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Approximation via statistical relative uniform convergence of sequences of functions at a point with respect to power series method

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Abstract
In the present paper, we generalize the notion of $P-$statistical convergence and we first define the notion of $P-$statistical relative uniform convergence of sequences of functions at a point. We demonstrate an approximation theorem for a sequence of functions. Also, we give an example, showing that our result is strict generalization of the corresponding classical ones. In the final section, we study the rates of convergence.

Keywords Power series method · Statistical convergence · Korovkin theorem · Positive linear operator

Mathematics Subject Classification 40G10 · 41A36 · 47B38

1 Introduction

The Korovkin-type approximation ([13, 20]) has emerged as a wide field of study in recent years. This type of approximation establishes the uniform convergence for a given sequence of positive linear operators in space of all continuous real valued function on compact interval assuming the classical convergence by the test functions. The Korovkin theorem was studied for a lot of classes of integral or discrete positive operators, acting over continuous functions. There are many variants of Korovkin-type approximation theorems with respect to convergence generated by summability matrices, statistical convergence (see e.g., [1, 3, 7, 17, 22, 24]) and " statistical relative uniform convergence" that is an extension of statistical

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uniform convergence of functions, associated with a scale function (see e.g., [6, 9–11, 15, 18]). Another interesting convergence method is $P-$statistical convergence, which recently introduced by Ünver and Orhan [27] and is a different type of statistical convergence given via power series methods. Thus it makes sense to ask: "Do the statistical convergence and $P-$ statistical convergence contain each other?". Ünver and Orhan answered this question as these two notions are incompatible via striking examples. In this work, we first give our new convergence named $P-$ statistical relative uniform convergence of sequences of functions at a point. Then, we establish an approximation theorem for a sequence of functions via this new convergence. We give an example, showing that our result is a strict generalization of the corresponding classical ones and we present some graphs. Finally, we are interested in the rates of convergence.

2 Preliminaries

Now, we begin with recalling the statistical convergence and the convergence in the sense of power series method of a sequence x :

Let $\mathbb{N}_0$ be the set of all non-negative integers and S be a subset of $\mathbb{N}_0$, then *the natural density of* S, denoted by $\delta(S)$, is given by, if the following limit exists:

$$\delta(S) := \lim_{n} \frac{1}{n+1} \left|\{\lambda \leq n : \lambda \in S\}\right|$$

where $|.|$ stands for the cardinality of the set ([23]).

A sequence $x = \{x_\lambda\}$ of numbers is *statistically convergen* t to L provided that, for every $\varepsilon > 0$,

$$\lim_{n} \frac{1}{n+1} \left|\{\lambda \leq n : |x_\lambda - L| \geq \varepsilon\}\right| = 0$$

that is,

$$S := S_n(\varepsilon) := \{\lambda \leq n : |x_\lambda - L| \geq \varepsilon\}$$

has natural density zero. This is denoted by $st - \lim_\lambda x_\lambda = L$ ([16, 26]). It is worth noting that, every convergent sequence (in the usual sense) is statistically convergent to the same number and unlike a convergent sequence, a statistically convergent sequence need not be convergent.

Now let's examine the definitions and properties of the power series method.

Let $\{p_\lambda\}$ be a non-negative real sequence such that $p_0 > 0$ and the following power series

$$p(s) := \sum_{\lambda=0}^{\infty} p_\lambda s^\lambda$$

has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{0<s\to R^-} \frac{1}{p(s)} \sum_{\lambda=0}^{\infty} p_\lambda s^\lambda x_\lambda = L$$

exists then we say that $x = \{x_\lambda\}$ is convergent in the sense of power series method ([21, 25]). Note that the method is regular iff $\lim_{0<s\to R^-} \dfrac{p_\lambda s^\lambda}{p(s)} = 0$ for every λ (see, e.g. [5]).

Remark 1 Let us notice first that in case of $R = 1$, the power series methods coincide with Abel summability method and logarithmic summability method when $p_\lambda = 1$ and $p_\lambda = \frac{1}{\lambda+1}$, respectively. In the case of $R = \infty$, the power series method coincides with Borel summability method when $p_\lambda = \frac{1}{\lambda!}$.

Here and in the sequel, power series method is always assumed to be regular.

It is worthwhile to point out that, Ünver and Orhan [27] have recently introduced $P-$statistical convergence. They show that statistical convergence and this new notion of convergence are incompatible via striking examples. Now, we recall the next definitions:

Definition 1 [27] Let $S \subset \mathbb{N}_0$. If the limit

$$\delta_P(S) := \lim_{0<s\to R^-} \frac{1}{p(s)} \sum_{\lambda\in S} p_\lambda s^\lambda$$

exists, then $\delta_P(S)$ is called the $P-$density of S. Note that, from the definition of a power series method and $P-$density it is obvious that $0 \leq \delta_P(S) \leq 1$ whenever it exists.

In a manner similar to the natural density, one can give some properties for the $P-$density:
(i) $\delta_P(\mathbb{N}_0) = 1$,
(ii) if $S \subset T$ then $\delta_P(S) \leq \delta_P(T)$,
(iii) if S has $P-$density then $\delta_P(\mathbb{N}_0/S) = 1 - \delta_P(S)$.

Definition 2 [27] Let $x = \{x_\lambda\}$ be a sequence. Then x is said to be statistically convergent with respect to power series method ($P-$ statistically convergent) to L if for any $\varepsilon > 0$

$$\lim_{0<s\to R^-} \frac{1}{p(s)} \sum_{\lambda\in S_\varepsilon} p_\lambda s^\lambda = 0$$

where $S_\varepsilon = \{\lambda \in \mathbb{N}_0 : |x_\lambda - L| \geq \varepsilon\}$, that is $\delta_P(S_\varepsilon) = 0$ for any $\varepsilon > 0$. This is denoted by $st_P - \lim x_\lambda = L$.

Demirci and Yıldız [12] have recently introduced statistical type convergences in the sense of power series method in modular spaces. In the following, we give $P-$statistical relative uniform convergence as a special case of these convergence methods:

Definition 3 Let $\{h_\lambda\}$ be a sequence of real functions defined on $G \subset \mathbb{R}$. We say that $\{h_\lambda\}$ converges $P-$statistically relatively uniformly to $h : G \to \mathbb{R}$ provided that for every $\varepsilon > 0$, there exists $S \subset \mathbb{N}_0$ with $\delta_P(S) = 0$ such that for every $\lambda \in \mathbb{N}_0 \backslash S$

$$|h_\lambda(u) - h(u)| < \varepsilon |\sigma(u)|$$

holds uniformly in u on the interval G.

Klippert and Williams defined the notion of uniform convergence of a sequence of functions at a point in [19]. This type of convergence is a stronger than the well known uniform convergence. Now we recall this interesting type of convergence:

Definition 4 [19] Let $G \subset \mathbb{R}$ and suppose that $\{h_\lambda\}$ is a sequence of real functions defined on G. Let $u_0 \in G$. We say that $\{h_\lambda\}$ converges uniformly at the point u_0 to $h : G \to \mathbb{R}$ provided that for every $\varepsilon > 0$, there exist $\eta > 0$ and $n \in \mathbb{N}$ such that for every $\lambda \geq n$, if $|u - u_0| < \eta$, then

$$|h_\lambda(u) - h(u)| < \varepsilon.$$

Recently, Demirci et al. [8] gave the notion of relative uniform convergence of a sequence of functions at a point and they proved the Korovkin type theorems (see also [4, 14, 28]). The definition rewrite by the authors. In the following, we give this notion:

Definition 5 [8] Let $\{h_\lambda\}$ be a sequence of real functions defined on $G \subset \mathbb{R}$. Let $u_0 \in G$. We say that $\{h_\lambda\}$ converges relatively uniformly at the point u_0 to $h : G \to \mathbb{R}$ provided that for every $\varepsilon > 0$, there exist a function σ, $|\sigma(u)| \neq 0$, called a scale function, $\eta > 0$ and $n \in \mathbb{N}_0$ such that for every $\lambda \geq n$, if $|u - u_0| < \eta$, then

$$|h_\lambda(u) - h(u)| < \varepsilon |\sigma(u)| .$$

This limit is denoted by $h_\lambda \rightrightarrows h(\{u_0\}; \sigma)$.

3 The notion of convergence at a point

Now, we can give the following definition which is our new type of convergence:

Definition 6 Let $G \subset \mathbb{R}$ and suppose that $\{h_\lambda\}$ is a sequence of real functions defined on G. Let $u_0 \in G$. We say that $\{h_\lambda\}$ converges $P-$statistically relatively uniformly at the point u_0 to $h : G \to \mathbb{R}$ provided that for every $\varepsilon > 0$, there exist a function σ, $|\sigma(u)| \neq 0$, called a scale function, $\eta > 0$ and $S \subset \mathbb{N}_0$ with $\delta_P(S) = 0$ such that for every $\lambda \in \mathbb{N}_0 \backslash S$, if $|u - u_0| < \eta$, then

$$|h_\lambda(u) - h(u)| < \varepsilon |\sigma(u)| .$$

This limit is denoted by $(st_P) - h_\lambda \rightrightarrows h(\{u_0\}; \sigma)$.

It will be observed that if $\{h_\lambda\}$ is $P-$statistically relatively uniformly convergent to a function h on G, then $\{h_\lambda\}$ converges $P-$statistically relatively uniformly at each point in G. Also, if the scale function is a non-zero constant then, the $P-$statistical uniform convergence of a sequence of functions at a point is the special case of $P-$statistical relative uniform convergence of a sequence of functions at a point. If σ is bounded then, the $P-$statistical relative uniform convergence at a point implies the $P-$statistical uniform convergence at a point. For all that, the $P-$statistical relative uniform convergence at a point does not imply the $P-$statistical uniform convergence at a point, when σ is unbounded. This is illustrated by the following example:

Example 1 Take $G = [0, 1]$ and define $g_\lambda : G \to \mathbb{R}$ by

$$g_\lambda(u) = \begin{cases} \lambda u, & \lambda = n^2, \\ \frac{2\lambda u}{3+\lambda u}, & \text{otherwise}, \end{cases} \tag{1}$$

$n = 0, 1, 2, ...$ Also, let the power series method is given with

$$p_\lambda = \begin{cases} 0, \ \lambda = n^2, \\ 1, \ \lambda \neq n^2, \end{cases} \tag{2}$$

$n = 0, 1, 2, ...$ It is clear that $\{g_\lambda\}$ converges $P-$ statistically relatively uniformly to $g = 0$ at $u_0 = 0$ to the scale function

$$\sigma(u) = \begin{cases} 1, & u = 0, \\ \frac{1}{u}, & 0 < u \leq 1. \end{cases} \tag{3}$$

Indeed, let $\varepsilon > 0$ be given and choose $\eta = \frac{\varepsilon}{2}$ and $S = \left\{\lambda : \lambda = n^2, n = 0, 1, 2, ...\right\}$. Then $\delta_P(S) = 0$. Let $\lambda \in \mathbb{N}_0 \backslash S$ and $u \in [0, 1]$ with $|u| \leq \eta$. Then,

$$\left|\frac{g_\lambda(u) - g(u)}{\sigma(u)}\right| \leq \left|\frac{2\lambda u^2}{3 + \lambda u}\right| \leq 2|u| < 2\eta = \varepsilon.$$

However, $\{g_\lambda\}$ does not converge $P-$statistically uniformly at 0. Indeed, for $\varepsilon = \frac{1}{3}$, $u = \frac{1}{\lambda} \in (0, 1]$ with $\frac{1}{\lambda} < \eta$ and $\lambda \in \mathbb{N}_0 \backslash S$, we get

$$\frac{2\lambda u}{3 + \lambda u} = \frac{1}{2} > \frac{1}{3}.$$

Also, it is neither uniformly nor relatively uniformly convergent to $g = 0$.

In the remaining three sections we deal with Korovkin type approximation theorems, an example which shows that our result is more sense, and the rate of convergence.

4 Korovkin type approximation

Several mathematicians have studied on extending Korovkin's theorems in many ways and to several convergence methods. Such studies described a theory which is called Korovkin-type approximation theory. Even today, the Korovkin-type approximation theory still continues to be studied in new types of convergence and in different spaces.

In this section, we apply the notion of $P-$statistical relative uniform convergence of sequences of functions at a point to prove a Korovkin type approximation theorem. Suppose that $C(G)$ is the space of all functions h continuous on G. We know that $C(G)$ is a Banach space with norm $\|h\| = \sup_{u \in G} |h(u)|$. We denote the value of $L_\lambda(h)$ at a point $u \in G$ by $L_\lambda(h(v); u)$ or briefly, $L_\lambda(h; u)$ and we use the test functions

$$e_0(u) = 1, e_1(u) = u \text{ and } e_2(u) = u^2.$$

We first recall the following Korovkin type approximation theorem rewritten by Demirci et al. [8].

Theorem 1 [8] *Let $\{L_\lambda\}$ be a sequence of positive linear operators acting from $C(G)$ into itself. Then for all $h \in C(G)$*

$$L_\lambda(h) \rightrightarrows h(\{u_0\}; \sigma)$$

if and only if

$$L_\lambda(e_i) \rightrightarrows e_i(\{u_0\}; \sigma_i), i = 0, 1, 2$$

where σ, σ_i possibly unbounded and $|\sigma(u)| > 0$, $|\sigma_i(u)| > 0$, $i = 0, 1, 2$.

Now, we can observe the following approximation result which is our main theorem.

Theorem 2 *Let $\{L_\lambda\}$ be a sequence of positive linear operators acting from $C(G)$ into itself. Then for all $h \in C(G)$*

$$(st_P) - L_\lambda(h) \rightrightarrows h(\{u_0\}; \sigma)$$

if and only if

$$(st_P) - L_\lambda(e_i) \rightrightarrows e_i(\{u_0\}; \sigma_i), i = 0, 1, 2$$

where σ, σ_i possibly unbounded and $|\sigma(u)| > 0$, $|\sigma_i(u)| > 0$, $i = 0, 1, 2$.

Proof Since $e_i \in C(G)$, $i = 0, 1, 2$, the necessary condition is obvious. Now, let $h \in C(G)$ and $u \in G$ be fixed. Let $F = \max\left\{|u|, |u|^2\right\}$. Also, by the continuity of h on G, we can write $|h(u)| \leq M$. Hence,

$$|h(v) - h(u)| \leq |h(v)| + |h(u)| \leq 2M.$$

Moreover, since h is uniformly continuous on G, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|h(v) - h(u)| < \varepsilon$ holds for all $v \in G$ satisfying $|v - u| < \delta$. Hence, we get

$$|h(v) - h(u)| < \frac{\varepsilon}{4} + \frac{2M}{\delta^2}(v - u)^2, \tag{4}$$

we can express it

$$-\frac{\varepsilon}{4} - \frac{2M}{\delta^2}(v - u)^2 < h(v) - h(u) < \frac{\varepsilon}{4} + \frac{2M}{\delta^2}(v - u)^2.$$

Now, ε can be chosen such that $0 < \varepsilon \leq 1$. By hypothesis, in correspondence with $\min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4M}, \frac{\varepsilon\delta^2}{32MF}\right\}$ and $i = 0, 1, 2$, there are σ_i, scale functions, $\eta_i > 0$ and $S_i \subseteq \mathbb{N}_0$ with $\delta_P(S_i) = 0$ such that

$$|L_\lambda(e_i; u) - e_i(u)| \leq \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4M}, \frac{\varepsilon\delta^2}{32MF}\right\}|\sigma_i(u)|$$

whenever $\lambda \in \mathbb{N}_0 \backslash S_i$, $|u - u_0| \leq \eta_i$. Choose

$$\sigma(u) = \max\{|\sigma_i(u)|; i = 0, 1, 2\}.$$

Then, it is straightforward to see that

$$|L_\lambda(e_i; u) - e_i(u)| \leq \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{4M}, \frac{\varepsilon\delta^2}{32MF}\right\}\sigma(u)$$

whenever $\lambda \in \mathbb{N}_0 \backslash S$ and $|u - u_0| \leq \eta$, where

$$S = \bigcup_{i=0}^{2} S_i \text{ with } \delta_P(S) = 0 \text{ and } \eta = \min\{\eta_i : i = 0, 1, 2\}.$$

We write

$$\begin{aligned} L_\lambda\left((. - u)^2; u\right) &\leq |L_\lambda(e_2; u) - e_2(u)| \\ &+2|u||L_\lambda(e_1; u) - e_1(u)| + \left|u^2\right||L_\lambda(e_0; u) - e_0(u)| \leq \frac{\varepsilon\delta^2}{8M}\sigma(u) \end{aligned}$$

for $\lambda \in \mathbb{N}_0 \backslash S$ and $u \in G$ with $|u - u_0| \leq \eta$. On the other hand, by the linearity and positivity of the operators L_λ and from (4), we have

$$\begin{aligned} |L_\lambda(h; u) - h(u)| &\leq |L_\lambda(h; u) - h(u) L_\lambda(e_0; u)| + |h(u)||L_\lambda(e_0; u) - e_0(u)| \\ &\leq L_\lambda\left(\frac{\varepsilon}{4} + \frac{2M}{\delta^2}\left\{(. - u)^2\right\}; u\right) + M|L_\lambda(e_0; u) - e_0(u)| \\ &= \frac{\varepsilon}{4}L_\lambda(e_0; u) + \frac{2M}{\delta^2}L_\lambda\left((. - u)^2; u\right) + M|L_\lambda(e_0; u) - e_0(u)| \\ &\leq \left\{\frac{\varepsilon}{4} + M\right\}|L_\lambda(e_0; u) - e_0(u)| + \frac{\varepsilon}{4} + \frac{2M}{\delta^2}L_\lambda\left((. - u)^2; u\right) \\ &\leq \varepsilon\sigma(u) \end{aligned}$$

whenever $\lambda \in \mathbb{N}_0 \backslash S$ and $u \in G$ with $|u - u_0| \leq \eta$, which yields the proof. □

It should be noted that we immediately get the following result from the above theorem whenever the scale function replaced by a non-zero constant:

Corollary 1 *Let $\{L_\lambda\}$ be the same as Theorem 2. Then $\{L_\lambda(e_i)\}$ $(i = 0, 1, 2)$ converges $P-$statistically uniformly at u_0 to e_r iff for each $h \in C(G)$, $\{L_\lambda(h)\}$ converges $P-$statistically uniformly at u_0 to h.*

5 An application

In this section, we deal with an example that shows our main Korovkin type approximation theorem is stronger than the corresponding classical ones.

Denote by $C_b(G)$ the space of all real-valued continuous and bounded functions defined on G, and by $C_c(G)$ we denote the subspace of $C_b(G)$ of all functions with compact support on G.

Example 2 Let $G = [0, 1]$, and consider the following linear positive operators:

$$M_\lambda(h; u) = \int_G K_\lambda(v)\, h(vu)\, dv, \lambda \in \mathbb{N}, u \in G,$$

for every $h \in C(G)$, where $K_\lambda(v) = (\lambda + 1)\, v^\lambda$ (see [20]). Then, using the operators M_λ, we define the sequence of positive linear operators $L := \{L_\lambda\}$ on $C_c(G)$ as follows:

$$L_\lambda(h; u) = (1 + g_\lambda(u))\, M_\lambda(h; u),$$

where $\{g_\lambda\}$ is the same as in (1). Let p_λ be the same as in (2) and $\sigma_i = \sigma$ $(i = 0, 1, 2)$, is the same as in (3). We now claim that $\{L_\lambda(e_i)\}$ converges $P-$statistically uniformly at u_0 to e_i with respect to the scale function σ_i. Now, observe that

$$|L_\lambda(e_0; u) - e_0(u)| = g_\lambda(u),$$

$$|L_\lambda(e_1; u) - e_1(u)| \leq \frac{1 + g_\lambda(u)}{\lambda + 2} + g_\lambda(u),$$

$$|L_\lambda(e_2; u) - e_2(u)| \leq \frac{2(1 + g_\lambda(u))}{\lambda + 3} + g_\lambda(u).$$

As we know from Example 1, $\{g_\lambda\}$ converges $P-$ statistically uniformly to $g = 0$ at $u_0 = 0$ with respect to the scale function σ. Hence our claim is true for $i = 0, 1, 2$. Then, from our Theorem 2, we get that $\{L_\lambda(h)\}$ converges $P-$statistically uniformly at u_0 to h with respect to the scale function σ. However, $\{L_\lambda(e_0)\}$ is neither $P-$statistically uniformly nor relatively uniformly convergent at $u_0 = 0$ to e_0. Hence Theorem 1 and Corollary 1 do not work for our operators (It is illustrated for the function $f(u) = u^4 + 3u + 2$ in Fig. 1).

6 Rate of convergence

We present some estimates of rates of $P-$statistical relative uniform convergence at a point for Korovkin-type theorems.

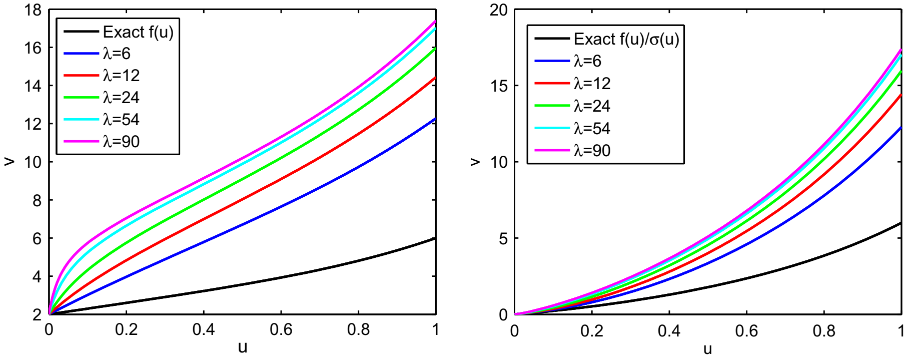

Fig. 1 (Left) The operators $L_\lambda(f;u)$ do not $P-$ statistically uniformly at 0, and hence $L_\lambda(f;u)$ do not $P-$statistically uniformly on $[0,1]$, but (Right) the operators $L_\lambda(f;u)$ with the scale function $\sigma(u)$, converges $P-$statistically uniformly to $f(u)=u^4+3u+2$ at $u_0=0$

As a tool, we use the usual modulus of continuity $\omega(h;\delta)$ defined as follows:

$$\omega(h;\delta) := \sup\{|h(u)-h(v)| : u, v \in G, |u-v| \le \delta\}.$$

where $h \in C_b(G)$ and $\delta > 0$. Observe that $\omega(h;\delta)$ is an increasing function of δ, $|h(u)-h(v)| \le \omega(h;|u-v|)$ for each $u, v \in G$ and $\omega(h;\delta) \le 2M$ for every δ, where $M = \sup_{u\in G}|h(u)|$ and

$$\omega(h;\gamma\delta) \le (1+\gamma)\,\omega(h;\delta)$$

for every $\gamma, \delta > 0$ (see also [2]).

Let's now present the following theorem.

Theorem 3 *Let $\{L_\lambda\}$ be a sequence of positive linear operators acting from $C(G)$ into itself. Assume that the following conditions hold:*

(a) $\{L_\lambda(e_0)\}$ converges $P-$statistically uniformly to e_0 at u_0 with respect to the scale function σ_0,

(b) $st_P - \lim \frac{\omega(h;\delta_\lambda)}{|\sigma_1(u)|} = 0$ for each $u \in G$ where $\delta_\lambda := \sqrt{L_\lambda\left((.-u)^2;u\right)}$.

Then we have, for all $h \in C(G)$, $\{L_\lambda(h)\}$ converges statistically uniformly to h at u_0 with respect to the scale function σ, where $\sigma(u) = \max\{|\sigma_i(u)| : i = 0, 1\}$.

Proof Let $u \in G$ and $h \in C(G)$ be fixed. By the linearity and the positivity of the operators L_λ, for all $\lambda \in \mathbb{N}_0$ and any $\delta > 0$, we have

$$\begin{aligned}
|L_\lambda(h;u) - h(u)| &\le |L_\lambda(h;u) - h(u)L_\lambda(e_0;u)| + |h(u)|\,|L_\lambda(e_0;u) - e_0(u)| \\
&\le L_\lambda\left(\left(1+\frac{(.-u)^2}{\delta^2}\right)\omega(h;\delta);u\right) \\
&\quad + |h(u)|\,|L_\lambda(e_0;u) - e_0(u)| \\
&= \omega(h;\delta)\,L_\lambda(e_0;u) + \frac{\omega(h;\delta)}{\delta^2}L_\lambda\left((.-u)^2;u\right) \\
&\quad + |h(u)|\,|L_\lambda(e_0;u) - e_0(u)|.
\end{aligned}$$

Put $\delta := \delta_\lambda = \sqrt{L_\lambda\left((. - u)^2 ; u\right)}$. Hence, we get

$$\frac{|L_\lambda(h; u) - h(u)|}{\sigma(u)} \leq [\omega(h; \delta_\lambda) + |h(u)|] \left| \frac{L_\lambda(e_0; u) - e_0(u)}{\sigma_0(u)} \right| + 2\frac{\omega(h; \delta_\lambda)}{|\sigma_1(u)|}.$$

By using (a) and (b) the proof is completed. □

Availability of data and material Not applicable

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