Rendiconti del Circolo Matematico di Palermo Series 2 (2023) 72:1125–1141
https://doi.org/10.1007/s12215-021-00681-z

Korovkin theory via P_p–statistical relative modular convergence for double sequences

Sevda Yıldız[1] · Kamil Demirci[1] · Fadime Dirik[1]

Received: 10 August 2021 / Accepted: 1 October 2021 / Published online: 7 February 2022

Abstract
In this paper, we introduce a new type of statistical relative modular convergence for the first time and we obtain Korovkin theorems for double sequences of positive linear operators on modular spaces in the sense of this new statistical type convergence with respect to power series method. Finally, we present an interesting application that satisfies our new approximation theorem but not satisfies the classical ones. So, we show that our results are meaningful.

Keywords Power series method · Double sequence · Korovkin theorem · Modular space · Statistical convergence

Mathematics Subject Classification 40A35 · 40G10 · 41A36 · 46E30

1 Introduction

The classical version of Bohman-Korovkin theorem establishes the uniform convergence in the space $C([a, b])$ of all continuous real functions defined on the interval $[a, b]$, for a sequence of positive linear operators (T_n) acting on $C([a, b])$, assuming the convergence only on the test functions $1, x, x^2$ ([1–3]). The list of variations and extensions of this result is very long. General approaches to the Korovkin type approximation theorems were studied by using summability methods and also, different convergence methods, mostly in the direction of statistical convergence, which was introduced independently by Fast [4] and Steinhaus [5]. The statistical Korovkin theorem studied by Gadjiev and Orhan [6], for the first time, and then Dirik and Demirci [7] proved the statistical Korovkin theorem for double sequences. Korovkin theorems were first studied on modular spaces (these spaces generalize the classical Lebesgue and Orlicz spaces) in [8] and studied by researchers (e.g., [9–12]). The definition of statistical relative convergence and the use of this convergence in the Korovkin theorem were first studied by Demirci and Orhan [13]. After this paper, many authors study this type of convergence and its generalization (e.g., [14–18]). The power series methods, including Abel and

✉ Sevda Yıldız
sevdaorhan@sinop.edu.tr

[1] Department of Mathematics, Sinop University, Sinop, Turkey

Borel methods, are well known. These methods are also members of the class of all continuous summability methods and they are considered in the Korovkin type approximation theory (e.g., [19–21]). In the present paper, we introduce the statistical relative modular convergence with respect to power series method and we obtain Korovkin theorems for double sequences of positive linear operators on modular spaces in the sense of this convergence. Finally, we present an interesting application that satisfies our new approximation theorem but not satisfies the classical ones.

2 Preliminaries

We begin with the concepts of Pringsheim convergence and statistical convergence for double sequences. The most basic statements reads as follows.

A double sequence $x = \left(x_{mn}\right)$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon)$, belonging to the set of all natural numbers, such that $\left|x_{mn} - L\right| < \varepsilon$ whenever $m, n > N$, where L is called the Pringsheim limit of x and denoted by $P - \lim_{m,n} x_{mn} = L$ (see [22]). We shall call such an x, briefly, "$P-$ convergent". A double sequence is called bounded if there exists a positive number M such that $\left|x_{mn}\right| \leq M$ for all $(m, n) \in \mathbb{N}^2$. Unlike a convergent single sequence, a convergent double sequence need not be bounded.

Statistical convergence of single sequences was introduced by Fast [4] and Steinhaus [5], independently and studied by many authors. This concept was extended to the double sequences by Moricz [23]. If $F \subset \mathbb{N}^2$ be a two-dimensional subset of positive integers and let $|.|$ denote the cardinality of the set. The double natural density of F is given by

$$\delta_2(F) := P - \lim_{k,l} \frac{|\{m \leq k,\ n \leq l : (m, n) \in F\}|}{kl},$$

if it exists. The number sequence $x = \left(x_{mn}\right)$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$F := F_{kl}(\varepsilon) := \left\{m \leq k,\ n \leq l : \ \left|x_{mn} - L\right| \geq \varepsilon\right\}$$

has natural density zero; in that case we write $st_2 - \lim_{m,n} x_{mn} = L$. It is a simple matter to see that, a P–convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, as regards statistical convergence, note that a statistically convergent double sequence need not to be bounded. For more details we refer the reader to [23].

Also, remember that the statistical convergence for double sequences was characterized in [23] as given below:

A double sequence $x = \left(x_{mn}\right)$ is statistically convergent to L iff there exists a set $F \subset \mathbb{N}^2$ such that the double natural density of F is 1 and

$$P - \lim_{m,n \to \infty \text{ and } (m,n) \in F} x_{mn} = L.$$

Let us turn our attention to power series method.

Let $\left(p_{mn}\right)$ be a double sequence of nonnegative numbers with $p_{00} > 0$ and such that the following power series

$$p(t,s) := \sum_{m,n=0}^{\infty} p_{mn} t^m s^n$$

has radius of convergence R with $R \in (0, \infty]$ and $t, s \in (0, R)$. If for all $t, s \in (0, R)$,

$$\lim_{t,s \to R^-} \frac{1}{p(t,s)} \sum_{m,n=0}^{\infty} p_{mn} t^m s^n x_{mn} = L$$

then we say that the double sequence $x = (x_{mn})$ is convergent to L in the sense of power series method and denoted by $P_p^2 - \lim x_{mn} = L$ ([24]). The power series method for double sequences is regular if and only if

$$\lim_{t,s \to R^-} \frac{\sum_{m=0}^{\infty} p_{m\nu} t^m}{p(t,s)} = 0 \text{ and } \lim_{t,s \to R^-} \frac{\sum_{n=0}^{\infty} p_{\mu n} s^n}{p(t,s)} = 0, \text{ for any } \mu, \upsilon, \tag{1}$$

hold [24].

Remark 1 Note that in case of $R = 1$, if $p_{mn} = 1$ and $p_{mn} = \frac{1}{(m+1)(n+1)}$, the power series methods coincide with Abel summability method and logarithmic summability method, respectively. In the case of $R = \infty$ and $p_{mn} = \frac{1}{m!n!}$, the power series method coincides with Borel summability method.

Here and throughout the paper, power series method is always assumed to be regular.

Prior to introducing the next definitions, it is worthwhile to point out that, Ünver and Orhan [25] have recently introduced P_p-density of $E \subset \mathbb{N}_0$ and the definition of P_p-statistical convergence for single sequences. Hence, they showed that statistical convergence and P_p- statistical convergence are incompatible. In view of their work, we introduce the definitions of P_p^2-density of $F \subset \mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$ and P_p^2-statistical convergence for double sequences:

Definition 1 Let $F \subset \mathbb{N}_0^2$. If the limit

$$\delta_{P_p}^2(F) := \lim_{t,s \to R^-} \frac{1}{p(t,s)} \sum_{(m,n) \in F} p_{mn} t^m s^n$$

exists, then $\delta_{P_p}^2(F)$ is called the P_p^2- density of F. Note that, from the definition of a power series method and P_p^2-density it is obvious that $0 \le \delta_{P_p}^2(F) \le 1$ whenever it exists.

Definition 2 Let $x = (x_{mn})$ be a double sequence. Then x is said to be statistically convergent with respect to power series method (P_p^2- statistically convergent) to L if for any $\varepsilon > 0$

$$\lim_{t,s \to R^-} \frac{1}{p(t,s)} \sum_{(m,n) \in F_\varepsilon} p_{mn} t^m s^n = 0$$

where $F_\varepsilon = \left\{(m,n) \in \mathbb{N}_0^2 : |x_{mn} - L| \ge \varepsilon\right\}$, that is $\delta_{P_p}^2(F_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case we write $st_{P_p}^2 - \lim x_{mn} = L$.

The statistical superior limit and inferior limit for single sequences have been first given by Fridy and Orhan in [26]. Then, Çakan and Altay [27] have generalized these concepts to double sequences.

Now, in view of these studies, we can introduce the concepts of P_p^2- *statistical superior limit* and *inferior limit* for double sequences. The P_p^2−statistical superior limit of $x = \left(x_{mn}\right)$ is

$$st_{P_p}^2 - \limsup x_{mn} = \begin{cases} \sup G_x, & \text{if } G_x \neq \emptyset, \\ -\infty, & \text{if } G_x = \emptyset, \end{cases}$$

where $G_x := \left\{c \in \mathbb{R} : \delta_{P_p}^2\left(\left\{(m,n) : x_{mn} > c\right\}\right) > 0 \text{ or does not exist in } \mathbb{R}\right\}$ and $\emptyset$ denotes the empty set. Similarly, the P_p^2−statistical inferior limit of x is

$$st_{P_p}^2 - \liminf x_{mn} = \begin{cases} \inf H_x, & \text{if } H_x \neq \emptyset, \\ \infty, & \text{if } H_x = \emptyset, \end{cases}$$

where $H_x := \left\{d \in \mathbb{R} : \delta_{P_p}^2\left(\left\{(m,n) : x_{mn} < d\right\}\right) > 0 \text{ or does not exist in } \mathbb{R}\right\}$. As in the statistical superior or inferior limit, it is obtained that

$$st_{P_p}^2 - \liminf x_{mn} \leq st_{P_p}^2 - \limsup x_{mn}$$

and also that, for any double sequence $x = \left(x_{mn}\right)$ satisfying
$\delta_{P_p}^2\left(\{(m,n) : \left|x_{mn}\right| > M\}\right) = 0$ for some $M > 0$,

$$st_{P_p}^2 - \lim x_{mn} = L \text{ iff } st_{P_p}^2 - \liminf x_{mn} = st_{P_p}^2 - \limsup x_{mn} = L.$$

Now, according to our aim for this work, we shall focus our attention on modular spaces and convergence methods in these spaces. Let us mention some well known notations and properties of modular spaces.

Let $Z = [a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, by $X\left(Z^2\right)$ we denote the space of all real-valued measurable functions on Z^2 provided with equality *a.e.* As usual, let $C\left(Z^2\right)$ denote the space of all continuous real-valued functions, and $C^\infty\left(Z^2\right)$ denote the space of all infinitely differentiable functions on Z^2. Then, we say that a functional $\rho : X\left(Z^2\right) \to [0, +\infty]$ is called a *modular* on $X\left(Z^2\right)$ if (i) $\rho(f) = 0$ if and only if $f = 0$*a.e.* in Z^2, (ii) $\rho(-f) = \rho(f)$ for every $f \in X\left(Z^2\right)$ and (iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $f, g \in X(Z^2)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular ρ is said to be $N-$*quasi convex* if there exists a constant $N \geq 1$ such that $\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng)$ holds for every $f, g \in X\left(Z^2\right)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*. A modular ρ is said to be $N-$*quasi semiconvex* if there exists a constant $N \geq 1$ such that $\rho(af) \leq Na\rho(Nf)$ holds for every $f \in X\left(Z^2\right)$ and $a \in (0, 1]$. It is clear that every $N-$quasi convex modular is $N-$quasi semiconvex. Please see details in [10, 28].

Some vector subspaces of $X(Z^2)$ are considered by means of a modular ρ as follows:

$$L^\rho\left(Z^2\right) := \left\{f \in X(Z^2) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0\right\}$$

and

$$E^\rho\left(Z^2\right) := \left\{f \in L^\rho\left(Z^2\right) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0\right\}$$

are called the *modular space* generated by ρ and the space of the finite elements of $L^\rho(Z^2)$, respectively. It is worthwhile to point out that when ρ is $N-$quasi semiconvex, then the space

$$\{f \in X(Z^2) \,:\, \rho(\lambda f) < +\infty \ \text{ for some } \lambda > 0\}$$

coincides with $L^\rho(Z^2)$. It is worth noting that, the main features of modulars can be found in [28, 29] (see also [30, 31]).

Now, we recall the notion of statistical modular convergence for double sequences.

Definition 3 [12] Let (f_{mn}) be a double function sequence whose terms belong to $L^\rho(Z^2)$. Then, (f_{mn}) is said to be*statistically modularly convergent* to a function $f \in L^\rho(Z^2)$ iff

$$st_2 - \lim_{m,n} \rho\left(\lambda_0\left(f_{mn} - f\right)\right) = 0 \ \text{ for some } \lambda_0 > 0. \tag{2}$$

In addition, if (2) is provided for all $\lambda > 0$, then it is called *statistical F−norm convergence* (or, *statistical strong convergence*), i.e.,

$$st_2 - \lim_{m,n} \rho\left(\lambda\left(f_{mn} - f\right)\right) = 0 \ \text{ for every } \lambda > 0.$$

The two notions of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \le M\rho(f)$ for every $f \in X(Z^2)$ (see [29]).

Now we introduce the convergence methods in the sense of power series method in modular spaces for double sequences:

Definition 4 Let (f_{mn}) be a double function sequence whose terms belong to $L^\rho(Z^2)$. Then, (f_{mn}) is said to be P_p^2-statistically relatively modularly convergent to a function $f \in L^\rho(Z^2)$ iff there exists a scale function $\sigma \in L^0(Z^2), |\sigma(x,y)| \neq 0$ such that

$$st^2_{P_p} - \lim \rho\left(\lambda_0\left(\frac{f_{mn} - f}{\sigma}\right)\right) = 0, \ \text{ for some } \lambda_0 > 0. \tag{3}$$

In addition, if (3) is satisfied for all $\lambda > 0$, then it is called P_p^2*−statistical relative F−norm convergence* (or, P_p^2*−statistical relative strong convergence*), i.e.,

$$st^2_{P_p} - \lim \rho\left(\lambda\left(\frac{f_{mn} - f}{\sigma}\right)\right) = 0 \ \text{ for every } \lambda > 0. \tag{4}$$

It can be immediately seen that (3) and (4) are equivalent if and only if the modular ρ satisfies the Δ_2-condition. Indeed, P_p^2-statistical relative strong convergence of the double sequence (f_{mn}) to f is equivalent to the condition $st^2_{P_p} - \lim_{m,n} \rho\left(2^N\lambda\left(\frac{f_{mn}-f}{\sigma}\right)\right) = 0$, for all $N = 1, 2, \ldots$ and some $\lambda > 0$. Let (f_{mn}) be P_p^2-statistically relatively modularly convergent to f, hence, there exists a $\lambda > 0$ such that $st^2_{P_p} - \lim_{m,n} \rho\left(\lambda\left(\frac{f_{mn}-f}{\sigma}\right)\right) = 0$. Also, Δ_2-condition implies by induction that $\rho\left(2^N\lambda\left(\frac{f_{mn}-f}{\sigma}\right)\right) \le M^N\rho\left(\lambda\left(\frac{f_{m,}-f}{\sigma}\right)\right)$. Then, we get

$$st^2_{P_p} - \lim \rho\left(2^N\lambda\left(\frac{f_{mn} - f}{\sigma}\right)\right) = 0.$$

Also, it will be observed that P_p^2–statistical modular convergence is the special case of P_p^2–statistical relative modular convergence in which the scale function is a non-zero constant.

Below we will present a striking example that our new definition makes more sense.

Example 1 *Take $Z^2 = [0,1]^2$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous function for which the following conditions hold:*

- *φ is convex,*
- *$\varphi(0) = 0, \varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.*

Hence, consider the functional ρ^φ on $L^0\left(Z^2\right)$ defined by

$$\rho^\varphi(f) := \int_0^1 \int_0^1 \varphi(|f(x,y)|)dxdy \quad for\, f \in X(Z^2).$$

In this case, ρ^φ is a convex modular on $X(Z^2)$, which satisfies all assumptions listed in this section. Consider the Orlicz space generated by φ as follows:

$$L_\varphi^\rho(Z^2) := \left\{ f \in X(Z^2) : \rho^\varphi(\lambda f) < +\infty \text{ for some } \lambda > 0 \right\}.$$

For each $m, n \in \mathbb{N}\, (m \geq 2, n \geq 2)$, define $g_{mn} : Z^2 \to \mathbb{R}$ by

$$g_{mn}(x,y) = \begin{cases} m^2n^2xy, & (x,y) \in \left[0, \frac{1}{m}\right] \times \left[0, \frac{1}{n}\right]; \\ & m = 2k \text{ or } n = 2l, \\ 4mn - m^2n^2xy, & (x,y) \in \left(\frac{1}{m}, \frac{2}{m}\right] \times \left(\frac{1}{n}, \frac{2}{n}\right]; \\ & m = 2k \text{ or } n = 2l, \\ 0, & (x,y) \in \left(\frac{2}{m}, 1\right] \times \left(\frac{2}{n}, 1\right]; \\ & m = 2k \text{ or } n = 2l, \\ 1, & \text{otherwise}, \end{cases} \tag{5}$$

$k, l = 1, 2, ...$

If $\varphi(x) = x^p$ for $1 \leq p < \infty, x \geq 0$, then $L_\varphi^\rho\left(Z^2\right) = L_p\left(Z^2\right)$. Moreover, we have for any function $f \in L_\varphi^\rho\left(Z^2\right)$ that $\rho^\varphi(f) = \|f\|_{L_p}^p$. Also, let the power series method be given with

$$p_{mn} = \begin{cases} 0, & m = 2k+1 \text{ and } n = 2l+1 \\ 1, & m = 2k \text{ or } n = 2l \end{cases}, \; k, l = 1, 2, ...$$

It is clear that $\left(g_{mn}\right)$ converges neither statistically modularly nor modularly. Also, $\left(g_{mn}\right)$ does not converge P_p^2–statistically modularly to $g = 0$ but converges P_p^2–statistically modularly relative to a scale function

$$\sigma(x,y) = \begin{cases} 1, & x = 0 \text{ or } y = 0 \\ \frac{1}{xy}, & (x,y) \in (0,1] \times (0,1] \end{cases}$$

on $L_1\left(Z^2\right)$. Indeed, for some $\lambda_0 > 0$, with the choice of $p = 1$ we have $\rho^\varphi(g) = \|g\|_{L_1}$,

$$\rho\left(\lambda_0\left(g_{mn}-g\right)\right)=\left\|\lambda_0\left(g_{mn}-g\right)\right\|_{L_1}$$
$$=\lambda_0\begin{cases}1, & m=2k+1 \text{ and } n=2l+1\\ 2, & m=2k \text{ or } n=2l\end{cases}, k,l=1,2,.... \tag{6}$$

Thus, we can easily have that, $\left(g_{mn}\right)$ does not converge statistically modularly and P_p^2- statistically modularly. Now, using the scale function σ,

$$\rho\left(\lambda_0\left(\frac{g_{mn}-g}{\sigma}\right)\right)=\left\|\lambda_0\left(\frac{g_{mn}-g}{\sigma}\right)\right\|_{L_1}=\lambda_0\begin{cases}\frac{1}{4}, & m=2k+1 \text{ and } n=2l+1\\ \frac{11}{3mn}, & m=2k \text{ or } n=2l\end{cases}, \ k,l=1,2,...,$$

then, we get

$$st_{P_p}^2-\lim\rho\left(\lambda_0\left(\frac{g_{mn}-g}{\sigma}\right)\right)=0.$$

On the other hand, thanks to (6), we can easily see that $\left(g_{mn}\right)$ neither converges to $g=0$ statistically modularly nor P_p^2- statistically modularly on $L_1\left(Z^2\right)$.

We conclude the section by mentioning the following assumptions concerning modular functionals:

A modular ρ is *monotone* if $\rho(f)\leq\rho(g)$ for $|f|\leq|g|$, ρ is said to be *finite* if $\chi_A\in L^\rho\left(Z^2\right)$ whenever A is measurable subset of Z^2 such that $\mu(A)<\infty$. If ρ is finite and, for every $\varepsilon>0$, $\lambda>0$, there exists a $\delta>0$ such that $\rho\left(\lambda\chi_B\right)<\varepsilon$ for any measurable subset $B\subset Z^2$ with $\mu(B)<\delta$, then ρ is *absolutely finite* and if $\chi_{G^2}\in E^\rho\left(Z^2\right)$, then ρ is *strongly finite*. A modular ρ is *absolutely continuous* provided that there exists an $\alpha>0$ such that, for every $f\in X\left(Z^2\right)$ with $\rho(f)<+\infty$, the following condition holds:

- for every $\varepsilon>0$ there is $\delta>0$ such that $\rho\left(\alpha f\chi_B\right)<\varepsilon$ whenever B is any measurable subset of Z^2 with $\mu(B)<\delta$.

Based upon above concepts (see [8, 10]) if a modular ρ is monotone and finite, then we have $C(Z^2)\subset L^\rho\left(Z^2\right)$. Similarly, if ρ is monotone and strongly finite, then $C(Z^2)\subset E^\rho\left(Z^2\right)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $C^\infty\left(Z^2\right)=L^\rho\left(Z^2\right)$. Some details of above properties may be found in [28, 31–33].

The following section is devoted to present Korovkin type approximation theorems via the statistical relative modular convergence with respect to power series method and to give some related results.

3 Korovkin theorems via power series method

Let ρ be a monotone and finite modular on $X\left(Z^2\right)$. Assume that D is a set satisfying $C^\infty\left(Z^2\right)\subset D\subset L^\rho\left(Z^2\right)$. We will assume that $\mathbb{T}:=\left(T_{mn}\right)$ is a sequence of positive linear operators from D into $X\left(Z^2\right)$ for which there exists a subset $X_{\mathbb{T}}\subset D$ with $C^\infty\left(Z^2\right)\subset X_{\mathbb{T}}$ and an unbounded function $\sigma\in X\left(Z^2\right)$ satisfying $\sigma(x,y)\neq 0$ such that

$$st_{P_p}^2-\limsup\rho\left(\lambda\left(\frac{T_{mn}(h)}{\sigma}\right)\right)\leq R\rho(\lambda h), \tag{7}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R.

Throughout the paper we use the test functions f_r $(r = 0, 1, 2, 3)$ defined by

$$f_0(x, y) = 1, f_1(x, y) = x, f_2(x, y) = y \text{ and } f_3(x, y) = x^2 + y^2.$$

Theorem 1 *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular on $X\left(Z^2\right)$. Let $\mathbb{T} := \left(T_{mn}\right)$ be a sequence of positive linear operators from D into $X\left(Z^2\right)$ satisfying (7) holds for each $f \in D$. Moreover, suppose that σ and σ_r are unbounded scale functions satisfying $|\sigma(x, y)| \geq a > 0$ and $\left|\sigma_r(x, y)\right| \geq a_r > 0$ $(r = 0, 1, 2, 3)$. Suppose that*

$$st^2_{P_p} - \lim \rho\left(\lambda\left(\frac{T_{mn}\left(f_r\right) - f_r}{\sigma_r}\right)\right) = 0, \text{ for every } \lambda > 0 \text{ and } r = 0, 1, 2, 3. \tag{8}$$

Now let f be any function belonging to $L^\rho\left(Z^2\right)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty\left(Z^2\right)$. Then we have

$$st^2_{P_p} - \lim \rho\left(\lambda_0\left(\frac{T_{mn}(f) - f}{\sigma}\right)\right) = 0, \quad \text{for some } \lambda_0 > 0. \tag{9}$$

Proof We first claim that

$$st^2_{P_p} - \lim \rho\left(\eta\left(\frac{T_{mn}(g) - g}{\sigma}\right)\right) = 0 \tag{10}$$

for every $g \in C(Z^2) \cap D$ and $\eta > 0$ where $\sigma(x, y) = \max\left\{\left|\sigma_r(x, y)\right|; r = 0, 1, 2, 3\right\}$. To see this it is enough to assume that g belongs to $C\left(Z^2\right) \cap D$ and η is any positive number. By the continuity of g on Z^2 and by the linearity and positivity of the operators T_{mn}, we easily see that (see, for instance [34]), for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $(u, v), (x, y) \in Z^2$

$$|g(u, v) - g(x, y)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u - x)^2 + (v - y)^2\right\}$$

namely

$$-\varepsilon - \frac{2M}{\delta^2}\left\{(u - x)^2 + (v - y)^2\right\} < g(u, v) - g(x, y) < \varepsilon + \frac{2M}{\delta^2}\left\{(u - x)^2 + (v - y)^2\right\}$$

where $M := \sup_{(x,y)\in Z^2} |g(x, y)|$. Let $\max\{|x|, |y|\} \leq |a| + |b|$, and put $c := \max\{|x|, |y|\}$. Since T_{mn} is linear and positive, we have

$$\begin{aligned}
&\left|T_{mn}(g; x, y) - g(x, y)\right| \\
&\leq \varepsilon + K\left\{\left|T_{mn}\left(f_0; x, y\right) - f_0(x, y)\right| + \left|T_{mn}\left(f_1; x, y\right) - f_1(x, y)\right|\right. \\
&\quad \left. + \left|T_{mn}\left(f_2; x, y\right) - f_2(x, y)\right| + \left|T_{mn}\left(f_3; x, y\right) - f_3(x, y)\right|\right\}
\end{aligned}$$

where $K := \max\left\{\varepsilon + M + \frac{4Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$. Now multiplying both sides of the above inequality by $\frac{1}{|\sigma(x,y)|}$, we get

$$\left|\frac{T_{mn}(g;x,y)-g(x,y)}{\sigma(x,y)}\right| \le \frac{\varepsilon}{|\sigma(x,y)|} + \frac{K}{|\sigma(x,y)|}\Big\{ \left|T_{mn}\left(f_0;x,y\right)-f_0(x,y)\right| + \left|T_{mn}\left(f_1;x,y\right)-f_1(x,y)\right| + \left|T_{mn}\left(f_2;x,y\right)-f_2(x,y)\right| + \left|T_{mn}\left(f_3;x,y\right)-f_3(x,y)\right| \Big\}.$$

Hence, the last inequality gives, for any $\eta > 0$, that

$$\begin{aligned} &\eta\left|\frac{T_{mn}(g;x,y)-g(x,y)}{\sigma(x,y)}\right| \\ &\le \frac{\eta\varepsilon}{|\sigma(x,y)|} + \eta K\left\{ \left|\frac{T_{mn}\left(f_0;x,y\right)-f_0(x,y)}{\sigma_0(x,y)}\right| \right. \\ &+ \left|\frac{T_{mn}\left(f_1;x,y\right)-f_1(x,y)}{\sigma_1(x,y)}\right| + \left|\frac{T_{mn}\left(f_2;x,y\right)-f_2(x,y)}{\sigma_2(x,y)}\right| \\ &\left. + \left|\frac{T_{mn}\left(f_3;x,y\right)-f_3(x,y)}{\sigma_3(x,y)}\right| \right\}. \end{aligned}$$

Thanks to the above inequality, since ρ is monotone, we get

$$\begin{aligned} \rho\left(\eta\left(\frac{T_{mn}(g)-g}{\sigma}\right)\right) \le{}& \rho\left(\frac{\eta\varepsilon}{\sigma} + \eta K\left(\frac{T_{mn}\left(f_0\right)-f_0}{\sigma_0}\right) + \eta K\left(\frac{T_{mn}\left(f_1\right)-f_1}{\sigma_1}\right)\right. \\ &\left. + \eta K\left(\frac{T_{mn}\left(f_2\right)-f_2}{\sigma_2}\right) + \eta K\left(\frac{T_{mn}\left(f_3\right)-f_3}{\sigma_3}\right)\right). \end{aligned}$$

Since ρ is N–quasi semiconvex and strongly finite, we may write that, assuming $0 < \varepsilon \le 1$

$$\begin{aligned} \rho\left(\eta\left(\frac{T_{mn}(g)-g}{\sigma}\right)\right) \le{}& N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right) + \rho\left(5\eta K\left(\frac{T_{mn}\left(f_0\right)-f_0}{\sigma_0}\right)\right) \\ &+ \rho\left(5\eta K\left(\frac{T_{mn}\left(f_1\right)-f_1}{\sigma_1}\right)\right) \\ &+ \rho\left(5\eta K\left(\frac{T_{mn}\left(f_2\right)-f_2}{\sigma_2}\right)\right) \\ &+ \rho\left(5\eta K\left(\frac{T_{mn}\left(f_3\right)-f_3}{\sigma_3}\right)\right). \end{aligned}$$

For a given $\varepsilon^* > 0$, choose an $\varepsilon \in (0, 1]$ such that $N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right) < \varepsilon^*$. Now we define the following sets:

$$E_\eta : \quad = \left\{ (m,n) : \rho\left(\eta\left(\frac{T_{mn}(g)-g}{\sigma}\right)\right) \geq \varepsilon^* \right\},$$

$$E_{\eta,r} : \quad = \left\{ (m,n) : \rho\left(4\eta K\left(\frac{T_{mn}\left(f_r\right)-f_r}{\sigma_r}\right)\right) \geq \frac{\varepsilon^* - N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)}{4} \right\},$$

$r = 0,1,2,3$. Then, it is easy to see that $E_\eta \subseteq \bigcup_{r=0}^{3} E_{\eta,r}$. So, we can write that

$$\delta^2_{P_p}\left(E_\eta\right) \leq \sum_{r=0}^{3} \delta^2_{P_p}\left(E_{\eta,r}\right).$$

It is evident from hypothesis (8), we get

$$\delta^2_{P_p}\left(E_\eta\right) = 0,$$

which proves our claim (10). Observe that (10) also holds for every $g \in C^\infty(Z^2)$. Now let $f \in L^\rho\left(Z^2\right)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty\left(Z^2\right)$. It is evident from $\mu\left(Z^2\right) < \infty$ and strong finitenessand absolute continuousness of ρ that ρ is also absolutely finite on $X(Z^2)$ (see [32]). Using these properties of the modular ρ, it is known from [28, 33] that the space $C^\infty(Z^2)$ is modularly dense in $L^\rho\left(Z^2\right)$, i.e., there exists a sequence $\left(g_{mn}\right) \subset C^\infty\left(Z^2\right)$ such that

$$P - \lim_{m,n} \rho\left(3\lambda_0^*\left(g_{mn} - f\right)\right) = 0 \text{ for some } \lambda_0^* > 0.$$

This means that, for every $\varepsilon > 0$, there are positive numbers $m_0 = m_0(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*\left(g_{mn} - f\right)\right) < \varepsilon \quad \text{for every } m \geq m_0 \text{ and } n \geq n_0. \tag{11}$$

On the other hand, by the linearity and positivity of the operators T_{mn}, we may write that

$$\begin{aligned}
\lambda_0^*\left|T_{mn}(f;x,y) - f(x,y)\right| \leq & \lambda_0^*\left|T_{mn}(f - g_{m_0n_0};x,y)\right| \\
& + \lambda_0^*\left|T_{mn}(g_{m_0n_0};x,y) - g_{m_0n_0}(x,y)\right| \\
& + \lambda_0^*\left|g_{m_0n_0}(x,y) - f(x,y)\right|,
\end{aligned}$$

holds for every $x, y \in Z$. Multiplying the last inequality by $\frac{1}{|\sigma(x,y)|}$, applying the modular ρ and moreover considering the monotonicity of ρ, we have

$$\begin{aligned}
\rho\left(\lambda_0^*\left(\frac{T_{mn}(f) - f}{\sigma}\right)\right) \leq & \rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(f - g_{m_0n_0}\right)}{\sigma}\right)\right) \\
& + \rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(g_{m_0n_0}\right) - g_{m_0n_0}}{\sigma}\right)\right) \\
& + \rho\left(3\lambda_0^*\left(\frac{g_{m_0n_0} - f}{\sigma}\right)\right).
\end{aligned}$$

Hence, assuming that $|\sigma| \geq a > 0$ ($a = \max\{a_r : r = 0, 1, 2, 3\}$) and taking into consideration that ρ is monotone, we may write that

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right) \leq &\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(f-g_{m_0n_0}\right)}{\sigma}\right)\right)\\ &+\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(g_{m_0n_0}\right)-g_{m_0n_0}}{\sigma}\right)\right)\\ &+\rho\left(\frac{3\lambda_0^*}{a}\left(g_{m_0n_0}-f\right)\right).\end{aligned} \tag{12}$$

Then, it follows from (11) and (12) that

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right) \leq &\varepsilon+\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(f-g_{m_0n_0}\right)}{\sigma}\right)\right)\\ &+\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(g_{m_0n_0}\right)-g_{m_0n_0}}{\sigma}\right)\right).\end{aligned} \tag{13}$$

So, taking P_p^2–statistical limit superior in both sides of (13) and also using the facts that $g_{m_0n_0} \in C^\infty(Z^2)$ and $f - g_{m_0n_0} \in X_{\mathbb{T}}$, we obtained from (7) that

$$\begin{aligned}&st_{P_p}^2-\limsup\rho\left(\lambda_0^*\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right)\\ &\leq \varepsilon+R\rho\left(3\lambda_0^*(f-g_{k_0,l_0})\right)\\ &\quad+st_{P_p}^2-\limsup\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(g_{m_0n_0}\right)-g_{m_0n_0}}{\sigma}\right)\right),\end{aligned}$$

which gives that

$$\begin{aligned}&st_{P_p}^2-\limsup\rho\left(\lambda_0^*\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right)\\ &\quad\leq \varepsilon(R+1)+st_{P_p}^2-\limsup\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(g_{m_0n_0}\right)-g_{m_0n_0}}{\sigma}\right)\right).\end{aligned} \tag{14}$$

By (10), since

$$st_{P_p}^2-\lim\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(g_{m_0n_0}\right)-g_{m_0n_0}}{\sigma}\right)\right)=0,$$

we get

$$st_{P_p}^2-\limsup\rho\left(3\lambda_0^*\left(\frac{T_{mn}\left(g_{m_0n_0}\right)-g_{m_0n_0}}{\sigma}\right)\right)=0. \tag{15}$$

Combining (14) with (15), we conclude that

$$st^2_{P_p} - \limsup \rho\left(\lambda_0^*\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right) \leq \varepsilon(P+1).$$

Since $\varepsilon > 0$ was arbitrary and $\rho\left(\lambda_0^*\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right)$ is non-negative for all $m, n \in \mathbb{N}$, we can easily see that

$$st^2_{P_p} - \lim \rho\left(\lambda_0^*\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right) = 0,$$

that is, the assertion. □

If the modular ρ satisfies the Δ_2-condition, then one can get immediately the following result from Theorem 1.

Theorem 2 *Let $\mathbb{T} := \left(T_{mn}\right)$, ρ and σ be the same as in Theorem 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent*:

(a) $st^2_{P_p} - \lim \rho\left(\lambda\left(\frac{T_{mn}(f_r)-f_r}{\sigma_r}\right)\right) = 0$, for every $\lambda > 0$ and $r = 0, 1, 2, 3$,

(b) $st^2_{P_p} - \lim \rho\left(\lambda\left(\frac{T_{mn}(f)-f}{\sigma}\right)\right) = 0$, *for every $\lambda > 0$ provided that f is any function belonging to $L^\rho(Z^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty\left(Z^2\right)$.*

We conclude this section by following Corollaries.

If one replaces the scale function by a nonzero constant, then the condition (7) reduces to

$$st^2_{P_p} - \limsup \rho\left(\lambda\left(T_{mn}h\right)\right) \leq R\rho(\lambda h) \tag{16}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, we obtain the next results from our main theorems, Theorems 1 and 2.

Corollary 1 *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular on $X\left(Z^2\right)$. Let $\mathbb{T} := \left(T_{mn}\right)$ be a double sequence of positive linear operators from D into $X\left(Z^2\right)$ satisfying (16). If $\left(T_{mn}f_r\right)$ is P_p^2-statistically strongly convergent to f_r for each $r = 0, 1, 2, 3$, then $\left(T_{mn}f\right)$ is P_p^2-statistically modularly convergent to f provided that f is any function belonging to $L^\rho\left(Z^2\right)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty\left(Z^2\right)$.*

Corollary 2 *Let $\mathbb{T} := \left(T_{mn}\right)$ and ρ be the same as in Corollary 1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent*:

(a) $\left(T_{mn}f_r\right)$ is P_p^2-statistically strongly convergent to f_r for each $r = 0, 1, 2, 3$,
(b) $\left(T_{mn}f\right)$ is P_p^2-statistically strongly convergent to f provided that f is any function belonging to $L^\rho(Z^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty\left(Z^2\right)$.

4 An example

In this section, we give an interesting example of positive linear operators that satisfies the conditions of Theorem 1 but not satisfies the conditions of Corollary 1 and also theorems given before.

Example 2 *Take* $Z = [0,1]$ *and let* $\varphi : [0,\infty) \to [0,\infty)$ *be a continuous function for which the following conditions hold:*

- φ *is convex,*
- $\varphi(0) = 0, \varphi(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to\infty} \varphi(u) = \infty$.

Hence, consider the functional ρ^{φ} *on* $X(Z^2)$ *defined by*

$$\rho^{\varphi}(f) := \int_0^1 \int_0^1 \varphi(|f(x,y)|)dxdy \text{ for } f \in X(Z^2). \tag{17}$$

In this case, as it is known, ρ^{φ} *is a convex modular on* $X(Z^2)$. *Consider the Orlicz space generated by* φ *as follows:*

$$L^{\rho}_{\varphi}(Z^2) := \left\{ f \in X(Z^2) : \rho^{\varphi}(\lambda f) < +\infty \quad \text{for some } \lambda > 0 \right\}.$$

Then consider the following bivariate Bernstein-Kantorovich operator $\mathbb{U} := (U_{mn})$ *on the space* $L^{\rho}_{\varphi}(Z^2)$ *which is defined by:*

$$U_{mn}(f;x,y) = \sum_{k=0}^{m}\sum_{l=0}^{n} p^{(m,n)}_{k,l}(x,y)(m+1)(n+1) \int_{k/(m+1)}^{(k+1)/(m+1)} \int_{l/(n+1)}^{(l+1)/(n+1)} f(u,v)dvdu \tag{18}$$

for $x, y \in Z$, *where* $p^{(m,n)}_{k,l}(x,y)$ *defined by*

$$p^{(m,n)}_{k,l}(x,y) = \binom{m}{k}\binom{n}{l} x^k y^l (1-x)^{m-k}(1-y)^{n-l}.$$

Also, it is clear that,

$$\sum_{k=0}^{m}\sum_{l=0}^{n} p^{(m,n)}_{k,l}(x,y) = 1. \tag{19}$$

Observe that the operators U_{mn} *map the Orlicz space* $L^{\rho}_{\varphi}(Z^2)$ *into itself. Because of* (19), *as in the proof of Lemma 5.1* ([8]) *and similarly to Example 1* ([12]), *we can use the Jensen inequality and obtain that for every* $f \in L^{\rho}_{\varphi}(Z^2)$ *and* $m, n \in \mathbb{N}$ *there is an absolute constant* $M > 0$ *such that*

$$\rho^{\varphi}(U_{mn}f) \le M\rho^{\varphi}(f).$$

Then, we know that, for any function $f \in L^{\rho}_{\varphi}(Z^2)$ *such that* $f - g \in X_{\mathbf{U}}$ *for every* $g \in C^{\infty}(Z^2)$, $(U_{mn}f)$ *is modularly convergent to* f, *with the choice of* $X_{\mathbf{U}} := L^{\rho}_{\varphi}(Z^2)$. *If* $\varphi(x) = x^p$ *for* $1 \le p < \infty$, $x \ge 0$, *then* $L^{\rho}_{\varphi}(Z^2) = L_p(Z^2)$. *Moreover we have* $\rho^{\varphi}(f) = \|f\|^p_{L_p}$.

Then, using the operators U_{mn}, *we define the sequence of positive linear operators* $\mathbb{V} := (V_{mn})$ *on* $L_1(Z^2)$ *as follows:*

$$V_{mn}(f;x,y) = \left(1 + g_{mn}(x,y)\right)U_{mn}(f;x,y) \ \ for \mathrm{f} \in \mathrm{L}_1\left(\mathrm{Z}^2\right), \ \mathrm{x}, \mathrm{y} \in [0,1], \tag{20}$$

where (g_{mn}) *is the same as in* (5) *and we choose* $\sigma_r(x,y) = \sigma(x,y)$ $(r = 0,1,2,3)$, where $\sigma(x,y) = \begin{cases} 1, & x = 0 \ or \ y = 0 \\ \frac{1}{xy}, & (x,y) \in (0,1] \times (0,1] \end{cases}$. *As in the proof of Lemma 5.1* ([8]) *and similarly to Example 2* ([34]), *we get, for every* $f \in L_1(Z^2)$, $\lambda > 0$ *and for positive constant* R, *that*

$$st^2_{P_p} - \limsup \left\| \lambda \frac{V_{mn}(f)}{\sigma} \right\|_{L_1} \leq R \|\lambda f\|_{L_1}.$$

We now claim that

$$st^2_{P_p} - \lim \left\| \lambda \left(\frac{V_{mn}(f_r) - f_r}{\sigma} \right) \right\|_{L_1} = 0, \ r = 0,1,2,3, \tag{21}$$

where $\sigma_r(x,y) = \sigma(x,y)$ *for* $r = 0,1,2,3$. *Observe that* $U_{mn}(f_0;x,y) = f_0$, $U_{mn}(f_1;x,y) = \frac{mx}{m+1} + \frac{1}{2(m+1)}$, $U_{mn}(f_2;x,y) = \frac{ny}{n+1} + \frac{1}{2(n+1)}$ *and* $U_{mn}(f_3;x,y) = \frac{m(m-1)x^2}{(m+1)^2} + \frac{2mx}{(m+1)^2} + \frac{1}{3(m+1)^2} + \frac{n(n-1)y^2}{(n+1)^2} + \frac{2ny}{(n+1)^2} + \frac{1}{3(n+1)^2}$. *So, we can see,*

$$\begin{aligned} \left\| \lambda \left(\frac{V_{mn}(f_0) - f_0}{\sigma} \right) \right\|_{L_1} &= \left\| \lambda \frac{(1 + g_{mn}) - 1}{\sigma} \right\|_{L_1} \\ &\leq \left\| \lambda \frac{g_{mn}}{\sigma} \right\|_{L_1} \end{aligned}$$

then, from Example 1, *we know that*

$$st^2_{P_p} - \lim \left\| \lambda \frac{g_{mn}}{\sigma} \right\|_{L_1} = 0 \tag{22}$$

and hence

$$st^2_{P_p} - \lim \left\| \lambda \left(\frac{V_{mn}(f_0) - f_0}{\sigma} \right) \right\|_{L_1} = 0,$$

*which guarantees that (*21*) holds true for* $r = 0$. *Also, we get*

$$\left\| \lambda \left(\frac{V_{mn}(f_1) - f_1}{\sigma} \right) \right\|_{L_1} \leq \frac{1}{m+1} \left(\|\lambda xy\|_{L_1} + \left\|\lambda x^2 y\right\|_{L_1} \right) + \left\| \lambda \frac{g_{mn}}{\sigma} \right\|_{L_1}.$$

Since the power series method is regular and from (22) *we have,*

$$st^2_{P_p} - \lim \left\| \lambda \left(\frac{V_{mn}(f_1) - f_1}{\sigma} \right) \right\|_{L_1} = 0.$$

So (21) holds true for $r = 1$. *Similarly, we have*

$$st^2_{P_p} - \lim \left\| \lambda \left(\frac{V_{mn}(f_2) - f_2}{\sigma} \right) \right\|_{L_1} = 0.$$

Finally, since

$$\begin{aligned} &\left\| \lambda \left(\frac{V_{mn}(f_3) - f_3}{\sigma} \right) \right\|_{L_1} \\ &\leq \frac{3m+1}{(m+1)^2} \left\| \lambda x^3 y \right\|_{L_1} + \frac{2m}{(m+1)^2} \left\| \lambda x^2 y \right\|_{L_1} + + \frac{3n+1}{(n+1)^2} \left\| \lambda x y^3 \right\|_{L_1} \\ &+ \frac{2n}{(n+1)^2} \left\| \lambda x y^2 \right\|_{L_1} + \left(\frac{1}{3(m+1)^2} + \frac{1}{3(n+1)^2} \right) \| \lambda x y \|_{L_1} + 6 \left\| \lambda \frac{g_{mn}}{\sigma} \right\|_{L_1}. \end{aligned}$$

Hence we can easily see that

$$st^2_{P_p} - \lim \left\| \lambda \left(\frac{V_{mn}(f_3) - f_3}{\sigma} \right) \right\|_{L_1} = 0.$$

So, our claim (21) *holds true for each* $r = 0, 1, 2, 3$. (V_{mn}) *satisfies all hypotheses of Theorem 1 and we immediately see that,*

$$st^2_{P_p} - \lim \left\| \lambda \left(\frac{V_{mn}(f) - f}{\sigma} \right) \right\|_{L_1} = 0,$$

on $Z = [0, 1]$ *for all* $f \in L_1(Z^2)$. *Also,* (g_{mn}) *does not converge statistically modularly and* P^2_p*−statistically modularly, thus* (V_{mn}) *does not satisfy statistical modular Korovkin type approximation theorem* ([12]) *and Corollary 1.*

Acknowledgements We would like to thank the referees for reading carefully and making valuable suggestions.

Author contributions All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors read and approved the final manuscript.

Funding The authors have no received any financial support for the research, authorship, or publication of this study.

Data availability statement Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Consent to participate The authors declare that they voluntarily agree to participate in this study.

References

1. Bohman, H.: On approximation of continuous and of analytic functions. Arkiv for Matematik **2**, 43–56 (1952)
2. Korovkin, P.P.: On convergence of linear positive operators in the space of continuous functions. Doklady Akademii Nauk **90**, 961–964 (1953). ((**Russian**))
3. Korovkin, P.P.: Linear Operators and Approximation Theory. Hindustan Publ. Co., Delhi (1960)
4. Fast, H.: Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)
5. Steinhaus, H.: Sur la convergence ordinaire et la convergence asymtotique. Colloq. Math. **2**, 73–74 (1951)
6. Gadjiev, A.D., Orhan, C.: Some approximation theorems via statistical convergence. Rocky Mt. J. Math. **32**, 129–138 (2002)
7. Dirik, F., Demirci, K.: Korovkin type approximation theorem for functions of two variables in statistical sense. Turk. J. Math. **34**(1), 73–83 (2010)
8. Bardaro, C., Mantellini, I.: Korovkin's theorem in modular spaces. Comment. Math. **47**, 239–253 (2007)
9. Acar, T., Dirik, F.: Korovkin-type theorems in weighted-spaces via summation process. Sci. World J. **2013**, 1–6 (2013)
10. Bardaro, C., Mantellini, I.: A Korovkin theorem in multivariate modular function spaces. J. Funct. Spaces Appl. **7**(2), 105–120 (2009)
11. Karakuş, S., Demirci, K., Duman, O.: Statistical approximation by positive linear operators on modular spaces. Positivity **14**(2), 321–334 (2010)
12. Orhan, S., Demirci, K.: Statistical approximation by double sequences of positive linear operators on modular spaces. Positivity **19**, 23–36 (2015)
13. Demirci, K., Orhan, S.: Statistically relatively uniform convergence of positive linear operators. Results Math. **69**, 359–367 (2016)
14. Demirci, K., Boccuto, A., Yıldız, S., Dirik, F.: Relative uniform convergence of a sequence of functions at a point and Korovkin-type approximation theorems. Positivity **24**(1), 1–11 (2020)
15. Demirci, K., Orhan ve, S.., Kolay, B..: Relative Hemen Hemen Yakınsaklık ve Yaklaşım Teoremleri. Sinop Üniversitesi Fen Bilimleri Dergisi **1**(2), 114–122 (2016)
16. Dirik, F., Şahin, P.O.: Statistical relatively equal convergence and Korovkin-type approximation theorem. Results Math. **72**(3), 1613–1621 (2017)
17. Kadak, U.: On relative weighted summability in modular function spaces and associated approximation theorems. Positivity **21**(4), 1593–1614 (2017)
18. Mohiuddine, S.A., Hazarika, B., Alghamdi, M.A.: Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems. Filomat **33**(14), 4549–4560 (2019)
19. Dirik, F., Yildiz, S., Demirci, K.: Abstract Korovkin theory for double sequences via power series method in modular spaces. Operators Matrices **13**(4), 1023–1034 (2019)
20. Tas, E., Yurdakadim, T.: Approximation by positive linear operators in modular spaces by power series method. Positivity **21**(4), 1293–1306 (2017)
21. Ünver, M.: Abel transforms of positive linear operators on weighted spaces. Bull. Belg. Math. Society-Simon Stevin **21**(5), 813–822 (2014)
22. Pringsheim, A.: Zur theorie der zweifach unendlichen zahlenfolgen. Math. Ann. **53**, 289–321 (1900)
23. Moricz, F.: Statistical convergence of multiple sequences. Arch. Math. (Basel) **81**, 82–89 (2004)
24. Baron, S., Stadtmüller, U.: Tauberian theorems for power series methods applied to double sequences. J. Math. Anal. Appl. **211**(2), 574–589 (1997)
25. Ünver, M., Orhan, C.: Statistical convergence with respect to power series methods and applications to approximation theory. Numer. Funct. Anal. Optim. **40**(5), 535–547 (2019)
26. Fridy, J., Orhan, C.: Statistical limit superior and limit inferior. Proc. Am. Math. Soc. **125**(12), 3625–3631 (1997)
27. Çakan, C., Altay, B.: Statistically boundedness and statistical core of double sequences. J. Math. Anal. Appl. **317**, 690–697 (2006)
28. Bardaro, C., Musielak, J., Vinti, G.: Nonlinear Integral Operators and Applications. de Gruyter Series in Nonlinear Analysis and Applications, vol. 9. Walter de Gruyter Publ, Berlin (2003)
29. Musielak, J.: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. Springer, Berlin (1983)
30. Kozlowski, W.M.: Modular Function Spaces, Pure Appl Math, vol. 122. Marcel Dekker Inc, New York (1988)
31. Musielak, J.: Nonlinear approximation in some modular function spaces I. Math. Japon. **38**, 83–90 (1993)

32. Bardaro, C., Mantellini, I.: Approximation properties in abstract modular spaces for a class of general sampling-type operators. Appl. Anal. **85**, 383–413 (2006)
33. Mantellini, I.: Generalized sampling operators in modular spaces. Comment. Math. **38**, 77–92 (1998)
34. Demirci, K., Orhan, S.: Statistical relative approximation on modular spaces. Results Math. **71**, 1167–1184 (2017)