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ORIGINAL PAPER

# Approximation via equi-statistical convergence in the sense of power series method

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**Abstract**
In this study, we define a new type of convergence using the notions of statistical convergence in the sense of the power series method and equi-statistical convergence then, we give an example that supports this new definition and after that we use it to prove a Korovkin-type approximation theorem. This theorem is a non-trivial generalization of Korovkin-type approximation theorems that have been studied in earlier papers. Also, we present an example that satisfies our approximation theorem which hasn't satisfied the one studied before. Moreover, we calculate the rate of equi-statistical convergence in the sense of the power series method and then, we prove a Voronovskaya-type approximation theorem. Finally, we summarize our results in the conclusion section.

**Keywords** Statistical convergence with respect to power series method · Equi-statistical convergence · Rate of convergence · Voronovskaya-type approximation theorem · Korovkin-type approximation theorem

**Mathematics Subject Classification** 41A25 · 41A36

## 1 Introduction and preliminaires

As it is known, Korovkin [19] proved an approximation theorem which is called Korovkin type approximation theorem in such a way that a sequence $(S_m)$ of positive linear operator converges uniformly to the function to be approximated (see, for instance, [2]). Steinhaus [28] and Fast [12] independently gave the notion of statistical convergence for real number sequences in 1951. Later, Gadjiev and Orhan [13] used this convergence in approximation theory and hence, it gives many advantages as it is stronger than the classical one. Recently,

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Korovkin type theorems have been studied using many new types of statistical convergence ([5–11,17,26]). Also, Balcerzak et al. [3] have introduced the definition of equi-statistical convergence which is stronger than uniform statistical convergence. Then, several works have been done by many researchers based on equi-statistical convergence (see [1,6,17,20–23]).

In this paper, the Korovkin type approximation theorem has been proved via a new type of statistical convergence which is defined using the notions of equi-statistical convergence and statistical convergence in the sense of power series method. After, an example has been presented that satisfies the new approximation theorem which hasn't satisfied the one studied before. Also, the rate of equi-statistical convergence in the sense of the power series method has been calculated. Finally, the Voronovskaya-type approximation theorem has been proved.

Let's remind some necessary definitions.

Let $\mathbb{N}$ be the set of natural numbers and let $E \subseteq \mathbb{N}$. Also let

$$E_m := \{k : k \leq m \text{ and } k \in E\} \tag{1}$$

and suppose that the symbol $|.|$ shows the cardinality of the set. The natural density of $E$ is defined by

$$\delta(E) = \lim_m \frac{|E_m|}{m} = \lim_m \frac{1}{m} |\{k : k \leq m \text{ and } k \in E\}| \tag{2}$$

if the limit exists.

Throughout this work, let $x = (x_m)$ be a real sequence.

The sequence $x$ is said to be statistically convergent to $\gamma$, if, for every $\varepsilon > 0$, the following set:

$$E := E(\varepsilon) := |\{m \in \mathbb{N} : \ |x_m - \gamma| \geq \varepsilon\}|$$

has natural density zero [12], i.e., for every $\varepsilon > 0$, we have

$$\delta(E(\varepsilon)) = \lim_m \frac{|E(\varepsilon)|}{m} = \lim_m \frac{1}{m} |\{k \leq m : |x_k - \gamma| \geq \varepsilon\}| = 0.$$

In that case, we write $st - \lim x_m = \gamma$.

Let $C(U)$ be the space of all continuous real valued functions on a compact subset $U$ of $\mathbb{R}$ (the real numbers). Let $g \in C(U)$ and $\|g\|_{C(U)}$ indicates the usual supremum norm of $g$ in $C(U)$.

Let $g_m \in C(U)$. If

(a) for every $\varepsilon > 0$ and for each $x \in U$,

$$\lim_m \frac{|K_m(x, \varepsilon)|}{m} = 0$$

where $K_m(x, \varepsilon) := \{m \in \mathbb{N} : \ |g_m(x) - g(x)| \geq \varepsilon\}$, $x \in U$, then $(g_m)$ is called *statistically pointwise convergent* to $g$ on $U$ and denoted by $g_m \to g \ (st)$ on $U$ (see [9]).

(b)

$$\lim_m \frac{|D_m(\varepsilon)|}{m} = 0$$

where $D_m(\varepsilon) := \left\{m \in \mathbb{N} : \ ||g_m - g||_{C(U)} \geq \varepsilon\right\}$, then $(g_m)$ is called *statistically uniformly convergent* to $g$ on $U$ and shown as $g_m \rightrightarrows g \ (st)$ on $U$ (see [9]).

(c) for every $\varepsilon > 0$,

$$\lim_m \frac{|K_m(x, \varepsilon)|}{m} = 0, \text{ uniformly in } x,$$

then $(g_m)$ is called *equi-statistically convergent* to $g$ on $U$ and shown as $g_m \twoheadrightarrow g(st)$ on $U$ (see [3]).

Let $(p_m)$ be a non-negative real sequence with $p_0 > 0$ and the corresponding power series

$$p(s) := \sum_{m=0}^{\infty} p_m s^m$$

has radius of convergence $R_p \in (0, \infty]$. If the limit

$$\lim_{0<s\to R^-} \frac{1}{p(s)} \sum_{j=0}^{\infty} x_m p_m s^m$$

exists, then $x = (x_m)$ is said to be convergent in the sense of power series method ([18,27]). It is worth noting that the power series method is regular if and only if $\lim_{0<s\to R_p^-} \dfrac{p_m s^m}{p(s)} = 0$ for every $m \in \mathbb{N}_0$ (see [4]).

Ünver and Orhan [29] have recently introduced $P-$density of $U \subset \mathbb{N}_0$ and the definition of statistical convergence in the sense of power series method for single sequences. Now, we recall this convergence.

Let $H \subset \mathbb{N}_0$. If the limit

$$\delta_P(H) := \lim_{0<s\to R_p^-} \frac{1}{p(s)} \sum_{m\in H} p_m s^m$$

exists, then $\delta_P(H)$ is called the $P-$density of $H$. Then, it is clear that $0 \leq \delta_P(H) \leq 1$ provided it exists.

The sequence $x = (x_m)$ is said to be statistically convergent in the sense of power series method to $\gamma$ if for any $\varepsilon > 0$

$$\lim_{0<s\to R_p^-} \frac{1}{p(s)} \sum_{m\in H_\varepsilon} p_m s^m = 0$$

where $H_\varepsilon = \{m \in \mathbb{N}_0 : |x_m - \gamma| \geq \varepsilon\}$, that is $\delta_P(H_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case, we write $st_P - \lim x_m = \gamma$.

Here and in the sequel, power series method is regular.

With the example below it can be seen that statistical convergence and statistical convergence in the sense of power series method cannot be compared.

***Example 1*** Let

$$p_m = \begin{cases} 1, & m \text{ is a square,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$x_m = \begin{cases} 0, & m \text{ is a square,} \\ m+1, & \text{otherwise.} \end{cases}$$

From the definition of $p_m$, we have

$$\lim_{0<s\to R_p^-} \frac{1}{p(s)} \sum_{m\in\{m\in\mathbb{N}_0 : |x_m|\geq\varepsilon\}} p_m s^m = 0.$$

Then, $x = (x_m)$ is statistically convergent in the sense of power series method to 0. It is obvious that $x$ is not statistically convergent.

Now let $x=(x_m)$ be a sequence defined by

$$x_j=\begin{cases} m^2, & m \text{ is a square,}\\ 0, & \text{otherwise.}\end{cases}$$

It is not difficult to observe that $x$ is statistically convergent to 0 however, since

$$\lim_{0<s\to R_p^-}\frac{1}{p\,(s)}\sum_{m\in\{m\in\mathbb{N}_0:|x_m|\geq\varepsilon\}} p_m s^m\neq 0$$

then, it is not statistically convergent in the sense of power series method.

Now, following [29] we define the statistical uniform convergence and statistical pointwise convergence in the sense of power series method and also, we introduce the concept of equi-statistical convergence in the sense of power series method.

**Definition 1** If, for every $\varepsilon>0$ and for each $x\in U$,

$$\delta_P\,(K_m(x,\varepsilon))=\lim_{0<s\to R^-}\frac{1}{p\,(s)}\sum_{m\in K_m(x,\varepsilon)} p_m s^m=0,$$

then $(g_m)$ is said to be statistically pointwise convergent in the sense of power series method to $g$ on $U$ and denoted by $g_m\to g\ (st_p)$ on $U$.

**Definition 2** If, for every $\varepsilon>0$,

$$\delta_P\,(D_m(\varepsilon))=\lim_{0<s\to R^-}\frac{1}{p\,(s)}\sum_{m\in D_m(\varepsilon)} p_m s^m=0,$$

then $(g_m)$ is said to be statistically uniformly convergent in the sense of power series method to $g$ on $U$ and denoted by $g_m\rightrightarrows g\ (st_p)$ on $U$.

**Definition 3** If, for every $\varepsilon>0$,

$$\delta_P\,(K_m(x,\varepsilon))=\lim_{0<s\to R^-}\frac{1}{p\,(s)}\sum_{m\in K_m(x,\varepsilon)} p_m s^m=0,\ \text{uniformly in } x,$$

then $(g_m)$ is said to be equi-statistically convergent in the sense of power series method to $g$ on $U$ and denoted by $g_m\twoheadrightarrow g(e_{st_P})$ on $U$.

Now, let us state the following result.

**Lemma 1** *If $g_m\rightrightarrows g\ (st_p)$ on $U$, then $g_m\twoheadrightarrow g(e_{st_P})$ on $U$. Also $g_m\twoheadrightarrow g(e_{st_P})$ on $U$, then $g_m\to g\ (st_p)$ on $U$.*

Now, we give some examples. The first one satisfies that equi-statistical convergence in the sense of power series method is stronger than statistical uniform convergence in the sense of power series method. The second one satisfies that statistical pointwise convergence in the sense of power series method is stronger than equi-statistical convergence in the sense of power series method.

***Example 2*** Let $U = [0, 1]$, for each $x \in U$, $q(x) = 0$ and $(q_m)$ is a function sequence on $U$ defined by

$$q_m(x) = \begin{cases} -3^m(x - \frac{1}{3^{m-1}}), & m \text{ is square and } x \in A, \\ 3^m(x - \frac{1}{3^m}), & m \text{ is square and } x \in B, \\ 0, & m \text{ is square and } x \notin A \cup B, \\ m, & m \text{ is not square,} \end{cases} \tag{3}$$

where $A = \left[\frac{1}{3^{m-1}} - \frac{1}{3^m}, \frac{1}{3^{m-1}}\right]$ and $B = \left[\frac{1}{3^m}, \frac{1}{3^{m-1}} - \frac{1}{3^m}\right]$. Consider the power series method that is given with

$$p_m = \begin{cases} 1 & m \text{ is square,} \\ 0 & m \text{ is not square.} \end{cases}$$

From the definition of $p_m$, we have $\delta_P(\{m : m \text{ is not square}\}) = 0$. Also, if $m$ is square then, for any $x \in U$,

$$\begin{aligned} |\{m \in \mathbb{N} : |q_m(x) - q(x)| \geq \varepsilon\}| &\leq |\{m \in \mathbb{N} : q_m(x) \neq 0\}| \\ &\leq 1. \end{aligned}$$

Hence, for any $x \in U$,

$$\begin{aligned} &\frac{1}{p(s)} \sum_{m \in \{m \in \mathbb{N}:\, |q_m(x) - q(x)| \geq \varepsilon\}} p_m s^m \\ &\leq \frac{1}{p(s)} p_{m_0} s^{m_0} \to 0 \ \left(0 < s \to R^-\right), \end{aligned}$$

$m_0 \in \mathbb{N}_0$.

Therefore, we get $q_m \twoheadrightarrow q\ (e_{st_P})$ on $U$. But since $\|q_m - q\|_{C(U)} = m$, then $(q_m)$ is not statistically uniformly convergent, and statistically uniformly convergent in the sense of power series method to the function $q = 0$ on $U$.

***Example 3*** Let $U = [0, 1]$, $(q'_m)$ is a function sequence on $U$ defined by

$$q'_m(x) = \begin{cases} x^m, & m \text{ is not square,} \\ 0, & m \text{ is square.} \end{cases}$$

and suppose also that $\lim_m q'_m(x) = q'(x)\ (x \in U)$. Consider the power series method that is given with

$$p'_m = \begin{cases} 1, & m \text{ is not square,} \\ 0, & m \text{ is square.} \end{cases}$$

It is easy to see that, $q'_m \to q'\ (st_p)$ on $U$. Thanks to definition of $p'_m$, we have $\delta_P(\{m : m \text{ is square}\}) = 0$. Assume that, $m$ is not square. If we take $\varepsilon = \frac{1}{3}$ and for any $x \in \left(\sqrt[m]{\frac{1}{3}}, 1\right)$,

$$\left|q'_m(x)\right| = \left|x^m\right| > \left|\left(\sqrt[m]{\frac{1}{3}}\right)^m\right| = \frac{1}{3}.$$

Hence, $q'_m \twoheadrightarrow 0(e_{st_P})$ on $U$ does not hold true.

## 2 A Korovkin-type approximation theorem

In this section, we apply the notion of equi-statistical convergence in the sense of power series method of a sequence of functions to state and prove a Korovkin type approximation theorem.

Let $S$ be a linear operator maps on $C(U)$. As we know, $S$ is called the positive linear operator on condition that $h \geq 0$ implies $S(h) \geq 0$. Also, we show the value of $S(h)$ at a point $x \in U$ by $S(h(t); x)$ or, briefly, $S(h; x)$.

Throughout the paper, we use the following test functions

$$h_j(x) = x^j, (j = 0, 1, 2).$$

First we recall the classical Korovkin-type theorem and statistical Korovkin-type theorem in the sense of power series method:

**Theorem 1** ([19]) *Let $(S_m)$ be a sequence of positive linear operators maps on $C(U)$. Then, for every $h \in C(U)$,*

$$\lim \|S_m(h) - h\|_{C(U)} = 0$$

*if and only if*

$$\lim \left\| S_m\left(h_j\right) - h_j \right\|_{C(U)} = 0, \ j = 0, 1, 2.$$

**Theorem 2** ([29]) *Let $(S_m)$ be a sequence of positive linear operators maps on $C(U)$. For all $h \in C(U)$,*

$$st_P - \lim \|S_m(h) - h\|_{C(U)} = 0$$

*if and only if*

$$st_P - \lim \left\| S_m\left(h_j\right) - h_j \right\|_{C(U)} = 0, \ j = 0, 1, 2.$$

Now, we give the following approximation theorem which is our main result:

**Theorem 3** *Let $(S_m)$ be a sequence of positive linear operators maps on $C(U)$. For all $h \in C(U)$,*

$$S_m(h) \twoheadrightarrow h(e_{st_P}) \text{ on } U, \tag{4}$$

*if and only if*

$$S_m(h_j) \twoheadrightarrow h_j(e_{st_P}) \text{ on } U, \ j = 0, 1, 2. \tag{5}$$

***Proof*** Since each of the functions $h_j(x) = x^j$, $j = 0, 1, 2$ is in $C(U)$, condition (5) can be obtained from condition (4). Now let's prove the opposite. Let $h \in C(U)$ and $x \in U$. By the continuity of $h$ on $U$, we have

$$|h(u) - h(x)| \leq 2\kappa$$

where $\kappa := \|h\|_{C(U)}$. By means of the continuity of $h$ on $U$, we can write that for $\forall \varepsilon > 0$, there exists a number $\delta > 0$ such that $|h(t) - h(x)| < \varepsilon$ for $\forall t \in U$ satisfying $|t - x| < \delta$. Hence, if $g(t) = (t - x)^2$ is taken, we get

$$|h(s) - h(x)| < \varepsilon + \frac{2\kappa}{\delta^2} g(s). \tag{6}$$

Since $S_m$ is positive linear operator, we write

$$\begin{aligned} |S_m(h; x) - h(x)| &= |S_m(h(t) - h(x); x) + h(x)(S_m(h_0; x) - h_0(x))| \\ &\leq S_m(|h(t) - h(x)|; x) + \kappa |S_m(h_0; x, y) - h_0(x, y)| \end{aligned} \tag{7}$$

Then, we compute the term of "$S_m(|h(t)-h(x)|;x)$" in (7),

$$S_m(|h(t)-h(x)|;x) \leq S_m\left(\varepsilon+\frac{2\kappa}{\delta^2}g(t);x\right)$$
$$= \varepsilon\,|S_m(h_0;x)-h_0(x)| + \frac{2\kappa}{\delta^2}S_m(g(t);x) \tag{8}$$

We calculate the term of "$S_m(g(t);x)$" in (8),

$$\begin{aligned} S_m(g(t);x) &= S_m((t-x)^2;x) \\ &= S_m(t^2-2xt+x^2;x) \\ &\leq |S_m(h_2;x)-h_2(x)| + 2\|h_1\|_{C(U)}|S_m(h_1;x)-h_1(x)| \\ &\quad + \|h_2\|_{C(U)}|S_m(h_0;x)-h_0(x)|. \end{aligned} \tag{9}$$

Using (9) in (7), we get

$$\begin{aligned} |S_m(h;x)-h(x)| &\leq \varepsilon + \left(\varepsilon+\kappa+\frac{2\kappa\|h_2\|_{C(U)}}{\delta^2}\right)|S_m(h_0;x)-h_0(x)| \\ &\quad + \frac{4\kappa\|h_1\|_{C(U)}}{\delta^2}|S_m(h_1;x)-h_1(x)| \\ &\quad + \frac{2\kappa}{\delta^2}|S_m(h_2;x)-h_2(x)| \\ &\leq \lambda\{|S_m(h_0;x)-h_0(x)| + |S_m(h_1;x)-h_1(x)| \\ &\quad + |S_m(h_2;x)-h_2(x)|\} + \varepsilon, \end{aligned} \tag{10}$$

where $\lambda = \max\left\{\varepsilon+\kappa+\frac{2\kappa\|h_2\|_{C(U)}}{\delta^2}, \frac{4\kappa\|h_1\|_{C(U)}}{\delta^2}, \frac{2\kappa}{\delta^2}\right\}$. Then, setting, for any $x \in U$,

$$R_m\left(x,\varepsilon'\right) = \left\{m \in \mathbb{N}_0 : \ |S_m(h;x)-h(x)| \geq \varepsilon'\right\}$$
$$R_{j,m}\left(x,\frac{\varepsilon'-\varepsilon}{3\lambda}\right) = \left\{m \in \mathbb{N}_0 : \ \left|S_m(h_j;x)-h_j(x)\right| \geq \frac{\varepsilon'-\varepsilon}{3\lambda}\right\}, \ j=0,1,2.$$

It follows from (10) that

$$R_m\left(x,\varepsilon'\right) \subset \bigcup_{j=0}^{2} R_{j,m}\left(x,\frac{\varepsilon'-\varepsilon}{3\lambda}\right)$$

and so

$$\frac{1}{p(s)}\sum_{m\in R_m(x,\varepsilon')} p_m s^m \leq \sum_{j=0}^{2}\frac{1}{p(s)}\sum_{m\in R_{j,m}\left(x,\frac{\varepsilon'-\varepsilon}{3\lambda}\right)} p_m s^m.$$

Then using the hypotheses (5), we get

$$S_m(h) \twoheadrightarrow h(e_{st_P}) \text{ on } U.$$

The proof of the theorem is completed. □

Now, an example that provides the above result is given.

***Example 4*** Let $U=[0,1]$ and take the Bernstein polynomials

$$B_m(h;x)=\sum_{s=0}^{m} h\left(\frac{s}{m}\right)\binom{m}{s} x^s (1-x)^{m-s}$$

on $C(U)$. Now, we introduce positive linear operators on $C(U)$ as follows:

$$S_m(h;x)=(1+q_m(x))B_m(h;x),\quad x\in U \text{ and } h\in C(U), \tag{11}$$

where $(q_m)$ is given by (3). Let

$$p_m=\begin{cases}1 & m \text{ is square,}\\ 0 & m \text{ is not square}\end{cases}$$

Then, we calculate

$$\begin{aligned}S_m(h_0;x)&=(1+q_m(x))h_0(x),\\ S_m(h_1;x)&=(1+q_m(x))h_1(x),\\ S_m(h_2;x)&=(1+q_m(x))\left[h_2(x)+\frac{x(1-x)}{m}\right].\end{aligned}$$

Since $q_m \twoheadrightarrow q=0\ (e_{st_P})$ on $U$, we conclude that

$$S_m\left(h_j\right)\twoheadrightarrow h_j\ (e_{st_P}) \text{ on } U,\ \text{ for each } j=0,1,2.$$

So, by Theorem 3, we see that

$$S_m(h)\twoheadrightarrow h\ \ (e_{st_P}) \text{ on } U \text{ for all } h\in C(U).$$

Since $(q_m)$ is not statistically uniformly convergent in the sense of power series method to the function $q=0$ on $U$, Theorem 2 does not provide for operators $(S_m)$ given by (11). Similarly, since $(q_m)$ is not uniformly convergent (in the ordinary sense) to $q=0$ on $U$, Theorem 1 does not work either. This shows us that our Theorem 3 is a non-trivial generalization of Theorem 1 and Theorem 2, respectively.

## 3 Rate of $P-$equi statistical convergence

In this section, we will study the corresponding rates of $P-$equi statistical convergence for sequences with the help of modulus of continuity.

We remind that the modulus of continuity of a function $h\in C(U)$ is defined by

$$w(h,\delta)=\sup\{|h(t)-h(x)|:|t-x|\le\delta,\ x,t\in U\},\ \ (\delta>0).$$

**Theorem 4** *Let $(S_m)$ be a sequence of positive linear operators maps on $C(U)$. The following conditions are provided:*

*(i) $S_m(h_0)\twoheadrightarrow h_0(e_{st_P})$ on $U$,*

*(ii) $w(h,\delta_m)\twoheadrightarrow 0(e_{st_P})$ on $U$, where $\delta_m:=\delta_m(x)=\sqrt{S_m(g;x)}$ with $g(t)=(t-x)^2$.*
*Then for every $h\in C(U)$,*

$$S_m(h)\twoheadrightarrow h\ (e_{st_P}),\ \text{on } U.$$

***Proof*** Let $h\in C(U)$ and $x\in U$. By means of the property of $S_m$ and the property of modulus of continuity $w$, we calculate

$$|S_m(h;x)-h(x)|\le S_m\left(|h(t)-h(x)|;x\right)+|h(x)|\,|S_m(h_0;x)-h_0(x)|$$

$$\le S_m\left(\left(\left(1+\frac{g(t)}{\delta^2}\right)w(g,\delta);x,y\right)+\kappa\left|S_m(e_0;x,y)-e_0(x,y)\right|\right.$$
$$= w(h,\delta)\,S_m(h_0;x)+\frac{w(h,\delta)}{\delta^2}S_m(g(t);x)$$
$$+\kappa\left|S_m(h_0;x)-h_0(x)\right|$$

where $\kappa := \|h\|_{C(U)}$. If we choose $\delta := \delta_m := \delta_m(x) = \sqrt{S_m(g;x)}$, this yield that

$$\begin{aligned}|S_m(h;x)-h(x)| \le \kappa\left|S_m(h_0;x)-h_0(x)\right| + 2w(h,\delta_m)\\ +w(h,\delta_m)\left|S_m(h_0;x)-h_0(x)\right|.\end{aligned} \tag{12}$$

Then, setting, for any $x \in U$,

$$R_m(x,\varepsilon^{'}) = \left\{m\in\mathbb{N}_0 : |S_m(h;x)-h(x)| \ge \varepsilon^{'}\right\},$$
$$R_{1,m}\left(x,\frac{\varepsilon^{'}}{3\kappa}\right) = \left\{m\in\mathbb{N}_0 : |S_m(h_0;x)-h_0(x)| \ge \frac{\varepsilon^{'}}{3\kappa}\right\},$$
$$R_{2,m}\left(x,\frac{\varepsilon^{'}}{6}\right) = \left\{m\in\mathbb{N}_0 : w(h,\delta_m) \ge \frac{\varepsilon^{'}}{6}\right\},$$
$$R_{3,m}\left(x,\frac{\varepsilon^{'}}{3}\right) = \left\{m\in\mathbb{N}_0 : w(h,\delta_m)\left|S_m(h_0;x)-h_0(x)\right| \ge \frac{\varepsilon^{'}}{3}\right\}.$$

It follows from (12) that

$$R_m\left(x,\varepsilon^{'}\right) \subset R_{1,m}\left(x,\frac{\varepsilon^{'}}{3\kappa}\right)\cup R_{2,m}\left(x,\frac{\varepsilon^{'}}{6}\right)\cup R_{3,m}\left(x,\frac{\varepsilon^{'}}{3}\right)$$

and define

$$R_{4,m}\left(x,\sqrt{\frac{\varepsilon^{'}}{3}}\right) = \left\{m\in\mathbb{N}_0 : w(h,\delta_m) \ge \sqrt{\frac{\varepsilon^{'}}{3}}\right\},$$
$$R_{5,m}\left(x,\sqrt{\frac{\varepsilon^{'}}{3}}\right) = \left\{m\in\mathbb{N}_0 : |S_m(h_0;x)-h_0(x)| \ge \sqrt{\frac{\varepsilon^{'}}{3}}\right\}.$$

Then, we get

$$R_m\left(x,\varepsilon^{'}\right) \subset R_{1,m}\left(x,\frac{\varepsilon^{'}}{3\kappa}\right)\cup R_{2,m}\left(x,\frac{\varepsilon^{'}}{6}\right)\cup R_{4,m}\left(x,\sqrt{\frac{\varepsilon^{'}}{3}}\right)\cup R_{5,m}\left(x,\sqrt{\frac{\varepsilon^{'}}{3}}\right),$$

so

$$\begin{aligned}\frac{1}{p(s)}\sum_{m\in R\left(x,\varepsilon^{'}\right)} p_m s^m \le \frac{1}{p(s)}\sum_{m\in R_{1,m}\left(x,\frac{\varepsilon^{'}}{3\kappa}\right)} p_m s^m\\ +\frac{1}{p(s)}\sum_{m\in R_{2,m}\left(x,\frac{\varepsilon^{'}}{6}\right)} p_m s^m\end{aligned}$$

$$+\frac{1}{p(s)}\sum_{m\in R_{4,m}\left(x,\sqrt{\frac{\varepsilon'}{3}}\right)} p_m s^m$$

$$+\frac{1}{p(s)}\sum_{m\in R_{5,m}\left(x,\sqrt{\frac{\varepsilon'}{3}}\right)} p_m s^m.$$

Then using the hypotheses (i) and (ii), we get

$$S_m(h) \twoheadrightarrow h(e_{st_P}) \text{ on } U.$$

□

***Remark 1*** Replace the conditions (i) and (ii) in Theorem 4 with

$$S_m(h_j) \twoheadrightarrow h_j(e_{st_P}) \text{ on } U, \text{ for } j=0,1,2. \tag{13}$$

Since

$$S_m\left(g(t);x\right) = S_m\left(h_2;x\right) - 2xS_m\left(h_1;x\right) + x^2 S_m\left(h_0;x\right)$$

we can write that

$$S_m\left(g(t);x\right) \le \sum_{j=0}^{2}\left|S_m(h_j;x) - h_j(x)\right| \tag{14}$$

where $\lambda = \max\left\{\varepsilon+\kappa+\frac{2\kappa\|h_2\|_{C(U)}}{\delta^2}, \frac{4\kappa\|h_1\|_{C(U)}}{\delta^2}, \frac{2\kappa}{\delta^2}\right\}$. Now it follows from (13) and (14) that

$$\delta_m := \delta_m(x) = \sqrt{S_m(g;x)} \twoheadrightarrow 0(e_{st_P}) \text{ on } U.$$

So, it yields that

$$w(h,\delta_m) \twoheadrightarrow 0(e_{st_P}) \text{ on } U. \tag{15}$$

Using (15) in Theorem 4, we see, for all $h\in C(U)$, that

$$S_m(h) \twoheadrightarrow h\ (e_{st_P}), \text{ on } U.$$

If we replace the conditions (i) and (ii) in Theorem 4 by the condition (13) , then we can obtain the rate of equi-statistical convergence in the sense of power series method for the sequence of positive linear operators in Theorem 3. □

## 4 A Voronovskaya-type approximation theorem

In this section, we prove a Voronovskaya-type approximation theorem via equi-statistical convergence in the sense of power series method for $\{S_m\}$ given by (11).

**Lemma 2** *Let* $x\in[0,1]$. *Then, we get*

$$m^2 S_m\left((t-x)^4;x\right) \twoheadrightarrow 3h_2\left(1-h_1\right)^2\left(e_{st_P}\right) \text{ on } U. \tag{16}$$

***Proof*** After some calculations, we can write the following form (16) that

$$m^2 S_m\left((t-x)^4;x\right) = \left(1+q_m(x)\right)\left[3x^2\left(1-x\right)^2 + \frac{x(-6x^3+12x^2-7x+1)}{m}\right].$$

So, we get

$$\left|m^2 S_m\left((t-x)^4; x\right) - 3x^2(1-x)^2\right| \leq 12q_m(x) + (1+q_m(x))\frac{26}{m} \tag{17}$$

for every $x \in [0,1]$. Since $q_m \twoheadrightarrow q = 0\ (e_{st_P})$ on $U$, it is easy to see that

$$12q_m(x) + (1+q_m(x))\frac{26}{m} \twoheadrightarrow 0(e_{st_P}) \text{ on } U. \tag{18}$$

Combining (17) and (18), the proof is complete. □

**Theorem 5** *For every $h \in C(U)$ such that $h^{'}, h^{''} \in C(U)$, we have*

$$m\{S_m(h) - h\} \twoheadrightarrow \frac{h_1 - h_2}{2} h^{''}(e_{st_P}) \text{ on } U.$$

***Proof*** Let $x \in U$ and $h^{'}, h^{''} \in C(U)$. Define the function $f_x$ by

$$f_x(t) = \begin{cases} \frac{h(t)-h(x)-h^{'}(x)(t-x)-\frac{1}{2}h^{''}(x)(t-x)^2}{(u-x)^2} & , t \neq x, \\ 0 & , t = x. \end{cases}$$

Then by assumption, we get $f_x(x) = 0$ and $f_x(.) \in C(U)$. By the Taylor formula for $h \in C(U)$, we have

$$\begin{aligned} h(t) = h(x) + h^{'}(x)(t-x) + \frac{1}{2}h^{''}(x)(t-x)^2 \\ + f_x(t)(t-x)^2. \end{aligned}$$

From linearity of $S_m$, we see that

$$\begin{aligned} S_m(h;x) &= h(x)(1+q_m(x)) + h^{'}(x) S_m(t-x;x) \\ &+ \frac{1}{2}h^{''}(x) S_m((t-x)^2;x) \\ &+ S_m\left(f_x(t)(t-x)^2;x\right). \end{aligned}$$

Now, we have

$$\begin{aligned} S_m(h;x) &= h(x)(1+q_m(x)) + \frac{(x-x^2)(1+q_m(x))}{2m}h^{''}(x) \\ &+ S_m\left(f_x(u)(u-x)^2;x\right) \end{aligned}$$

which yields

$$\begin{aligned} m\{S_m(h;x) - h(x)\} &= mq_m(x)h(x) + \frac{(x-x^2)(1+q_m(x))}{2m}h''(x) \\ &+ mS_m\left(f_x(t)(t-x)^2;x\right). \end{aligned} \tag{19}$$

Now, we apply the Cauchy-Schwarz inequality for the third term to the right of (19), we have

$$m\left|S_m\left(f_x(t)(t-x)^2;x\right)\right| \leq \left(S_m\left(f_x^2(t);x\right)\right)^{1/2} \cdot \left(m^2 S_m\left((t-x)^4;x\right)\right)^{1/2}$$

Let $\phi_x(t) = f_x^2(t)$. We can show that $\phi_x(x) = 0$ and $\phi_x(.) \in C(U)$. From Theorem 3,

$$S_m\left(f_x^2(t);x\right) = S_m(\phi_x(t);x) = \phi_x(x) \twoheadrightarrow 0(e_{st_P}) \text{ on } U. \tag{20}$$

Using (20) and Lemma 2, we obtain from (19)

$$mS_m\left(f_x(t)(t-x)^2;x\right) \twoheadrightarrow 0(e_{st_P}) \text{ on } U. \tag{21}$$

Considering (21), (19) and also $q_m \twoheadrightarrow q = 0\ (e_{st_P})$ on $U$, we have

$$m\left\{S_m(h) - h\right\} \twoheadrightarrow \frac{h_1 - h_2}{2} h''(e_{st_P}) \text{ on } U.$$

This completes the proof. □

## 5 Concluding Remarks

In this section, we give some remarks concerning to the results which we have studied here.

***Remark 2*** As we state in Example 4, the sequence $(S_m)$ given by (11) satisfies Theorem 3 while it does not satisfy Theorem 1 and Theorem 2. Thus, this application clearly shows that our Theorem 3 is a non-trivial extension of the classical Korovkin-type approximation theorem in [19] and also, statistical Korovkin-type approximation theorem with respect to power series method in [29].

***Remark 3*** Throughout the paper, we consider the statistical type convergence which are defined by power series methods. It is worth noting that, these methods are in general non-matrix methods. Many researchers are interested in matrix summability methods (for example, Cesàro methods, deferred Cesàro methods, deferred Nörlund methods) and they have proved Korovkin-type approximation theorems via these methods (see [14–16,24,25]). It is known that the mentioned matrix methods and power series methods are incompatible. For this reason, our proposed Korovkin-type approximation theorem and the mentioned Korovkin-type approximation theorems are incompatible. Thus, the obtained results here are meaningful.

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