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RESEARCH PAPER

Weighted Convergence and Applications to Approximation Theorems for Sequences of Monotone and Sublinear Operators

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Abstract
In this paper, we utilize a convergence termed weighted convergence, which is expressed via weighted density. This extends asymptotic (also known in the literature as natural density and linear density) and logarithmic densities. We employ this to express and prove Korovkin-type theorems for sequences of operators that are monotone and sublinear (msLOs). To demonstrate the practical relevance of our results, illustrative examples that satisfy the conditions of our theorems are included. The rate of convergence for these msLOs is analysed using the modulus of continuity.

Keywords Weighted density · Monotone and sublinear operators · Choquet integral

Mathematics Subject Classification 11B05 · 41A35 · 47H05

1 Introduction and Preliminaries

Densities have long been foundational in advancing such as probabilistic and additive number theory, analysis, and ergodic theory, as demonstrated by extensive research on their properties and applications. The concept of weighted density, as introduced in Alexander (1967), Rohrbach and Volkmann (1955), offers a comprehensive framework that includes asymptotic and logarithmic densities. The weighted density provides a more comprehensive perspective, allowing convergence analysis in cases where the asymptotic density may not be sufficient. Importantly, while every set with asymptotic density also possesses logarithmic density, the reverse is not necessarily true, i.e., a set can exhibit logarithmic density without having asymptotic density. Despite its established role in number theory and analysis, weighted density has not yet been explored in the context of approximation theory for monotone and sublinear operators.

Korovkin-type theorems focus on the approximation behaviour of certain test functions (see, for the classical theorem of Korovkin (1953, 1960)), and have been extensively studied in the context of various convergence types, including uniform, weighted, and statistical convergence (see e.g. Ayman-Mursaleen and Serra-Capizzano (2022); Demirci et al. (2021); Duman et al. (2003); Demirci and Yıldız (2022); Gadjiev and Orhan (2002); Şahin Bayram and Yıldız (2022); Uluçay et al. (2023); Ünver and Orhan (2019)). Over the years, these theorems have been adapted to a wide range of modified operators, where the primary focus has been to verify whether these operators satisfy the conditions of the classical Korovkin-type theorems (see e.g. Alamer and Nasiruzzaman (2024); Ansari and Özger (2024); Cai et al. (2024); Gupta et al. (2021a, 2021b); Kumar (2021); Kumar et al. (2024); Rao et al. (2024); Senapati et al. (2024a); Senapatia et al. (2024b)). Furthermore, the traditional requirement of positivity and linearity for operators has been relaxed, leading to the exploration of more general classes of operators, such as monotone and sublinear operators. Recent advancements in this field have demonstrated that these generalized operators not only preserve the fundamental

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properties of approximation but also extend the applicability of Korovkin-type theorems to broader functional spaces and more complex settings. This shift has opened new avenues for research where the flexibility of monotone and sublinear operators proves to be highly advantageous (see e.g. Gal and Iancu (2023); Gal and Niculescu (2020, 2022); Gal et al. (2024); Iancu (2023); Demirci et al. (2025)).

The present work aims to address this gap by using weighted density to derive Korovkin-type theorems, ultimately extending the applicability of weighted density within approximation theory.

Let $\mathbb{N}$ denote the set of natural numbers. Also let $E \subseteq \mathbb{N}$. The well-known asymptotic and logarithmic densities of a subset E are given, respectively, as follows:

$$d(E) = \lim_i \frac{1}{i}\sum_{k=1}^{i} \chi_E(k), \quad \delta(E) = \lim_i \frac{1}{\log i}\sum_{k=1}^{i} \frac{\chi_E(k)}{k}$$

provided that the limit exists. Here χ_E denotes the characteristic function of the set E. By substituting the limits with limit inferior or limit superior, which are always well-defined, we obtain the corresponding lower densities $\underline{d}(E)$, $\underline{\delta}(E)$ and the upper densities $\overline{d}(E)$, $\overline{\delta}(E)$. These quantities are related by the well-known inequalities

$$0 \le \underline{d}(E) \le \underline{\delta}(E) \le \overline{\delta}(E) \le \overline{d}(E) \le 1$$

Halberstam and Roth (1983); Tenenbaum (2015).

The concept of weighted density, introduced in Alexander (1967); Rohrbach and Volkmann (1955), offers a generalization of asymptotic and logarithmic densities. It is well established that every set with asymptotic density also has logarithmic density, but a set can possess logarithmic density without having asymptotic density.

For example, take $E_k = \left\{k^{k^2}+1, k^{k^2}+2, \ldots, k^{k^2+1}\right\}$, $k \in \mathbb{N}$ and $E = \bigcup_{k=2}^{\infty} E_k$. We know that $\delta(E) = 0$ but $d(E)$ does not exist ($\underline{d}(E) = 0$ and $\overline{d}(E) = 1$) Alghamdi et al. (2013).

Let $f : \mathbb{N} \to (0, \infty)$ be a weight function. For E in the set of natural numbers, the weighted density $d_f(E)$ define as

$$d_f(E) := \lim_{i\to\infty} \frac{1}{\sum_{k=1}^{i} f(k)} \sum_{k \le i, k\in E} f(k) \tag{1}$$

with properties

$$\lim_{i\to\infty} \frac{f(i)}{\sum_{k=1}^{i} f(k)} = 0 \text{ and } \sum_{k=1}^{\infty} f(k) = \infty. \tag{2}$$

Note that, from this definition it is obvious that $0 \le d_f(E) \le 1$ whenever it exists. Also this density has the following properties:

(i) $d_f(\mathbb{N}) = 1$ and if E has a d_f-density, $d_f(\mathbb{N}/E) = 1 - d_f(E)$,

(ii) if $E \subset F$, then $d_f(E) \le d_f(F)$,

(iii) $d_f(E \cup F) \le d_f(E) + d_f(F)$,

(iv) If E is a finite element subset of $\mathbb{N}$, then $d_f(E) = 0$.

The primary method for comparing weighted densities is the classical result by Rajagopal (1948), which, in the context of weighted densities, is stated as follows:

Let $f_1, f_2 : \mathbb{N} \to (0, \infty)$ be weight functions satisfying (2). If $\frac{f_1(k)}{f_2(k)}$ is decreasing, then for any $E \subset \mathbb{N}$, we have

$$\underline{d}_{f_2}(E) \le \underline{d}_{f_1}(E) \le \overline{d}_{f_1}(E) \le \overline{d}_{f_2}(E).$$

Also, then we say that the weighted density d_{f_2} is called stronger than the weighted density d_{f_1} (d_{f_1} is weaker than d_{f_2}).

Remark 1 It is straightforward to observe that this density coincides with the asymptotic density if $f_2(k) = 1$ for all k and with the logarithmic density if $f_1(k) = \frac{1}{k}$ for all k. Hence, the asymptotic density is stronger than the logarithmic density. For the logarithmic density it is convenient to replace the denominator of (1) by $\log i$.

Let us now use this concept of weighted density to give the following new definition.

Definition 1 If

$$\lim_{i\to\infty} \frac{1}{\sum_{k=1}^{i} f(k)} \sum_{k \le i, k\in E_\epsilon} f(k) = 0$$

where $E_\epsilon := \{k \le i : |x_k - L| \ge \epsilon\}$, i.e.,

$$d_f(E_\epsilon) = d_f(\{k \le i : |x_k - L| \ge \epsilon\}) = 0,$$

for every $\epsilon > 0$ then we say that $x = \{x_k\}$ is weighted convergent to L. Symbolically, we write $d_f - \lim x_k = L$.

We now revisit several key definitions and concepts that are important for our study. These include monotone and sublinear operators, as well as Choquet integrals.

Let M be a metric measure space, denoted by the triple (M, d, μ) where M is a space equipped with the metric d and the measure μ defined on the sigma field of Borel subsets of M.

If we consider the vector lattice $VL(M)$ of all real-valued functions defined on M, endowed with the pointwise ordering, then important vector sublattices of $VL(M)$ are $C(M)$, which is the space of functions of all continuous and bounded, and $AC(M)$, which is the space of functions of all bounded and almost everywhere continuous, and $L^q(M)$ $(1 \leq q < \infty$), which is the space of functions of all Borel measurable and Lebesgue integrable.

On $C(M)$, we consider the uniform norm $\|h\| = \sup_{x \in M}|h(x)|$, while on $L^q(M)$, $1 \leq q < \infty$, we consider the usual q-norm $\|h\|_q = \left(\int_M |h(x)|^q dx\right)^{1/q}$.

Let M and N be two metric spaces and E and F two ordered vector subspaces (or the positive cones) of $VL(M)$ and $VL(N)$ respectively, which contain the unity. An operator $S : E \to F$ is called *a weakly nonlinear operator* (respectively *a weakly nonlinear functional* when $F = \mathbb{R}$) if it satisfies the following three conditions:

1. *(Sublinearity)* S is positively homogeneous and subadditive, that is,

   $S(\alpha h) = \alpha S(h)$ and $S(h+g) \leq S(h) + S(g)$

   for all h, g in E and $\alpha \geq 0$,
2. *(Monotonicity)* $h \leq g$ in E implies $S(h) \leq S(g)$,
3. *(Translatability)* $S(h + \alpha.1) = S(h) + \alpha S(1)$ for all $h \in E$ and $\alpha \geq 0$.

In this paper, we are also interested in operators that satisfy the following condition:

(Subunital property) $S(1) \leq 1$.

If E and F are closed vector sublattices of the Banach lattices $C(M)$ and $C(N)$, respectively, then every monotone and subadditive operator (functional when $F = \mathbb{R}$) $S : E \to F$ satisfies the inequality

$$|S(h) - S(g)| \leq S(|h - g|) \text{ for all } h, g. \tag{3}$$

If an operator (functional when $F = \mathbb{R}$) S is monotone and positively homogeneous, then we necessarily have $S(0) = 0$.

This paper examines the convergence properties of sequences of operators that are both monotone and sublinear.

Suppose now that (M, Σ) is a measurable space with $M \neq \emptyset$ and Σ is a σ-algebra of subsets of M. The function $m : \Sigma \to [0, +\infty]$ is said to be a capacity (or monotone set function) iff $m(\emptyset) = 0$ and $m(E) \leq m(F)$ for all $E, F \in \Sigma$ with $E \subset F$ (monotonicity). A capacity m is submodular if

$$m(E \cup F) + m(E \cap F) \leq m(E) + m(F), \text{ for all } E, F \in \Sigma.$$

If $m(M) = 1$, then a capacity m is said to be normalized. The capacities provide a non additive generalization of probability measures, that is, of capacities m having the property of σ-additivity,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

for every sequence $\{E_k\}$ of disjoint sets with $\bigcup_{k=1}^{\infty} E_k \in \Sigma$.

Let m be normailzed capacity on Σ. If $h : M \to \mathbb{R}$ is Σ-measurable, then for any $E \in \Sigma$, the Choquet integral is given by

$$(C)\int_E hdm = \int_0^{+\infty} m(F_\beta(h) \cap E)d\beta + \int_{-\infty}^0 [m(F_\beta(h) \cap E) - m(E)]d\beta,$$

where $F_\beta(h) = \{u \in M : h(u) \geq \beta\}$. If $(C)\int_E hdm \in \mathbb{R}$, then h is called Choquet integrable on E. Note that if $h \geq 0$, then $\int_{-\infty}^{0} . = 0$. Also, if m is σ -additive measure, then the Choquet integral coincides with Lebesgue integral. We now conclude by stating the following assumptions regarding Choquet integrals:

(*a*) if $h \geq 0$, then $(C)\int_M hdm \geq 0$,

(*b*) $h \leq g$ implies $(C)\int_M hdm \leq (C)\int_M gdm$,

(*c*) for all $c \geq 0$, we have $(C)\int_M chdm = c(C)\int_M hdm$,

(*d*) $(C)\int_M 1dm = m(M)$,

(*e*) if m is submodular, then $(C)\int_M (h+g)dm \leq (C)\int_M hdm + (C)\int_M gdm$.

The concept of capacity and the Choquet integral have played a foundational role in the development of non-additive measure theory, providing a versatile framework for modelling cases where traditional additive measures are inadequate. These tools have seen extensive application in various fields, including statistics, financial economics, probability theory, multicriteria decision-making and the study of cooperative games. For additional concepts related to capacity and the Choquet integral, we refer the reader to Adams (1998); Choquet (1954); Denneberg (1994); Grabisch and Roubens (2000); Wang and Klir (2009); Wang and Yan (2007) (see also Gal and Iancu (2023) and Gal and Niculescu (2021), as well as the references therein).

We assume throughout the paper that the test functions are

$$h_0(x) = 1, \mathrm{h}_1(x) = x, \mathrm{h}_2(x) = -x, \mathrm{h}_3(x) = x^2.$$

2 Approximation Results and Applications

In this section, we use the weighted convergence to express and prove Korovkin-type theorems for sequences of operators that are monotone and sublinear. To demonstrate the practical relevance of our results, illustrative examples that satisfy the conditions of our theorems are included.

2.1 Approximation via Weighted Convergence Almost Everywhere

Theorem 1 *Let M be a locally compact subset of $\mathbb{R}$ and we denote a vector sublattice of $VL(M)$ by E that contains the test functions: $h_i,\ i = 0, 1, 2, 3$. Suppose that $\{S_k\}$ is a sequence of monotone and sublinear operators from E into itself. If*

$$d_f - \lim S_k(h_i) = h_i \text{ a.e., } i = 0, 1, 2, 3, \tag{4}$$

then for all nonnegative functions h in $E \cap AC(M)$, we have

$$d_f - \lim S_k(h) = h \text{ a.e..} \tag{5}$$

Moreover, if S_k is translatable, then for all $h \in E \cap AC(M)$, we have

$$d_f - \lim S_k(h) = h \text{ a.e..}$$

Clarifying a point, it is essential to mention that if M is included in $[0, \infty)$, then the test functions, h_i, $i = 0, 1, 2, 3$, are reduced to $1, \ -x, \ x^2$.

Proof Let $h \in E \cap AC(M)$ and suppose that y is a point of continuity of h with

$$d_f - \lim S_k(h_i)(y) = h_i(y) \text{ a.e., } i = 0, 1, 2, 3.$$

Since y is a point of continuity of h, we can write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|h(x) - h(y)| < \varepsilon$ for all $x \in M$ satisfying $|x - y| < \delta$. If $|x - y| \geq \delta$, then $|h(x) - h(y)| < \frac{2\|h\|}{\delta^2}|x - y|^2$, hence, for any $x \in M$ we have

$$\begin{aligned} |h(x) - h(y)| &< \varepsilon + \frac{2\|h\|}{\delta^2}|x - y|^2 \\ &= \varepsilon + \frac{2\|h\|}{\delta^2}\left\{x^2 + 2x(C - y) + 2C(-x) + y^2\right\} \end{aligned} \tag{6}$$

where $C = \max\{y, 0\}$. Thanks to inequality (3) and (6), monotonicity and sublinearity of the operators S_k, we conclude in case $h \geq 0$ that

$$\begin{aligned} &|S_k(h) - h(y)| \\ &\quad = |S_k(h) - S_k(h(y).1) + h(y)S_k(1) - h(y)| \\ &\quad \leq S_k(|h - h(y)|) + h(y)|S_k(1) - 1| \\ &\quad \leq S_k\left(\varepsilon + \frac{2\|h\|}{\delta^2}\left\{x^2 + 2x(C - y) + 2C(-x) + y^2\right\}\right) \\ &\qquad + h(y)|S_k(h_0) - h_0(y)| \\ &\quad \leq \varepsilon + K\{(S_k(h_3) - h_3(y)) + 2(C - y)(S_k(h_1) - h_1(y)) \\ &\qquad + 2C(S_k(h_2) - h_2(y)) \\ &\qquad + |S_k(h_0) - h_0(y)|\} \end{aligned}$$

where $K = \max\left\{\frac{2\|h\|}{\delta^2}, \varepsilon + h(y)\right\}$. By our choice of y and in light of the inequality above, we obtain the following for every $\varepsilon < \varepsilon'$,

$$\begin{aligned} &\{k \in \mathbb{N}_0 : |S_k(h) - h(y)| \geq \varepsilon'\} \\ &\quad \subset \{k \in \mathbb{N}_0 : (S_k(h_3) - h_3(y)) + 2(C - y)(S_k(h_1) - h_1(y)) \\ &\quad + 2C(S_k(h_2) - h_2(y)) + |S_k(h_0) - h_0(y)| \geq \frac{\varepsilon' - \varepsilon}{K}\right\} \\ &\quad \subset \left\{k \in \mathbb{N}_0 : |S_k(h_3) - h_3(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\} \\ &\quad \cup \left\{k \in \mathbb{N}_0 : |S_k(h_1) - h_1(y)| \geq \frac{\varepsilon' - \varepsilon}{8K(C - y)}\right\} \\ &\quad \cup \left\{k \in \mathbb{N}_0 : |S_k(h_2) - h_2(y)| \geq \frac{\varepsilon' - \varepsilon}{8KC}\right\} \\ &\quad \cup \left\{k \in \mathbb{N}_0 : |S_k(h_0) - h_0(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &d_f(\{k \in \mathbb{N}_0 : |S_k(h) - h(y)| \geq \varepsilon'\}) \\ &\quad \leq d_f\left(\left\{k \in \mathbb{N}_0 : |S_k(h_3) - h_3(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\}\right) \\ &\quad + d_f\left(\left\{k \in \mathbb{N}_0 : |S_k(h_1) - h_1(y)| \geq \frac{\varepsilon' - \varepsilon}{8K(C - y)}\right\}\right) \\ &\quad + d_f\left(\left\{k \in \mathbb{N}_0 : |S_k(h_2) - h_2(y)| \geq \frac{\varepsilon' - \varepsilon}{8KC}\right\}\right) \\ &\quad + d_f\left(\left\{k \in \mathbb{N}_0 : |S_k(h_0) - h_0(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\}\right) \end{aligned}$$

which gives $d_f(\{k \in \mathbb{N}_0 : |S_k(h) - h(y)| \geq \varepsilon'\}) = 0$ with hypotheses, that is, $d_f - \lim S_k(h) = h(y)$. Now suppose in addition that S_k is translatable for all $k \in \mathbb{N}$. Since $h + \|h\| \geq 0$, in view of (5), we get

$$d_f - \lim S_k(h + \|h\|) = h + \|h\| \text{ a.e.}$$

and since S_k is also translatable, we can write

$$S_k(h + \|h\|) = S_k(h) + \|h\|S_k(1) = S_k(h) + \|h\|S_k(h_0).$$

Thanks to our hypotheses (4), we get, for all $h \in E \cap AC(M)$, $d_f - \lim S_k(h) = h$ a.e.. Notice that if $M \subset [0, \infty)$, the inequality (6) rewritten as follows:

$$|h(x) - h(y)| < \varepsilon + \frac{2\|h\|}{\delta^2}\left\{x^2 + 2C(-x) + y^2\right\}.$$

Hence, we get

$$\begin{aligned} |S_k(h) - h(y)| &\leq S_k(|h - h(y)|) + h(y)|S_k(h_0) - h_0(y)| \\ &\leq \varepsilon S_k(h_0) + \frac{2\|h\|}{\delta^2}\left\{S_k(h_3) + 2CS_k(h_2) + y^2 S_k(h_0)\right\} \\ &\quad + h(y)|S_k(h_0) - h_0(y)| \end{aligned}$$

where $h \geq 0$. Thus, the proofs proceed as outlined above, leading to the desired result. □

Remark 2 If we replace the space $AC(M)$ with $C(M)$, Theorem 1 still holds.

Corollary 1 *Let $\mathcal{R}([a, b])$ stands for the vector lattice of all Riemann integrable functions defined on one dimensional* *compact interval $[a,b]$. Suppose that $\{S_k\}$ is a sequence of monotone and sublinear operators from $\mathcal{R}([a,b])$ into $\mathcal{R}([a,b])$. If*

$$d_f - \lim S_k(h_i) = h_i \text{ a.e., } i = 0,1,2,3,$$

then for all nonnegative functions h in $\mathcal{R}([a,b])$, we have

$$d_f - \lim S_k(h) = h \text{ a.e..}$$

Moreover, if S_k is translatable, then for all $h \in \mathcal{R}([a,b])$, we have $d_f - \lim S_k(h) = h$ a.e..

2.2 Approximation via Weighted Convergence in Measure

Theorem 2 *Suppose that M be a compact subset of $\mathbb{R}$ endowed with positive Borel measure μ and let $\{S_k\}$ is a sequence of monotone, subunital and sublinear operators from $C(M)$ into itself. If*

$$d_f - \lim \mu(\{y \in M : |S_k(h_i)(y) - h_i(y)| \geq \varepsilon\}) = 0, \mathrm{i} = 0,1,2,3, \tag{7}$$

then for all nonnegative functions h in $C(M)$, we have

$$d_f - \lim \mu(\{y \in M : |S_k(h)(y) - h(y)| \geq \varepsilon\}) = 0. \tag{8}$$

Moreover, if S_k is translatable, then for all $h \in C(M)$, we have

$$d_f - \lim \mu(\{y \in M : |S_k(h)(y) - h(y)| \geq \varepsilon\}) = 0.$$

Proof Let h be nonnegative function in $C(M)$ and $\varepsilon > 0$. Thanks to the uniform continuity of h, by classical method, it can be written that for every $\varepsilon' < \varepsilon$ with $\varepsilon' > 0$, there is $\delta > 0$,such that

$$|h(x) - h(y)| < \varepsilon' + \frac{2\|h\|}{\delta^2}|x-y|^2 \text{ for all } x,y \in M.$$

Then as in Theorem 1, we have

$$\begin{aligned}
&|S_k(h) - h(y)| \leq \varepsilon' + \frac{2\|h\|}{\delta^2} S_k\left((x-y)^2\right) + h(y)|S_k(h_0) - h_0(y)| \\
&\leq \varepsilon' + \frac{2\|h\|}{\delta^2}\{S_k(h_3)(y) + 2(C-y)S_k(h_1)(y) + 2CS_k(h_2)(y) + y^2\} \\
&\quad + h(y)|S_k(h_0) - h_0(y)|
\end{aligned}$$

where $C = \max\{y, 0\}$. In view of the above inequality, we obtain

$$\begin{aligned}
&\{y \in M : |S_k(h)(y) - h(y)| \geq \varepsilon\} \\
&\quad \subset \left\{ y \in M : \varepsilon' + \frac{2\|h\|}{\delta^2}\{S_k(h_3)(y) + 2(C-y)S_k(h_1)(y) + 2CS_k(h_2)(y) + y^2\} \right. \\
&\quad \left. + h(y)|S_k(h_0) - h_0(y)| \geq \varepsilon \right\} \\
&\quad = \{y \in M : (S_k(h_3) - h_3(y)) + 2(C-y)(S_k(h_1) - h_1(y)) \\
&\quad \left. + 2C(S_k(h_2) - h_2(y)) + |S_k(h_0) - h_0(y)| \geq \frac{\varepsilon - \varepsilon'}{K} \right\} \\
&\quad \subset \left\{ y \in M : |S_k(h_3) - h_3(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \\
&\quad \cup \left\{ y \in M : |S_k(h_1) - h_1(y)| \geq \frac{\varepsilon - \varepsilon'}{8K(C-y)} \right\} \\
&\quad \cup \left\{ y \in M : |S_k(h_2) - h_2(y)| \geq \frac{\varepsilon - \varepsilon'}{8KC} \right\} \\
&\quad \cup \left\{ y \in M : |S_k(h_0) - h_0(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\}
\end{aligned}$$

where $K = \max\left\{\frac{2\|h\|}{\delta^2}, h(y)\right\}$. Hence, we get

$$\begin{aligned}
&\mu(\{y \in M : |S_k(h) - h(y)| \geq \varepsilon\}) \\
&\quad \leq \mu\left(\left\{ y \in M : |S_k(h_3) - h_3(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\}\right) \\
&\quad + \mu\left(\left\{ y \in M : |S_k(h_1) - h_1(y)| \geq \frac{\varepsilon - \varepsilon'}{8K(C-y)} \right\}\right) \\
&\quad + \mu\left(\left\{ y \in M : |S_k(h_2) - h_2(y)| \geq \frac{\varepsilon - \varepsilon'}{8KC} \right\}\right) \\
&\quad + \mu\left(\left\{ y \in M : |S_k(h_0) - h_0(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\}\right).
\end{aligned}$$

Now, for every $\eta > 0$,

$$
\begin{aligned}
&\{k\in\mathbb{N}_0:\mu(\{y\in M:|S_k(h)-h(y)|\geq\varepsilon\})\geq\eta\}\\
&\quad\subset\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_3)-h_3(y)|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}\right)\geq\frac{\eta}{4}\right\}\\
&\quad\cup\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_1)-h_1(y)|\geq\frac{\varepsilon-\varepsilon'}{8K(C-y)}\right\}\right)\geq\frac{\eta}{4}\right\}\\
&\quad\cup\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_2)-h_2(y)|\geq\frac{\varepsilon-\varepsilon'}{8KC}\right\}\right)\geq\frac{\eta}{4}\right\}\\
&\quad\cup\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_0)-h_0(y)|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}\right)\geq\frac{\eta}{4}\right\}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&d_f(\{k\in\mathbb{N}_0:\mu(\{y\in M:|S_k(h)-h(y)|\geq\varepsilon\})\geq\eta\})\\
&\quad\leq d_f\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_3)-h_3(y)|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}\right)\geq\frac{\eta}{4}\right\}\right)\\
&\quad+d_f\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_1)-h_1(y)|\geq\frac{\varepsilon-\varepsilon'}{8K(C-y)}\right\}\right)\geq\frac{\eta}{4}\right\}\right)\\
&\quad+d_f\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_2)-h_2(y)|\geq\frac{\varepsilon-\varepsilon'}{8KC}\right\}\right)\geq\frac{\eta}{4}\right\}\right)\\
&\quad+d_f\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(h_0)-h_0(y)|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}\right)\geq\frac{\eta}{4}\right\}\right).
\end{aligned}
$$

In view of our hypotheses (7), we obtain

$$d_f(\{k\in\mathbb{N}_0:\mu(\{y\in M:|S_k(h)-h(y)|\geq\varepsilon\})\geq\eta\})=0,$$

i.e.,

$$d_f-\lim\mu(\{y\in M:|S_k(h)-h(y)|\geq\varepsilon\})=0.$$

For the second part, suppose now that S_k is also translatable for all $k\in\mathbb{N}$. Since $h+\|h\|\geq 0$, in view of (8), we get

$$d_f-\lim\mu(\{y\in M:|S_k(h+\|h\|)-(h+\|h\|)(y)|\geq\varepsilon\})=0$$

and as in Theorem 1, we have

$$S_k(h+\|h\|)=S_k(h)+\|h\|S_k(h_0).$$

Thus, for all $h\in C(M)$, we get $d_f-\lim\mu(y\in M:|S_k(h)-h(y)|\geq\varepsilon)=0$. □

Remark 3 If M is considered to be locally compact, then Theorem 2 remains true under the following local convergence condition

$$d_f-\lim\mu(\{y\in E:|S_k(h)-h(y)|\geq\varepsilon\})=0$$

for every compact subset E of M.

2.3 Applications to Approximation Theorems

In this subsection, the application of the possibilistic Kantorovich operators and Bernstein-Kantorovich-Choquet type operators to approximation theorems is highlighted, with emphasis on their advantages over traditional methods. The Bernstein-Kantorovich-Choquet type operators were chosen due to their unique ability to approximate nonlinear functionals while preserving the monotonicity and continuity properties inherent in Choquet integrals. In contrast to linear operators, these operators provide a more comprehensive framework for handling complex, non-additive measures. Through these illustrative cases, we aim to bridge the gap between abstract theory and practical application, demonstrating the advantages of our approach over conventional methods Figs 1 and 2.

Example 1 Let f be a weight function defined by $f(k)=k$ for all k. Also, let $\{x_k\}$ be a sequence with general term

$$x_k:=\begin{cases}\sqrt{k}, & \text{if } k \text{ is a square,}\\ 0, & \text{otherwise.}\end{cases}\tag{9}$$

We know that $\sum_{k=1}^{i}f(k)=\sum_{k=1}^{i}k=\frac{i(i+1)}{2}$. Also, for every $\epsilon>0$, we have

$$
\begin{aligned}
\sum_{k\in\{k\leq i:|x_k-0|\geq\epsilon\}}k &\leq\sum_{k\in\{k\leq i:x_k\neq 0\}}k\\
&=\sum_{m=1}^{[|\sqrt{i}|]}m^2=\frac{[|\sqrt{i}|]([|\sqrt{i}|]+1)(2[|\sqrt{i}|]+1)}{6}\\
&\leq\frac{\sqrt{i}(\sqrt{i}+1)(2\sqrt{i}+1)}{6}
\end{aligned}
$$

and assume that the symbol $[|i|]$ indicates the greatest integer less than or equal to the real number i. Then, we obtain

$$
\begin{aligned}
0&\leq\lim_{i\to\infty}\frac{1}{\sum_{k=1}^{i}k}\sum_{k\in\{k\leq i:|x_k-0|\geq\epsilon\}}k=\lim_{i\to\infty}\frac{1}{\frac{i(i+1)}{2}}\sum_{k\in\{k\leq i:|x_k-0|\geq\epsilon\}}k\\
&\leq\lim_{i\to\infty}\frac{1}{\frac{i(i+1)}{2}}\frac{\sqrt{i}(\sqrt{i}+1)(2\sqrt{i}+1)}{6}=0,
\end{aligned}
$$

and that means $d_f-\lim x_k=0$. Now, let us consider

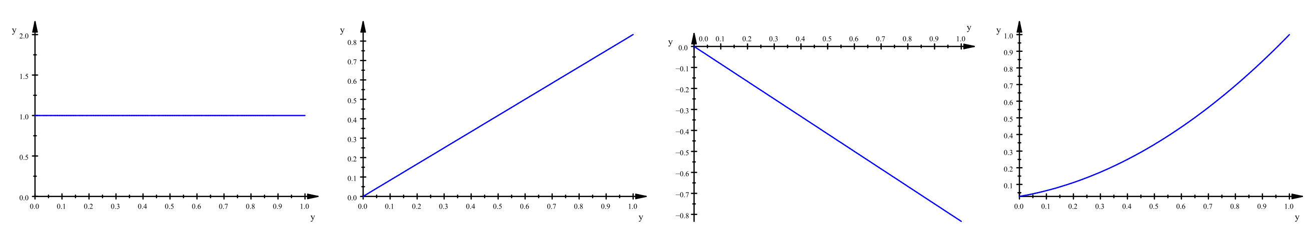

Fig. 1 Behavior of the possibilistic Kantorovich operators for the test functions 1, x, $-x$, x^2 and $k=5$ (is not perfect square)

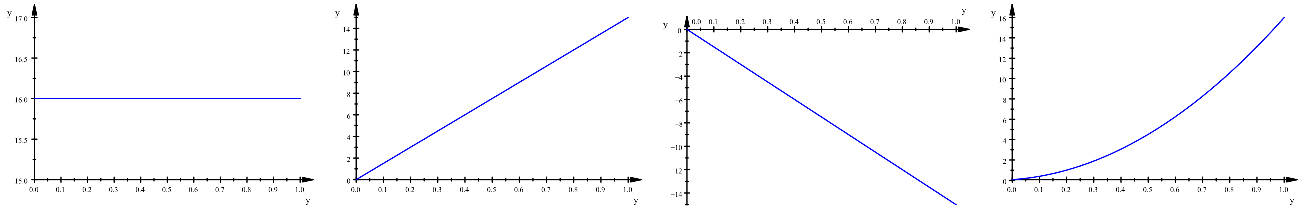

Fig. 2 Behavior of the possibilistic Kantorovich operators for the test functions 1, x, $-x$, x^2 and $k = 225$ (is perfect square)

monotone, sublinear and translatable operators S_k, the possibilistic Kantorovich operators Gal (2020), given by

$$S_k(h)(y) = \sum_{n=0}^{k} \binom{k}{n} y^n (1-y)^{k-n} \sup\left\{h(y) : y \in \left[\frac{n}{k+1}, \frac{n+1}{k+1}\right]\right\}.$$

As it is known, $S_k(h_0)(y) = h_0(y)$, $S_k(h_1)(y) = \frac{k}{k+1} h_1(y) + \frac{1}{k+1}$, $S_k(h_2)(y) = \frac{k}{k+1} h_2(y)$ and

$$S_k(h_3)(y) = \left(\frac{k}{k+1}\right)^2 h_3(y) + \frac{2k}{(k+1)^2} h_1(y) + \frac{1}{(k+1)^2}$$

(see, also Iancu (2023)). Using these operators and the sequence $\{x_k\}$, we introduce the following

$$T_k(h)(y) = (1 + x_k) S_k(h)(y), \text{ for all } h \in E \cap AC(M).$$

It is easy to see that each T_k is a monotone, sublinear and translatable operator. Applying now Theorem 1, it follows that

$$d_f - \lim T_k(h) = h(y) \text{ at each point } y \text{ of continuity of } h.$$

Example 2 Let us consider the Bernstein-Kantorovich-Choquet (BKC) polynomial operators $K_{k,m}^{(1)} : C([0,1]) \to C([0,1])$ by

$$K_{k,m}^{(1)}(h)(y) = \sum_{n=0}^{k} \frac{(C)\int_{\frac{n}{k+1}}^{\frac{n+1}{k+1}} h(u) dm(u)}{m\left(\left[\frac{n}{k+1}, \frac{n+1}{k+1}\right]\right)} \binom{k}{n} y^n (1-y)^{k-n}$$

where $m := \sqrt{\lambda}$with the Lebesgue measure λ. It is known from Gal and Niculescu (2022) that the operators monotone, translatable and sublinear. Because of the results given in Gal and Niculescu (2020), we can say that $K_{k,m}^{(1)}(h_0) \to h_0$, $K_{k,m}^{(1)}(h_1) \to h_1$, $K_{k,m}^{(1)}(h_2) \to h_2$ and $K_{k,m}^{(1)}(h_3) \to h_3$ uniformly on $[0,1]$. Now, define the sequence

$$K_{k,m}(h)(y) = \sum_{n=0}^{k} \frac{(C)\int_{\frac{n}{k+1}}^{\frac{n+1}{k+1}} h(x_k u) dm(u)}{m\left(\left[\frac{n}{k+1}, \frac{n+1}{k+1}\right]\right)} \binom{k}{n} y^n (1-y)^{k-n}$$

where $0 < x_k \le 1$ with $d_f - \lim x_k = 1$. Observe that $K_{k,m}(h_0) = h_0$, $K_{k,m}(h_1) = x_k K_{k,m}^{(1)}(h_1)$, $K_{k,m}(h_2) = x_k K_{k,m}^{(1)}(h_2)$ and $K_{k,m}(h_3) = x_k^2 K_{k,m}^{(1)}(h_3)$. Therefore, $K_{k,m}$ satisfies the conditions of Theorem 2, then for all $h \in C([0,1])$, we have

$$d_f - \lim \mu\left(\left\{y \in [0,1] : \left|K_{k,m}(h)(y) - h(y)\right| \ge \varepsilon\right\}\right) = 0.$$

Example 3 Let us consider the Bernstein-Chlodovsky-Kantorovich-Choquet (BCKC) polynomial operators $C_{k,m}^{(1)} : \mathcal{R}([0,\alpha_k]) \to \mathcal{R}([0,\alpha_k])$ by

$$C_{k,m}^{(1)}(h)(y) = \sum_{n=0}^{k} \frac{(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} h(u) dm(u)}{m\left(\left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right]\right)} \binom{k}{n} \left(\frac{y}{\alpha_{k+1}}\right)^n \left(1 - \frac{y}{\alpha_{k+1}}\right)^{k-n}$$

where $m := \sqrt{\lambda}$with the Lebesgue measure λ and $\{\alpha_n\}$ is a positive sequence with $\lim_n \alpha_n = \infty$, $\lim_n \frac{\alpha_n}{n} = 0$ (Demirci et al. (2025)). It is important to point out that if $m := \lambda$, then the operators become Bernstein-Chlodovsky-Kantorovich operators and if $m := \delta_k$ (the Dirac measure), then the operators become Bernstein-Chlodovsky operators. Now, thanks to Choquet integral's sublinearity, monotonity and comonotone additivity with a submodular capacity m, we have these properties for $C_{k,m}^{(1)}((a)-(e))$.

Clearly, $C_{k,m}^{(1)}(h_0)(y) = h_0(y)$. Also, we get

$$C_{k,m}^{(1)}(h_1)(y) = \frac{k}{k+1} y + \frac{2\alpha_{k+1}}{3(k+1)},$$

$$C_{k,m}^{(1)}(h_2)(y) = \frac{k}{k+1}(-y) + \frac{\alpha_{k+1}}{3(k+1)},$$

$$C_{k,m}^{(1)}(h_3)(y) = \frac{k(k-1)}{(k+1)^2} y^2 + \frac{7}{3} \frac{k\alpha_{k+1}}{(k+1)^2} y + \frac{4}{3}\left(\frac{\alpha_{k+1}}{k+1}\right)^2 - \frac{4}{5}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{5}{2}},$$

(see Demirci et al. (2025)). Using now $C_{k,m}^{(1)}$, we define the following

$$C_{k,m}(h)(y)=\sum_{n=0}^{k}\frac{(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}}h(x_k u)dm(u)}{m\left(\left[\frac{n\alpha_{k+1}}{k+1},\frac{(n+1)\alpha_{k+1}}{k+1}\right]\right)}\binom{k}{n}\left(\frac{y}{\alpha_{k+1}}\right)^n\left(1-\frac{y}{\alpha_{k+1}}\right)^{k-n}$$

where $0<x_k\leq 1$ with $d_f-\lim x_k=1$. It is very easy to see that each $C_{k,m}$ is a monotone, sublinear and translatable operator and observe that $C_{k,m}(h_0)=h_0$, $C_{k,m}(h_1)=x_k C^{(1)}_{k,m}(h_1)$, $C_{k,m}(h_2)=x_k C^{(1)}_{k,m}(h_2)$ and $C_{k,m}(h_3)=x_k^2 C^{(1)}_{k,m}(h_3)$. Consequently: Applying Corollary 1, for all $h\in\mathcal{R}([0,\alpha_k])$, we have $d_f-\lim C_{k,m}(h)=h$ a.e..

2.4 Rates of Weighted Convergence

In this section, we present some estimates of rates of convergence for our Korovkin-type theorems. Regarding the rates of convergence for these Korovkin-type theorems, we can consider the following modulus of continuity:

$$\omega(h,\delta):=\sup_{x,y\in M:\mathrm{d}(x,y)\leq\delta}|h(x)-h(y)|$$

where $h\in C(M)$ and $\delta>0$. It is readily seen that, $\omega(h,\delta)$ is an increasing function of δ, $|h(x)-h(y)|\leq\omega(h,d(x,y))$ for each $x,y\in M$, $\omega(h,\delta)\leq 2\|h\|$ for every $\delta>0$, and $\omega(h,\gamma\delta)\leq(1+\gamma)\omega(h,\delta)$ for every $\gamma,\delta>0$.

Theorem 3 *Let M, E and $\{S_k\}$ be as in Theorem 1. If*
(i) $d_f-\lim S_k(h_0)=h_0$ a.e.,
(ii) $d_f-\lim\omega(h,\delta_S)=0$ a.e., where $\delta_S:=\sqrt{S_k\left((.-y)^2\right)}$.

Then for all nonnegative functions h in $E\cap AC(M)$, we have

$$d_f-\lim S_k(h)=h \text{ a.e..}$$

Moreover, if S_k is translatable, then for all $h\in E\cap AC(M)$, we have $d_f-\lim S_k(h)=h$ a.e..

Proof Let $h\in E\cap AC(M)$ and suppose that y is a point of continuity of h with

$$d_f-\lim S_k(h_0)(y)=h_0(y) \text{ a.e. and } d_f-\lim\omega(h,\delta_S)=0 \text{ a.e..}$$

With the help of positive linear operators S_k and property of modulus of continuity ω, we get

$$\begin{aligned}&|S_k(h)-h(y)|\leq S_k(|h-h(y)|)+h(y)|S_k(h_0)-h_0|\\ &\leq S_k\left(\left(1+\frac{(x-y)^2}{\delta^2}\right)\omega(h,\delta)\right)+h(y)|S_k(h_0)-h_0(y)|\\ &\leq\omega(h,\delta)S_k(h_0)+\frac{\omega(h,\delta)}{\delta^2}S_k\left((x-y)^2\right)+h(y)|S_k(h_0)-h_0(y)|\end{aligned}$$

If we choose $\delta:=\delta_S:=\sqrt{S_k\left((x-y)^2\right)}$, this yield that

$$|S_k(h)-h(y)|\leq\omega(h,\delta_S)(S_k(h_0)+1)+h(y)|S_k(h_0)-h_0(y)|.$$

By our choice of y, we get

$$d_f-\lim S_k(h)=h(y).$$

Now suppose in addition that S_k is translatable for all $k\in\mathbb{N}$. Following a similar approach to the proof of Theorem 1, we can easily obtain, for all $h\in E\cap AC(M)$, $d_f-\lim S_k(h)=h$ a.e.. □

Theorem 4 *Suppose that M, $\{S_k\}$ be as in Theorem 2. If*
(i) $d_f-\lim\mu(\{y\in M:|S_k(h_0)(y)-h_0(y)|\geq\varepsilon\})=0$,
(ii) $d_f-\lim\mu(\{y\in M:\omega(h,\delta_S)\geq\varepsilon\})=0$ a.e., where $\delta_S:=\sqrt{S_k\left((.-y)^2\right)}$,

then for all nonnegative functions h in $C(M)$, we have

$$d_f-\lim\mu(\{y\in M:|S_k(h)(y)-h(y)|\geq\varepsilon\})=0.$$

Moreover, if S_k is translatable, then for all $h\in C(M)$, we have

$$d_f-\lim\mu(\{y\in M:|S_k(h)(y)-h(y)|\geq\varepsilon\})=0.$$

Proof Let h be nonnegative function in $C(M)$ and $\varepsilon>0$. As in Theorem 3, we have

$$|S_k(h)-h(y)|\leq\omega(h,\delta_S)(S_k(h_0)+1)+h(y)|S_k(h_0)-h_0(y)|.$$

In view of the above inequality, we obtain

$$\begin{aligned}&\{y\in M:|S_k(h)(y)-h(y)|\geq\varepsilon\}\\ &\subset\{y\in M:\omega(h,\delta_S)(S_k(h_0)+1)+h(y)|S_k(h_0)-h_0(y)|\geq\varepsilon\}.\end{aligned}$$

Now, considering the above inclusion and the hypotheses (i), (ii), the proof is completed immediately. Furthermore, suppose that S_k is translatable for all $k\in\mathbb{N}$. Following a similar approach to the proof of Theorem 3, we can easily get, for all $h\in C(M)$,

$$d_f-\lim\mu(\{y\in M:|S_k(h)(y)-h(y)|\geq\varepsilon\})=0.$$

□

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Conflict of interest The authors declare that they have no Conflict of interest.

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