respectively. After basic calculations one can see that when $\rho$ is $N$-quasi semiconvex, then the space $L^{\rho}\left(G^{2}\right)$ can be characterised as follows:

$$L^{\rho}\left(G^{2}\right)=\left\{f\in X\left(G^{2}\right):\rho\left(\lambda f\right)<+\infty \text{ for some } \lambda>0\right\}.$$

The main features and various aspects of modulars can be found in [4,18] (see also [14,19]).

Some approximation theorems were introduced by Bardaro and Mantellini [5] via modular convergence and strong convergence. After that Karakuş et al. [12] studied the modular Korovkin-type theorem via the concept of statistical convergence and then these type of approximations were extended to the spaces of double sequences by Orhan and Demirci [23]. Furthermore the notions of relative modular convergence and statistical relative modular convergence have been introduced in [8,27]. Most recently Orhan and Demirci [9] extend these notions to the modular space of double sequences of positive linear operators as follows:

**Definition 1** [9] Let $\left(f_{i,j}\right)$ be a function sequence whose terms belong to $L^{\rho}\left(G^{2}\right)$. Then, $\left(f_{i,j}\right)$ is said to be statistical *relatively modularly convergent* to a function $f\in L^{\rho}\left(G^{2}\right)$ if there exists a function $\sigma(x,y)$, called a scale function $\sigma\in X\left(G^{2}\right)$, $|\sigma(x,y)|\neq 0$ such that

$$st_{2}-\lim_{i,j}\rho\left(\lambda_{0}\left(\frac{f_{i,j}-f}{\sigma}\right)\right)=0 \quad \text{for some} \quad \lambda_{0}>0. \tag{1}$$

Also, $\left(f_{i,j}\right)$ is statistical relatively *$F$-norm convergent* (or, statistical relatively *strongly convergent* ) to $f$ iff

$$st_{2}-\lim_{i,j}\rho\left(\lambda\left(\frac{f_{i,j}-f}{\sigma}\right)\right)=0 \quad \text{for every} \quad \lambda>0. \tag{2}$$

As it is known from [18], (1) and (2) are equivalent if and only if the modular $\rho$ satisfies the $\Delta_2$-condition, i.e. there exists a constant $M>0$ such that $\rho(2f)\leq M\rho(f)$ for every $f\in X\left(G^{2}\right)$.

It will be observed that statistical modular convergence is the special case of statistical relative modular convergence in which the scale function is a non-zero constant. Moreover, if $\sigma(x,y)$ is bounded, statistical relative modular convergence implies statistical modular convergence. However, statistical relative modular convergence does not imply statistical modular convergence, when $\sigma(x,y)$ is unbounded [9].

As mentioned previously, some types of $\mathcal{A}$-summation processes in Korovkin-type approximation theory have been studied in various spaces like $L_p$ $(p>0)$ or more generally modular spaces. The notion of $\mathcal{A}$-summation process on the one dimensional modular space $X(G)$ was introduced in [22] and Kolay, Orhan and Demirci [15] improved this notion to relative $\mathcal{A}$-summation process and they obtained an extension of the classical Korovkin theorem on a modular space using this concept.

Now we introduce the notion of the *relative modular $\mathcal{A}$-summation process* for double sequences as follows:

A sequence $\mathbb{T}:=\left(T_{i,j}\right)$ of positive linear operators of $D$ into $X\left(G^{2}\right)$ is called a *relative $\mathcal{A}$-summation process* on $D$ if $\left(T_{i,j}f\right)$ is relatively $\mathcal{A}$-summable to $f$ (with respect to modular $\rho$) for every $f\in D$, i.e.,

$$P-\lim_{k,l}\rho\left[\lambda\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right]=0, \text{ uniformly in } m,n \tag{3}$$

for some $\lambda > 0$, where for all $k, l, m, n \in \mathbb{N}$, $f \in D$ the series

$$A^{\mathbb{T}}_{k,l,m,n}(f) := \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j} f \tag{4}$$

is absolutely convergent almost everywhere with respect to Lebesgue measure and we denote the value of $T_{i,j} f$ at a point $(x, y) \in G^2$ by $T_{i,j}(f(s,t); x, y)$ or briefly, $T_{i,j}(f; x, y)$. It will be observed that *$\mathcal{A}$-summation process* is the special case of *relative $\mathcal{A}$-summation process* in which the scale function is a non-zero constant.

In this respect a qualitative result on this new convergence of $\mathcal{A}$-summation process can be obtained by applying some Korovkin type theorems for double sequences of positive linear operators on a modular space.

The following assumptions concerning modular functionals will be used:

A modular $\rho$ is *monotone* if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$, $\rho$ is said to be *finite* if $\chi_A \in L^\rho(G^2)$ whenever $A$ is measurable subset of $G^2$ such that $\mu(A) < \infty$. If $\rho$ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset G^2$ with $\mu(B) < \delta$, then $\rho$ is *absolutely finite* and if $\chi_{G^2} \in E^\rho(G^2)$, then $\rho$ is *strongly finite.* A modular $\rho$ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X(G^2)$ with $\rho(f) < +\infty$, the following condition holds:

- for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f\chi_B) < \varepsilon$ whenever $B$ is any measurable subset of $G^2$ with $\mu(B) < \delta$.

Observe now that (see [5,6]) if a modular $\rho$ is monotone and finite, then we have $C(G^2) \subset L^\rho(G^2)$. In a similar manner, if $\rho$ is monotone and strongly finite, then $C(G^2) \subset E^\rho(G^2)$. Also, if $\rho$ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty(G^2)} = L^\rho(G^2)$. For some important relations between the above properties we refer to [3,4,16,19].

## 2 Korovkin type theorems

Let $\rho$ be a monotone and finite modular on $X(G^2)$. Assume that $D$ is a set satisfying $C^\infty(G^2) \subset D \subset L^\rho(G^2)$. We can construct such a subset $D$ when $\rho$ is monotone and finite (see [5]). Let $\mathbb{T} := (T_{i,j})$ be a sequence of positive linear operators from $D$ into $X(G^2)$. Assume that there exists a subset $X_{\mathbb{T}} \subset D$ with $C^\infty(G^2) \subset X_{\mathbb{T}}$ and an unbounded function $\sigma \in X(G^2)$ satisfying $\sigma(x, y) \neq 0$ such that

$$st_2 - \limsup_{k,l} \rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(h)}{\sigma}\right)\right) \leq S\rho(\lambda h), \quad \text{uniformly in } m, n, \tag{5}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant $S$.

Throughout the paper we use the test functions $e_r$ $(r = 0, 1, 2, 3)$ defined by

$$e_0(x, y) = 1, \quad e_1(x, y) = x, \quad e_2(x, y) = y \quad \text{and} \quad e_3(x, y) = x^2 + y^2.$$

**Theorem 1** *Let $\mathcal{A} = (A^{(m,n)})$ be a sequence of four dimensional infinite non-negative real matrices and let $\rho$ be a monotone, strongly finite, absolutely continuous and $N$-quasi semiconvex modular on $X(G^2)$. Let $\mathbb{T} := (T_{i,j})$ be a sequence of positive linear operators* *from $D$ into $X\left(G^2\right)$ satisfying* (5) *for each $f\in D$. Moreover, suppose that $\sigma_r(x,y)$ is an unbounded function satisfying $|\sigma_r(x,y)|\geq a_r>0$ $(r=0,1,2,3)$. Suppose that*

$$st_2-\lim_{k,l}\rho\left(\lambda\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(e_r)-e_r}{\sigma_r}\right)\right)=0,\quad \textit{uniformly in } m,n \tag{6}$$

*for every $\lambda>0$ and $r=0,1,2,3$. Now let $f$ be any function belonging to $L^\rho\left(G^2\right)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^\infty\left(G^2\right)$. Then we have*

$$st_2-\lim_{k,l}\rho\left(\lambda_0\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)=0,\quad \textit{uniformly in } m,n \tag{7}$$

*for some $\lambda_0>0$ where $\sigma(x,y)=\max\{|\sigma_r(x,y)|;r=0,1,2,3\}$.*

*Proof* We first claim that

$$st_2-\lim_{k,l}\rho\left(\eta\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(g)-g}{\sigma}\right)\right)=0\quad \text{uniformly in } m,n \tag{8}$$

for every $g\in C(G^2)\cap D$ and $\eta>0$ where $A_{k,l,m,n}^{\mathbb{T}}(g)=\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}g$. To see this assume that $g$ belongs to $C\left(G^2\right)\cap D$ and $\eta$ is any positive number. By the continuity of $g$ on $G^2$ and by the linearity and positivity of the operators $T_{i,j}$, we easily see that (see, for instance [23]), for a given $\varepsilon>0$, there exists a number $\delta>0$ such that for all $(s,t),(x,y)\in G^2$

$$|g(s,t)-g(x,y)|<\varepsilon+\frac{2M}{\delta^2}\left\{(s-x)^2+(t-y)^2\right\}$$

namely

$$-\varepsilon-\frac{2M}{\delta^2}\left\{(s-x)^2+(t-y)^2\right\}<g(s,t)-g(x,y)<\varepsilon+\frac{2M}{\delta^2}\left\{(s-x)^2+(t-y)^2\right\}$$

where $M:=\sup_{(x,y)\in G^2}|g(x,y)|$. Since $T_{i,j}$ is linear and positive, by similar calculations in [10], we get

$$\begin{aligned}
&\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(g;x,y)-g(x,y)\right|\\
&\quad\leq\varepsilon+K\left\{\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(e_0;x,y)-e_0(x,y)\right|\right.\\
&\qquad+\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(e_1;x,y)-e_1(x,y)\right|\\
&\qquad+\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(e_2;x,y)-e_2(x,y)\right|\\
&\qquad\left.+\left|\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}^{(m,n)}T_{i,j}(e_3;x,y)-e_3(x,y)\right|\right\}
\end{aligned}$$

where $K := \max\left\{\varepsilon + M + \frac{4Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$ and $c := \max\{|x|, |y|\}$. By multiplying both sides of the above inequality by $\frac{1}{|\sigma(x,y)|}$, we have

$$\begin{aligned}
&\left|\frac{\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g;x,y) - g(x,y)}{\sigma(x,y)}\right| \\
&\quad\leq \frac{\varepsilon}{|\sigma(x,y)|} + \frac{K}{|\sigma(x,y)|}\left\{\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_0;x,y) - e_0(x,y)\right|\right. \\
&\qquad + \left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_1;x,y) - e_1(x,y)\right| \\
&\qquad + \left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_2;x,y) - e_2(x,y)\right| \\
&\qquad \left. + \left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_3;x,y) - e_3(x,y)\right|\right\}.
\end{aligned}$$

So the last inequality gives, for any $\eta > 0$, that

$$\begin{aligned}
&\eta\left|\frac{\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g;x,y) - g(x,y)}{\sigma(x,y)}\right| \\
&\quad\leq \frac{\eta\varepsilon}{|\sigma(x,y)|} + \eta K\left\{\left|\frac{\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_0;x,y) - e_0(x,y)}{\sigma_0(x,y)}\right|\right. \\
&\qquad + \left|\frac{\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_1;x,y) - e_1(x,y)}{\sigma_1(x,y)}\right| \\
&\qquad + \left|\frac{\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_2;x,y) - e_2(x,y)}{\sigma_2(x,y)}\right| \\
&\qquad \left. + \left|\frac{\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(e_3;x,y) - e_3(x,y)}{\sigma_3(x,y)}\right|\right\}.
\end{aligned}$$

Applying the modular $\rho$ in the both-sides of the above inequality, since $\rho$ is monotone, we get

$$\begin{aligned}
\rho\left(\eta\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(g) - g}{\sigma}\right)\right) &\leq \rho\left(\frac{\eta\varepsilon}{\sigma} + \eta K\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(e_0) - e_0}{\sigma_0}\right) + \eta K\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(e_1) - e_1}{\sigma_1}\right)\right. \\
&\qquad \left. + \eta K\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(e_2) - e_2}{\sigma_2}\right) + \eta K\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(e_3) - e_3}{\sigma_3}\right)\right).
\end{aligned}$$

So, we may write that

$$\rho\left(\eta\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(g)-g}{\sigma}\right)\right)\leq\rho\left(\frac{5\eta\varepsilon}{\sigma}\right)+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_0)-e_0}{\sigma_0}\right)\right)$$
$$+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_1)-e_1}{\sigma_1}\right)\right)$$
$$+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_2)-e_2}{\sigma_2}\right)\right)$$
$$+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_3)-e_3}{\sigma_3}\right)\right).$$

Since $\rho$ is $N$-quasi semiconvex and strongly finite, we have, assuming $0<\varepsilon\leq 1$

$$\rho\left(\eta\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(g)-g}{\sigma}\right)\right)\leq N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_0)-e_0}{\sigma_0}\right)\right)$$
$$+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_1)-e_1}{\sigma_1}\right)\right)$$
$$+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_2)-e_2}{\sigma_2}\right)\right)$$
$$+\rho\left(5\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_3)-e_3}{\sigma_3}\right)\right).$$

For a given $\varepsilon^*>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)<\varepsilon^*$. Now we define the following sets:

$$E_\eta:=\left\{(k,l):\rho\left(\eta\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(g)-g}{\sigma}\right)\right)\geq\varepsilon^*\right\},$$
$$E_{\eta,r}:=\left\{(k,l):\rho\left(4\eta K\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_r)-e_r}{\sigma_r}\right)\right)\geq\frac{\varepsilon^*-N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)}{4}\right\},\quad r=0,1,2,3.$$

Then, it is easy to see that $E_\eta\subseteq\bigcup_{r=0}^{3}E_{\eta,r}$. So, we can write that

$$\delta_2\left(E_\eta\right)\leq\sum_{r=0}^{3}\delta_2\left(E_{\eta,r}\right).$$

Using the hypothesis (6), we get

$$\delta_2\left(E_\eta\right)=0,$$

which proves our claim (8). Observe that (8) also holds for every $g\in C^\infty(G^2)$. Now let $f\in L^\rho\left(G^2\right)$ satisfying $f-g\in X_{\mathbb{T}}$ for every $g\in C^\infty\left(G^2\right)$. Since $\mu\left(G^2\right)<\infty$ and $\rho$ is strongly finite and absolutely continuous, we can see that $\rho$ is also absolutely finite on $X(G^2)$ (see [3]). Using these properties of the modular $\rho$, it is known from [4,16] that the space $C^\infty(G^2)$ is modularly dense in $L^\rho\left(G^2\right)$, i.e., there exists a sequence $\left(g_{k,l}\right)\subset C^\infty\left(G^2\right)$ such that

$$P-\lim_{k,l}\rho\left(3\lambda_0^*\left(g_{k,l}-f\right)\right)=0\quad\text{for some }\lambda_0^*>0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*\left(g_{k,l} - f\right)\right) < \varepsilon \quad \text{for every } k, l \geq k_0. \tag{9}$$

On the other hand, by the linearity and positivity of the operators $T_{i,j}$, the inequality

$$\begin{aligned}
&\lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f; x, y) - f(x, y)\right| \\
&\leq \lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f - g_{k_0,k_0}; x, y)\right| \\
&\quad + \lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g_{k_0,k_0}; x, y) - g_{k_0,k_0}(x, y)\right| \\
&\quad + \lambda_0^*\left|g_{k_0,k_0}(x, y) - f(x, y)\right|,
\end{aligned}$$

holds for every $x, y \in G$ and $m, n \in \mathbb{N}$. Multiplying by $\frac{1}{|\sigma(x,y)|}$, applying the modular $\rho$ and moreover considering the monotonicity of $\rho$, we have

$$\begin{aligned}
\rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) &\leq \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(f - g_{k_0,k_0}\right)}{\sigma}\right)\right) \\
&+ \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{g_{k_0,k_0} - f}{\sigma}\right)\right).
\end{aligned}$$

Hence observing that $|\sigma| \geq a > 0$ ($a = \max\{a_r : r = 0, 1, 2, 3\}$) and taking into consideration that $\rho$ is monotone, we may write that

$$\begin{aligned}
\rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq\ & \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(f - g_{k_0,k_0}\right)}{\sigma}\right)\right) \\
&+ \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) \\
&+ \rho\left(\frac{3\lambda_0^*}{a}\left(g_{k_0,k_0} - f\right)\right).
\end{aligned} \tag{10}$$

Then, it follows from (9) and (10) that

$$\begin{aligned}
\rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq\ & \varepsilon + \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(f - g_{k_0,k_0}\right)}{\sigma}\right)\right) \\
&+ \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right).
\end{aligned} \tag{11}$$

So, taking statistical limit superior as $k, l \to \infty$ in the both-sides of (11) and also using the facts that $g_{k_0,k_0} \in C^{\infty}(G^2)$ and $f - g_{k_0,k_0} \in X_{\mathbb{T}}$, we obtain from (5) that

$$st_2 - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right)$$
$$\leq \varepsilon + S\rho\left(3\lambda_0^*(f - g_{k_0,k_0})\right)$$
$$+ st_2 - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right),$$

which gives

$$st_2 - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right)$$
$$\leq \varepsilon(S+1) + st_2 - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right). \tag{12}$$

By (8), since

$$st_2 - \lim_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) = 0, \quad \text{uniformly in } m, n,$$

we get

$$st_2 - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) = 0, \quad \text{uniformly in } m, n. \tag{13}$$

Combining (12) with (13), we conclude that

$$st_2 - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) \leq \varepsilon(S+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$st_2 - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) = 0 \quad \text{uniformly in } m, n.$$

Furthermore, since $\rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f)-f}{\sigma}\right)\right)$ is non-negative for all $k, l, m, n \in \mathbb{N}$, we can easily see that

$$st_2 - \lim_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{k,l,m,n}^{\mathbb{T}}(f) - f}{\sigma}\right)\right) = 0, \quad \text{uniformly in } m, n,$$

which completes the proof.

If the modular $\rho$ satisfies the $\Delta_2$-condition, then one can get immediately the following result from Theorem 1.

**Theorem 2** *Let $\mathcal{A} = \left(A^{(m,n)}\right)$ be a sequence of four dimensional infinite non-negative real matrices and $\mathbb{T} := \left(T_{i,j}\right)$, $\rho$ and $\sigma$ be the same as in Theorem 1. If $\rho$ satisfies the $\Delta_2$-condition, then the following statements are equivalent:*

(a) $st_2 - \lim_{k,l} \rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(e_r)-e_r}{\sigma_r}\right)\right) = 0$ *uniformly in* $m, n$, *for every* $\lambda > 0$ *and* $r = 0, 1, 2, 3$,

(b) $st_2 - \lim_{k,l} \rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{k,l,m,n}(f)-f}{\sigma}\right)\right) = 0$ *uniformly in* $m, n$, *for every* $\lambda > 0$ *provided that* $f$ *is any function belonging to* $L^{\rho}(G^2)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}\left(G^2\right)$.

If one replaces the matrices $A^{(m,n)}$ by the identity matrix and take the scale function as a non-zero constant, then condition (5) reduces to

$$st_2 - \limsup_{i,j} \rho\left(\lambda\left(T_{i,j}h\right)\right) \leq S\rho\left(\lambda h\right) \tag{14}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant $S$. In this case, the next results which were obtained by Orhan and Demirci [23] immediately follows from our Theorems 1 and 2.

**Corollary 1** ([23]) *Let* $\rho$ *be a monotone, strongly finite, absolutely continuous and* $N$*-quasi semiconvex modular on* $X\left(G^2\right)$. *Let* $\mathbb{T} := \{T_{i,j}\}$ *be a sequence of positive linear operators from* $D$ *into* $X\left(G^2\right)$ *satisfying* (14). *If* $\left(T_{i,j}e_r\right)$ *is statistically strongly convergent to* $e_r$ *for each* $r = 0, 1, 2, 3$, *then* $\left(T_{i,j}f\right)$ *is statistically modularly convergent to* $f$ *provided that* $f$ *is any function belonging to* $L^{\rho}\left(G^2\right)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}\left(G^2\right)$.

**Corollary 2** ([23]) *Let* $\mathbb{T} := \left(T_{i,j}\right)$ *and* $\rho$ *be the same as in Corollary* 1. *If* $\rho$ *satisfies the* $\Delta_2$*-condition, then the following statements are equivalent:*

(a) $\left(T_{i,j}e_r\right)$ *is statistically strongly convergent to* $e_r$ *for each* $r = 0, 1, 2, 3$,

(b) $\left(T_{i,j}f\right)$ *is statistically strongly convergent to* $f$ *provided that* $f$ *is any function belonging to* $L^{\rho}(G^2)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}\left(G^2\right)$.

## 3 An example

In this section, we give an example of positive linear operators which satisfies the conditions of Theorem 1 but not satisfies the conditions of Corollary 1.

*Example 1* Take $G = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- $\varphi$ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.

Hence, let us consider the functional $\rho^{\varphi}$ on $X(G^2)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \int_0^1 \varphi\left(|f(x, y)|\right) dxdy \quad \text{for } f \in X\left(G^2\right). \tag{15}$$

In this case, $\rho^{\varphi}$ is a convex modular on $X\left(G^2\right)$, which satisfies all assumptions listed in Sect. 1 (see [5]). Let us consider the Orlicz space generated by $\varphi$ as follows:

$$L^{\rho}_{\varphi}(G^2) := \left\{f \in X\left(G^2\right) : \rho^{\varphi}\left(\lambda f\right) < +\infty \text{ for some } \lambda > 0\right\}.$$

Then let us consider the following bivariate Bernstein–Kantorovich operator $\mathbb{U} := \left(U_{i,j}\right)$ on the space $L_{\varphi}^{\rho}\left(G^2\right)$ which is defined by:

$$U_{i,j}(f; x, y) = \sum_{k=0}^{i}\sum_{l=0}^{j} p_{k,l}^{(i,j)}(x, y)(i+1)(j+1)\int\limits_{k/(i+1)}^{(k+1)/(i+1)}\int\limits_{l/(j+1)}^{(l+1)/(j+1)} f(t, s)\, ds dt \tag{16}$$

for $x, y \in G$, where $p_{k,l}^{(i,j)}(x, y)$ is defined by

$$p_{k,l}^{(i,j)}(x, y) = \binom{i}{k}\binom{j}{l} x^k y^l (1-x)^{i-k}(1-y)^{j-l}.$$

It is clear that,

$$\sum_{k=0}^{i}\sum_{l=0}^{j} p_{k,l}^{(i,j)}(x, y) = 1. \tag{17}$$

Observe that the operators $U_{i,j}$ map the Orlicz space $L_{\varphi}^{\rho}\left(G^2\right)$ into itself. Because of (17), as in the proof of [5] Lemma 5.1 and similar to Example 1 [23], we can use the Jensen inequality and obtain that for every $f \in L_{\varphi}^{\rho}\left(G^2\right)$ and $i, j \in \mathbb{N}$ there is an absolute constant $M > 0$ such that

$$\rho^{\varphi}\left(\frac{U_{i,j} f}{\sigma}\right) \leq M\rho^{\varphi}(f).$$

Then, we know that, for any function $f \in L_{\varphi}^{\rho}\left(G^2\right)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}\left(G^2\right)$, $\left(U_{i,j} f\right)$ is relatively modularly convergent to $f$, with the choice of $X_{\mathbf{U}} := L_{\varphi}^{\rho}(G^2)$ (see for details [9]).

If $\varphi(x) = x^p$ for $1 \leq p < \infty$, $x \geq 0$, then $L_{\varphi}^{\rho}\left(G^2\right) = L_p\left(G^2\right)$. Moreover we have $\rho^{\varphi}(f) = \|f\|_{L_p(G^2)}^{p}$. Also for each $i, j \in \mathbb{N}$, define $g_{i,j} : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$g_{i,j}(x, y) = \begin{cases} 1, & i = t^2 \quad \text{and} \quad j = s^2 \\ i^3 j^3 xy, & (x, y) \in \left(0, \frac{1}{i}\right) \times \left(0, \frac{1}{j}\right); i \neq t^2 \quad \text{or} \quad j \neq s^2 \\ 0, & (x, y) \notin \left(0, \frac{1}{i}\right) \times \left(0, \frac{1}{j}\right); i \neq t^2 \quad \text{or} \quad j \neq s^2 \end{cases} \tag{18}$$

$t, s = 1, 2, \ldots$. Now observe that, $\left(g_{i,j}\right)$ does not converge statistically modularly but converges to $g = 0$ statistically modularly relative to the scale function $\sigma(x, y) = \begin{cases} 1, & x = 0 \text{ or } y = 0 \\ \frac{1}{x^2 y^2}, & (x, y) \in (0, 1] \times (0, 1] \end{cases}$ on $L_1\left(G^2\right)$. Indeed, for some $\lambda_0 > 0$, with the choice of $p = 1$ we have $\rho^{\varphi}(g) = \|g\|_{L_1(G^2)}$,

$$\begin{aligned} \rho\left(\lambda_0\left(g_{i,j} - g\right)\right) &= \left\|\lambda_0\left(g_{i,j} - g\right)\right\|_{L_1(G^2)} \\ &= \lambda_0 \begin{cases} 1, & i = t^2 \text{ and } j = s^2 \\ \frac{ij}{4}, & i \neq t^2 \text{ or } j \neq s^2, \end{cases} \quad t, s = 1, 2, \ldots, \end{aligned}$$

then we have $st_2 - \lim_{i,j} \left\|\lambda_0\left(g_{i,j} - g\right)\right\|_{L_1(G^2)} \neq 0$.Using the scale function $\sigma$,

$$\begin{aligned} &\rho\left(\lambda_0\left(\frac{g_{i,j} - g}{\sigma}\right)\right) \\ &= \lambda_0 \begin{cases} 1, & i = t^2 \quad \text{and} \quad j = s^2 \\ \frac{1}{16ij}, & i \neq t^2 \quad \text{or} \quad j \neq s^2, \end{cases} \quad t, s = 1, 2, \ldots, \end{aligned}$$

we get

$$st_2 - \lim_{i,j} \left\| \lambda_0 \left( \frac{g_{i,j} - g}{\sigma} \right) \right\|_{L_1(G^2)} = 0. \tag{19}$$

Also, assume that $\mathcal{A} := \left(A^{(m,n)}\right) = \left(a^{(m,n)}_{k,l,i,j}\right)$ is a sequence of four dimensional infinite matrices defined by $a^{(m,n)}_{k,l,i,j} = \frac{1}{kl}$ if $m \leq i \leq m+k-1, n \leq j \leq n+l-1, (m, n = 1, 2, \ldots)$ and $a^{(m,n)}_{k,l,i,j} = 0$ otherwise. Then, using the operators $U_{i,j}$, we define the sequence of positive linear operators $\mathbb{V} := \left(V_{i,j}\right)$ on $L_1\left(G^2\right)$ as follows:

$$V_{i,j}(f; x, y) = \left(1 + g_{i,j}(x, y)\right) U_{i,j}(f; x, y) \text{ for } f \in L_1\left(G^2\right), x, y \in [0, 1] \text{ and } i, j \in \mathbb{N}, \tag{20}$$

and we choose $\sigma_r(x, y) = \sigma(x, y)$ $(r = 0, 1, 2, 3)$. As in the proof of Lemma 5.1 [5] and similarly to Example 1 [23], we get, for every $f \in L_1\left(G^2\right)$, $\lambda > 0$ and for positive constant $C$, that

$$st_2 - \limsup_{k,l} \left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(f)}{\sigma} \right\|_{L_1(G^2)} \leq C \left\| \lambda f \right\|_{L_1(G^2)}, \text{ uniformly in } m, n.$$

where $A^{\mathbb{V}}_{k,l,m,n}(f) = \sum^{\infty}_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} V_{i,j} f$ as in (4).

We now claim that

$$st_2 - \lim_{k,l} \left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(e_r) - e_r}{\sigma} \right\|_{L_1(G^2)} = 0, \text{ uniformly in } m, n; r = 0, 1, 2, 3, \tag{21}$$

where $\sigma_r(x, y) = \sigma(x, y)$ for $r = 0, 1, 2, 3$. Observe that $U_{i,j}(e_0; x, y) = e_0$, $U_{i,j}(e_1; x, y) = \frac{ix}{i+1} + \frac{1}{2(i+1)}$, $U_{i,j}(e_2; x, y) = \frac{jy}{j+1} + \frac{1}{2(j+1)}$ and $U_{i,j}(e_3; x, y) = \frac{i(i-1)x^2}{(i+1)^2} + \frac{2ix}{(i+1)^2} + \frac{1}{3(i+1)^2} + \frac{j(j-1)y^2}{(j+1)^2} + \frac{2jy}{(j+1)^2} + \frac{1}{3(j+1)^2}$. So, we can see,

$$\begin{aligned} \left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(e_0) - e_0}{\sigma} \right\|_{L_1(G^2)} &= \left\| \lambda \frac{\sum^{m+k-1}_{i=m} \sum^{n+l-1}_{j=n} \frac{1}{kl}(1 + g_{i,j}) - 1}{\sigma} \right\|_{L_1(G^2)} \\ &\leq \frac{1}{kl} \sum^{m+k-1}_{i=m} \sum^{n+l-1}_{j=n} \left\| \lambda \frac{g_{i,j}}{\sigma} \right\|_{L_1(G^2)} \end{aligned}$$

because of $\frac{1}{kl} \sum^{m+k-1}_{i=m} \sum^{n+l-1}_{j=n} \left\| \lambda \frac{g_{i,j}}{\sigma} \right\|_{L_1(G^2)} = \lambda \begin{cases} \frac{1}{9}, & i = t^2 \text{ and } j = s^2 \\ \frac{1}{kl} \sum\limits^{m+k-1}_{i=m} \sum\limits^{n+l-1}_{j=n} \frac{1}{16ij}, & i \neq t^2 \text{ or } j \neq s^2 \end{cases}$,

$t, s = 1, 2, \ldots$, we get

$$st_2 - \lim_{k,l} \frac{1}{kl} \sum^{m+k-1}_{i=m} \sum^{n+l-1}_{j=n} \left\| \lambda \frac{g_{i,j}}{\sigma} \right\|_{L_1(G^2)} = 0, \text{ uniformly in } m, n \tag{22}$$

and hence

$$st_2 - \lim_{k,l} \left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(e_0) - e_0}{\sigma} \right\|_{L_1(G^2)} = 0, \text{ uniformly in } m, n,$$

which guarantees that (21) holds true for $r = 0$. Also, we get

$$\left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(e_1) - e_1}{\sigma} \right\|_{L_1(G^2)} < \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{i}{i+1} - 1 \right) \left\| \lambda f_{3,2} \right\|_{L_1(G^2)} + \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{1}{2(i+1)} \right) \left\| \lambda f_{2,2} \right\|_{L_1(G^2)} + \frac{2}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \left\| \lambda \frac{g_{i,j}}{\sigma} \right\|_{L_1(G^2)}$$

where $f_{u,v}(x, y) = x^u y^v \, for u, v = 2, 3, 4$.

Since

$$st_2 - \lim_{k,l} \left( \sup_{m,n} \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{i}{i+1} - 1 \right) \right) = 0,$$

$$\underset{k}{st_2} - \lim_{k,l} \left( \sup_{m,n} \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{1}{2(i+1)} \right) = 0$$

and from (22) we have,

$$st_2 - \lim_{k,l} \left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(e_1) - e_1}{\sigma} \right\|_{L_1(G^2)} = 0, \text{ uniformly in } m, n.$$

So (21) holds true for $r = 1$. Similarly, we have

$$st_2 - \lim_{k,l} \left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(e_2) - e_2}{\sigma} \right\|_{L_1(G^2)} = 0, \text{ uniformly in } m, n.$$

Finally, since

$$\left\| \lambda \frac{A^{\mathbb{V}}_{k,l,m,n}(e_3) - e_3}{\sigma} \right\|_{L_1(G^2)} < \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{i(i-1)}{(i+1)^2} - 1 \right) \left\| \lambda f_{4,2} \right\|_{L_1(G^2)} + \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{j(j-1)}{(j+1)^2} - 1 \right) \left\| \lambda f_{2,4} \right\|_{L_1(G^2)} + \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{2i}{(i+1)^2} \right) \left\| \lambda f_{3,2} \right\|_{L_1(G^2)} + \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \frac{2j}{(j+1)^2} \right) \left\| \lambda f_{2,3} \right\|_{L_1(G^2)} + \left( \frac{1}{kl} \sum_{i=m}^{m+k-1} \sum_{j=n}^{n+l-1} \left( \frac{1}{3(i+1)^2} + \frac{1}{3(j+1)^2} \right) \right) \left\| \lambda f_{2,2} \right\|_{L_1(G^2)}$$

$$+\frac{6}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\left\|\lambda\frac{g_{i,j}}{\sigma}\right\|_{L_1(G^2)}$$

where $f_{u,v}(x,y)=x^u y^v$ for $u,v=2,3,4$. Hence we can easily see that

$$st_2-\lim_{k,l}\left\|\lambda\frac{A^{\mathbb{V}}_{k,l,m,n}(e_3)-e_3}{\sigma}\right\|_{L_1(G^2)}=0,\ \text{uniformly in}\ m,n.$$

So, our claim (21) holds true for each $r=0,1,2,3$. $(V_{i,j})$ satisfies all hypothesis of Theorem 1 and we immediately see that,

$$st_2-\lim_{k,l}\left\|\lambda\frac{A^{\mathbb{V}}_{k,l,m,n}(f)-f}{\sigma}\right\|_{L_1(G^2)}=0,\ \text{uniformly in}\ m,n,$$

on $G^2=[0,1]\times[0,1]$ for all $f\in L_1(G^2)$. Also, since $(g_{i,j})$ does not converge modularly in statistical sense, $(V_{i,j})$ is not satisfy Corollary 1.

## 4 Concluding remarks

We give now some reduced results with the special choice of scale function and the matrices $A^{(m,n)}$ which show the importance of Theorem 1 in approximation theory:

- If one replaces the scale function by a non-zero constant, then from our Theorems 1 and 2 we immediately get the Korovkin type theorems of statistical $\mathcal{A}$-summation process for double sequences.
- If one replaces the matrices $A^{(m,n)}$ by the identity matrix, then from our Theorems 1 and 2 we immediately get the statistical relative Korovkin type theorems for double sequences given by Demirci and Orhan [9].
- As it is well-known $(X,\|.\|)$ is a normed space then, $\rho(.)=\|.\|$ is a convex modular in $X$. So, all of the theorems of this paper are considered valid on normed spaces as well.

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