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Results in Mathematics

Approximation Results via Power Series Method for Sequences of Monotone and Sublinear Operators

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Abstract. In this paper, we use the power series methods to study Korovkin-type approximation theorems for sequences of operators that are monotone and sublinear. It is important to point out that any positive linear operator is monotone sublinear but the converse is not true. We also calculate the rate of convergence in terms of modulus of continuity. Finally, we provide some examples that satisfy our theorems and give quantitative estimates. We also define Bernstein-Chlodovsky-Kantorovich-Choquet (BCKC) polynomial operators similar to those given before and obtain approximation estimate.

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Keywords. Power series method, monotone and sublinear operators, nonlinear Choquet integral.

1. Introduction

Korovkin-type theorems, a fundamental concept in the field of approximation theory, have an important place thanks to their applications in various mathematical disciplines. Based on the fundamental work of Korovkin [20,21], who established convergence conditions for sequences of positive linear operators on continuous functions, researchers have studied various generalizations. These investigations aimed to reduce the linearity and positivity requirements imposed by Korovkin's theorem (see e.g. [6,11,14,28,35]) by taking different conditions such as sublinearity, monotonicity and various convergence modes

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(see e.g. [2,10,13,14,17–19,23]). This pursuit has led to a deeper understanding of approximation processes and has been a source of inspiration for new work.

In the Korovkin-type approximation theory, the concept of uniform convergence, which has further increased the interest of researchers, has been replaced by a variety of convergence types, such as almost convergence, measurable convergence, and statistical convergence. Among these, the power series method emerges as an important type of convergence and has led to the characterization of new convergence theorems (see, for example, [5,7,8,24,25,27,29–32]). Utilizing these methods, researchers have extended this theorem to include alternative properties such as sublinearity or monotonicity instead of the traditional linearity or positivity conditions. This has enriched the field of approximation theory and opened new avenues for applications in various mathematical fields.

The focus of this paper is to advance the current line of research by examining approximation results for sequences of monotone and nonlinear operators using power series methods. In this way, our objective is to generalize the known Korovkin-type theorems and derive new convergence rates using the modulus of continuity. Furthermore, we introduce and analyse a new class of operators, the Bernstein-Chlodovsky-Kantorovich-Choquet (BCKC) operators, extending previously studied operators. Also, we provide quantitative estimates for the operators' approximation behaviour. By providing illustrative examples, we demonstrate the applicability of our results in concrete settings.

Our research not only advances existing investigations in the field of approximation theory but also introduces novel methodologies and operators. These contributions expand the field of approximation theory and enhance its potential applications across various mathematical disciplines.

2. Preliminaries

This section reviews some essential definitions and concepts related to our study, particularly those associated with Korovkin-type approximation theory, monotone and sublinear operators, and power series methods.

2.1. Sublinear and Monotone Operators

Let M be a metric measure space, denoted by the triple (M, d, μ) where M is a space equipped with the metric d and the measure μ defined on the sigma field of Borel subsets of M.

If we consider the vector lattice $VL(M)$ of all real-valued functions defined on M, endowed with the pointwise ordering, then important vector sublattices of $VL(M)$ are $C(M)$, which is the space of functions of all continuous and bounded, and $AC(M)$, which is the space of functions of all bounded and almost everywhere continuous, and $L^q(M)$ $(1 \leq q < \infty)$, which is the space of functions of all Borel measurable and Lebesgue integrable with modulo equality a.e..

On $C(M)$, we consider the uniform norm $\|f\| = \sup_{x\in M} |f(x)|$, while on $L^q(M)$, $1 \le q < \infty$, we consider the usual q-norm $\|f\|_q = \left(\int_M |f(x)|^q \, dx\right)^{1/q}$.

Let M and N be two metric spaces and E and F two respectively ordered vector subspaces (or the positive cones) of $VL(M)$ and $VL(N)$ that contain the unity. An operator $T : E \to F$ is called *a weakly nonlinear operator* (respectively *a weakly nonlinear functional* when $F = \mathbb{R}$) if it satisfies the following three conditions:

1. *(Sublinearity)* T is positively homogeneous and subadditive, that is,

$$T(\alpha f) = \alpha T(f) \text{ and } T(f+g) \le T(f) + T(g)$$

   for all f, g in E and $\alpha \ge 0$,
2. *(Monotonicity)* $f \le g$ in E implies $T(f) \le T(g)$,
3. *(Translatability)* $T(f + \alpha.1) = T(f) + \alpha T(1)$ for all $f \in E$ and $\alpha \ge 0$.

In this paper, we also interested in the operator which satisfies the following condition:

(Subunital property) $T(1) \le 1$.

If E and F are closed vector sublattices of the Banach lattices $C(M)$ and $C(N)$, respectively, then every monotone and subadditive operator (functional when $F = \mathbb{R}$) $T : E \to F$ satisfies the inequality

$$|T(f) - T(g)| \le T(|f - g|) \text{ for all } f, g. \tag{1}$$

If an operator (functional when $F = \mathbb{R}$) T is monotone and positively homogeneous, then we necessarily have $T(0) = 0$.

In this paper, we investigate the convergence behavior of sequences of operators that are both monotone and sublinear.

2.2. Power Series Methods

Power series methods have emerged as a highly effective tool in studying approximation and summability, especially when traditional approaches are insufficient. With this method, sequences or series of operators can be written in terms of power series and valuable information about their convergence can be obtained. In the study of Korovkin-type theorems, the power series methods offer a powerful method for investigating convergence properties beyond standard linear methods, thus extending our perspective on approximation processes.

Let $\{p_k\}$ be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{k=0}^{\infty} p_k t^k$$

has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k x_k$$

exists then we say that $x = \{x_k\}$ is convergent in the sense of power series method [22,26]. Note that the method is regular iff $\lim_{0<t\to R^-} \frac{p_k t^k}{p(t)} = 0$ for every k (see,e.g. [3]).

Remark 1. In the case of $R = 1$, if $p_k = 1$ and $p_k = \frac{1}{k+1}$, the power series methods give us Abel method and logarithmic method, respectively. Also, if $R = \infty$ and $p_k = \frac{1}{k!}$, then the power series method gives us Borel method.

Here and in the sequel, power series method is always assumed to be regular.

We assume throughout the paper that the test functions are

$$f_0(x) = 1, f_1(x) = x, f_2(x) = -x, f_3(x) = x^2.$$

3. Approximation via Power Series Method Almost Everywhere

The following result and its proof were suggested by Theorem 2 in [17].

Theorem 1. *Let M be a locally compact subset of $\mathbb{R}$ and we denote a vector sublattice of $VL(M)$ by E that contains the test functions: f_i, $i = 0,1,2,3$. Suppose that $\{T_k\}$ is a sequence of monotone and sublinear operators from E into itself such that*

$$\sup_{0<t<R} \frac{1}{p(t)} \sum_{k=0}^{\infty} |T_k(1)| p_k t^k < \infty \ a.e.. \tag{2}$$

If

$$\lim_{0<t\to R^-} S_t(f_i) = f_i \ a.e., \ i = 0,1,2,3, \tag{3}$$

then for all nonnegative functions f in $E \cap AC(M)$, we have

$$\lim_{0<t\to R^-} S_t(f) = f \ a.e. \tag{4}$$

where $S_t(.) := \frac{1}{p(t)} \sum_{k=0}^{\infty} T_k(.) p_k t^k$. Moreover, if T_k is translatable, then for all $f \in E \cap AC(M)$, we have $\lim_{0<t\to R^-} S_t(f) = f$ a.e..

If M is included in $[0,\infty)$, then the test functions, f_i, $i = 0,1,2,3$, are reduced to 1, $-x$, x^2.

Proof. Let $f \in E \cap AC(M)$ and suppose that y is a point of continuity of f with

$$\lim_{0<t\rightarrow R^{-}} S_t\left(f_i\right)(y) = f_i\left(y\right) \text{ a.e., } i = 0, 1, 2, 3.$$

Since y is a point of continuity of f, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $\left|f\left(x\right) - f\left(y\right)\right| < \varepsilon$ for all $x \in M$ satisfying $\left|x - y\right| < \delta$. If $\left|x - y\right| \geq \delta$, then $\left|f\left(x\right) - f\left(y\right)\right| < \frac{2\|f\|}{\delta^2}\left|x - y\right|^2$, hence, for any $x \in M$ we have

$$\begin{aligned} \left|f\left(x\right) - f\left(y\right)\right| &< \varepsilon + \frac{2\left\|f\right\|}{\delta^2}\left|x - y\right|^2 \\ &= \varepsilon + \frac{2\left\|f\right\|}{\delta^2}\left\{x^2 + 2x\left(C - y\right) + 2C\left(-x\right) + y^2\right\} \end{aligned} \tag{5}$$

where $C = \max\left\{y, 0\right\}$. Thanks to inequality (1) and using (5), monotonicity and sublinearity of the operators T_k, we conclude in case $f \geq 0$ that

$$\begin{aligned} \left|S_t\left(f\right) - f\left(y\right)\right| &= \left|\frac{1}{p\left(t\right)}\sum_{k=0}^{\infty} T_k\left(f\right) p_k t^k - f\left(y\right)\right| \\ &= \left|\frac{1}{p\left(t\right)}\sum_{k=0}^{\infty} T_k\left(f - f\left(y\right) + f\left(y\right)\right) p_k t^k - f\left(y\right)\right| \\ &\leq \left|\frac{1}{p\left(t\right)}\sum_{k=0}^{\infty} \left(T_k\left(f\right) - T_k\left(f\left(y\right).1\right) + f\left(y\right)T_k\left(1\right)\right) p_k t^k - f\left(y\right)\right| \\ &\leq \frac{1}{p\left(t\right)}\sum_{k=0}^{\infty} T_k\left(\left|f - f\left(y\right)\right|\right) p_k t^k + f\left(y\right)\left|\frac{1}{p\left(t\right)}\sum_{k=0}^{\infty} T_k\left(1\right) p_k t^k - 1\right| \\ &\leq S_t\left(\varepsilon + \frac{2\left\|f\right\|}{\delta^2}\left\{x^2 + 2x\left(C - y\right) + 2C\left(-x\right) + y^2\right\}\right) \\ &\quad + f\left(y\right)\left|S_t\left(f_0\right) - f_0\left(y\right)\right| \\ &\leq \varepsilon S_t\left(f_0\right) + \frac{2\left\|f\right\|}{\delta^2}\left\{S_t\left(f_3\right) + 2\left(C - y\right)S_t\left(f_1\right) + 2CS_t\left(f_2\right) + y^2 S_t\left(f_0\right)\right\} \\ &\quad + f\left(y\right)\left|S_t\left(f_0\right) - f_0\left(y\right)\right| \\ &\leq \varepsilon S_t\left(f_0\right) + K\left\{S_t\left(f_3\right) + 2\left(C - y\right)S_t\left(f_1\right) + 2CS_t\left(f_2\right) + y^2 S_t\left(f_0\right)\right. \\ &\quad \left. + \left|S_t\left(f_0\right) - f_0\left(y\right)\right|\right\} \end{aligned}$$

where $K = \max\left\{\frac{2\|f\|}{\delta^2}, f\left(y\right)\right\}$. By our choice of y, we get

$$\lim_{0<t\rightarrow R^{-}} \left|S_t\left(f\right) - f\left(y\right)\right| \leq \varepsilon,$$

since $\varepsilon > 0$ is arbitrary, we obtain that $\lim_{0<t\rightarrow R^{-}} S_t\left(f\right) = f\left(y\right)$. Now suppose in addition that T_k is translatable for all $k \in \mathbb{N}$. Since $f + \|f\| \geq 0$, in view of (4), we get

$$\lim_{0<t\rightarrow R^{-}} S_t\left(f + \|f\|\right) = f + \|f\| \text{ a.e.}$$

and since T_k is also translatable, we can write

$$\begin{aligned} S_t\left(f+\|f\|\right) &= \frac{1}{p(t)}\sum_{k=0}^{\infty} T_k\left(f+\|f\|\right) p_k t^k \\ &= \frac{1}{p(t)}\sum_{k=0}^{\infty}\left\{T_k(f)+\|f\|\,T_k(1)\right\} p_k t^k \\ &= \frac{1}{p(t)}\sum_{k=0}^{\infty} T_k(f)\, p_k t^k + \|f\|\,\frac{1}{p(t)}\sum_{k=0}^{\infty} T_k(1)\, p_k t^k \\ &= S_t(f) + \|f\|\, S_t(f_0). \end{aligned}$$

Thanks to our hypotheses (3), we get, for all $f \in E \cap AC(M)$, $\lim_{0<t\to R^-} S_t(f) = f$ a.e.. Notice now that if M is included in $[0,\infty)$, then one can rewrite the inequality (5) as

$$|f(x) - f(y)| < \varepsilon + \frac{2\|f\|}{\delta^2}\left\{x^2 + 2C(-x) + y^2\right\}.$$

Hence, we get

$$\begin{aligned} |S_t(f) - f(y)| &\le S_t\left(|f - f(y)|\right) + f(y)\left|S_t(f_0) - f_0(y)\right| \\ &\le \varepsilon S_t(f_0) + \frac{2\|f\|}{\delta^2}\left\{S_t(f_3) + 2C S_t(f_2) + y^2 S_t(f_0)\right\} \\ &\quad + f(y)\left|S_t(f_0) - f_0(y)\right| \end{aligned}$$

where $f \ge 0$. So the proofs continue as above and we get the desired result. $\square$

Remark 2. If we replace the space $AC(M)$ with $C(M)$, then Theorem 1 still works.

Corollary 1. *Let $\mathcal{R}([a,b])$ stands for the vector lattice of all Riemann integrable functions defined on one dimensional compact interval $[a,b]$. Suppose that $\{T_k\}$ is a sequence of monotone and sublinear operators from $\mathcal{R}([a,b])$ into $\mathcal{R}([a,b])$ such that*

$$\sup_{0<t<R}\frac{1}{p(t)}\sum_{k=0}^{\infty}|T_k(1)|\, p_k t^k < \infty \ \ a.e..$$

If

$$\lim_{0<t\to R^-} S_t(f_i) = f_i \ \ a.e., \ i = 0,1,2,3,$$

then for all nonnegative functions f in $\mathcal{R}([a,b])$, we have

$$\lim_{0<t\to R^-} S_t(f) = f \ \ a.e..$$

Moreover, if T_k is translatable, then for all $f \in \mathcal{R}([a,b])$, we have $\lim_{0<t\to R^-}$ $S_t(f) = f$ a.e..

4. Approximation via Power Series Method in Measure

The proof of the following result is inspired by Theorem 4 in [17].

Theorem 2. *Suppose that M be a compact subset of $\mathbb{R}$ endowed with positive Borel measure μ and let $\{T_k\}$ is a sequence of monotone, subunital and sublinear operators from $C(M)$ into itself such that*

$$\sup_{0<t<R} \frac{1}{p(t)} \sum_{k=0}^{\infty} |T_k(f_0)| p_k t^k < \infty. \tag{6}$$

If

$$\lim_{0<t\to R^-} \mu(\{y \in M : |S_t(f_i)(y) - f_i(y)| \geq \varepsilon\}) = 0, \;\; i = 0, 1, 2, 3, \tag{7}$$

then for all nonnegative functions f in $C(M)$, we have

$$\lim_{0<t\to R^-} \mu(\{y \in M : |S_t(f)(y) - f(y)| \geq \varepsilon\}) = 0 \tag{8}$$

where $S_t(.) := \frac{1}{p(t)} \sum_{k=0}^{\infty} T_k(.) p_k t^k$. Moreover, if T_k is translatable, then for all $f \in C(M)$, we have $\lim_{0<t\to R^-} \mu(\{y \in M : |S_t(f)(y) - f(y)| \geq \varepsilon\}) = 0$.

Proof. Let f be nonnegative function in $C(M)$ and $\varepsilon > 0$. Thanks to the uniform continuity of f, for every $\varepsilon' < \varepsilon$ with $\varepsilon' > 0$, there is $\delta > 0$, such that

$$|f(x) - f(y)| < \varepsilon' + \frac{2\|f\|}{\delta^2} |x - y|^2 \text{ for all } x, y \in M.$$

Then as in Theorem 1, we have

$$\begin{aligned} |S_t(f) - f(y)| &\leq \varepsilon' + \frac{2\|f\|}{\delta^2} S_t\left((x-y)^2\right) + f(y) |S_t(f_0) - f_0(y)| \\ &\leq \varepsilon' + \frac{2\|f\|}{\delta^2} \left\{S_t(f_3)(y) + 2(C-y) S_t(f_1)(y) + 2C S_t(f_2)(y) + y^2\right\} \\ &\quad + f(y) |S_t(f_0) - f_0(y)| \end{aligned}$$

where $C = \max\{y, 0\}$. In view of the above inequality, we obtain

$$\begin{aligned} &\{y \in M : |S_t(f)(y) - f(y)| \geq \varepsilon\} \\ &\subset \left\{ y \in M : \varepsilon' + \frac{2\|f\|}{\delta^2} \left\{S_t(f_3)(y) + 2(C-y) S_t(f_1)(y) + 2C S_t(f_2)(y) + y^2\right\} \right. \\ &\quad \left. + f(y) |S_t(f_0) - f_0(y)| \geq \varepsilon \right\} \\ &= \{y \in M : (S_t(f_3) - f_3(y)) + 2(C-y)(S_t(f_1) - f_1(y)) \\ &\quad \left. + 2C(S_t(f_2) - f_2(y)) + |S_t(f_0) - f_0(y)| \geq \frac{\varepsilon - \varepsilon'}{K} \right\} \\ &\subset \left\{ y \in M : |S_t(f_3) - f_3(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \\ &\quad \cup \left\{ y \in M : |S_t(f_1) - f_1(y)| \geq \frac{\varepsilon - \varepsilon'}{8K(C-y)} \right\} \end{aligned}$$

$$\cup\left\{y\in M:\left|S_t\left(f_2\right)-f_2\left(y\right)\right|\geq\frac{\varepsilon-\varepsilon'}{8KC}\right\}$$
$$\cup\left\{y\in M:\left|S_t\left(f_0\right)-f_0\left(y\right)\right|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}$$

where $K=\max\left\{\frac{2\|f\|}{\delta^2},f\left(y\right)\right\}$. Hence, we get

$$\begin{aligned}
&\mu\left(\left\{y\in M:\left|S_t\left(f\right)-f\left(y\right)\right|\geq\varepsilon\right\}\right)\\
&\quad\leq\mu\left(\left\{y\in M:\left|S_t\left(f_3\right)-f_3\left(y\right)\right|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}\right)\\
&\qquad+\mu\left(\left\{y\in M:\left|S_t\left(f_1\right)-f_1\left(y\right)\right|\geq\frac{\varepsilon-\varepsilon'}{8K\left(C-y\right)}\right\}\right)\\
&\qquad+\mu\left(\left\{y\in M:\left|S_t\left(f_2\right)-f_2\left(y\right)\right|\geq\frac{\varepsilon-\varepsilon'}{8KC}\right\}\right)\\
&\qquad+\mu\left(\left\{y\in M:\left|S_t\left(f_0\right)-f_0\left(y\right)\right|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}\right).
\end{aligned}$$

In view of our hypotheses (7), we obtain

$$\lim_{0<t\to R^-}\mu\left(\left\{y\in M:\left|S_t\left(f\right)-f\left(y\right)\right|\geq\varepsilon\right\}\right)=0.$$

For the second part, suppose now that T_k is also translatable for all $k\in\mathbb{N}$. Since $f+\|f\|\geq0$, in view of (8), we get

$$\lim_{0<t\to R^-}\mu\left(\left\{y\in M:\left|S_t\left(f+\|f\|\right)-\left(f+\|f\|\right)\left(y\right)\right|\geq\varepsilon\right\}\right)=0$$

and as in Theorem 1, we have

$$S_t\left(f+\|f\|\right)=S_t\left(f\right)+\|f\|\,S_t\left(f_0\right).$$

Thus, for all $f\in C(M)$, we get $\lim_{0<t\to R^-}\mu\left(y\in M:\left|S_t\left(f\right)-f\left(y\right)\right|\geq\varepsilon\right)=0$. □

Remark 3. If M is considered to be locally compact, then Theorem 2 remains true under the following local convergence condition

$$\lim_{0<t\to R^-}\mu\left(\left\{y\in E:\left|S_t\left(f\right)-f\left(y\right)\right|\geq\varepsilon\right\}\right)=0$$

for every compact subset E of M.

5. Rate of Convergence

In this section, we present some estimates of rates of convergence for our Korovkin-type theorems.

We consider the following usual modulus of continuity:

$$\omega\left(f,\delta\right):=\sup_{x,y\in M:\ d(x,y)\leq\delta}\left|f\left(x\right)-f\left(y\right)\right|$$

where $f \in C(M)$ and $\delta > 0$. It is readily seen that, $\omega(f,\delta)$ is an increasing function of δ, $|f(x) - f(y)| \leq \omega(f, d(x,y))$ for each $x, y \in M$, $\omega(f,\delta) \leq 2\|f\|$ for every $\delta > 0$, and $\omega(f,\gamma\delta) \leq (1+\gamma)\,\omega(f,\delta)$ for every $\gamma, \delta > 0$.

Theorem 3. *Let M, E and $\{T_k\}$ be as in Theorem 1. If*

(i) $\lim_{0<t\to R^-} S_t(f_0) = f_0$ *a.e.,*

(ii) $\lim_{0<t\to R^-} \omega(f,\delta_t) = 0$ *a.e., where* $\delta_t := \sqrt{S_t\left((. - y)^2\right)}$.

Then for all nonnegative functions f in $E \cap AC(M)$, we have

$$\lim_{0<t\to R^-} S_t(f) = f \ \text{a.e.}.$$

Moreover, if T_k is translatable, then for all $f \in E \cap AC(M)$, we have $\lim_{0<t\to R^-}$

$S_t(f) = f$ *a.e..*

Proof. Let $f \in E \cap AC(M)$ and suppose that y is a point of continuity of f with

$$\lim_{0<t\to R^-} S_t(f_0)(y) = f_0(y) \ \text{a.e. and} \lim_{0<t\to R^-} \omega(f,\delta_t) = 0 \ \text{a.e.}.$$

With the help of positive linear operators T_k and property of modulus of continuity ω, we get

$$\begin{aligned} |S_t(f) - f(y)| &\leq S_t(|f - f(y)|) + f(y)\,|S_t(f_0) - f_0| \\ &\leq S_t\left(\left(1 + \frac{(x-y)^2}{\delta^2}\right)\omega(f,\delta)\right) + f(y)\,|S_t(f_0) - f_0(y)| \\ &\leq \omega(f,\delta)\,S_t(f_0) + \frac{\omega(f,\delta)}{\delta^2} S_t\left((x-y)^2\right) + f(y)\,|S_t(f_0) - f_0(y)| \end{aligned}$$

If we choose $\delta := \delta_t := \sqrt{S_t\left((x-y)^2\right)}$, this yield that

$$|S_t(f) - f(y)| \leq \omega(f,\delta_t)\,(S_t(f_0) + 1) + f(y)\,|S_t(f_0) - f_0(y)|\,.$$

By our choice of y, we get

$$\lim_{0<t\to R^-} S_t(f) = f(y)\,.$$

Now suppose in addition that T_k is translatable for all $k \in \mathbb{N}$. Similar to the proof of Theorem 1, we can easily get, for all $f \in E \cap AC(M)$, $\lim_{0<t\to R^-} S_t(f)$ $= f$ a.e.. □

Theorem 4. *Suppose that M, $\{T_k\}$ be as in Theorem 2. If*

(i) $\lim_{0<t\to R^-} \mu(\{y \in M : |S_t(f_0)(y) - f_0(y)| \geq \varepsilon\}) = 0$,

(ii) $\lim_{0<t\to R^-} \mu(\{y \in M : \omega(f,\delta_t) \geq \varepsilon\}) = 0$ *a.e., where* $\delta_t :$ $= \sqrt{S_t\left((. - y)^2\right)}$,

then for all nonnegative functions f in $C(M)$, we have

$$\lim_{0<t\to R^-} \mu\left(\left\{y\in M : \left|S_t\left(f\right)\left(y\right)-f\left(y\right)\right|\geq\varepsilon\right\}\right)=0.$$

Moreover, if T_k is translatable, then for all $f\in C(M)$, we have

$$\lim_{0<t\to R^-} \mu\left(\left\{y\in M : \left|S_t\left(f\right)\left(y\right)-f\left(y\right)\right|\geq\varepsilon\right\}\right)=0.$$

Proof. Let f be nonnegative function in $C(M)$ and $\varepsilon>0$. As in Theorem 3, we have

$$\left|S_t\left(f\right)-f\left(y\right)\right|\leq\omega\left(f,\delta_t\right)\left(S_t\left(f_0\right)+1\right)+f\left(y\right)\left|S_t\left(f_0\right)-f_0\left(y\right)\right|.$$

In view of the above inequality, we obtain

$$\begin{aligned}&\left\{y\in M : \left|S_t\left(f\right)\left(y\right)-f\left(y\right)\right|\geq\varepsilon\right\}\\ &\quad\subset\left\{y\in M : \omega\left(f,\delta_t\right)\left(S_t\left(f_0\right)+1\right)+f\left(y\right)\left|S_t\left(f_0\right)-f_0\left(y\right)\right|\geq\varepsilon\right\}.\end{aligned}$$

Now considering the above inclusion, the hypotheses (i), (ii), the proof is completed at once. Also, suppose now in addition that T_k is translatable for all $k\in\mathbb{N}$. Similar to the proof of Theorem 3, we can easily get, for all $f\in C(M)$,

$$\lim_{0<t\to R^-} \mu\left(\left\{y\in M : \left|S_t\left(f\right)\left(y\right)-f\left(y\right)\right|\geq\varepsilon\right\}\right)=0.$$

□

6. Applications to Approximation Theorems

In this section we provide some examples that satisfy our theorems and give quantitative estimate. We also define Bernstein-Chlodovsky-Kantorovich-Choquet (BCKC) polynomial operators similar to given before and obtain approximation estimate.

Example 1. Firstly, let us consider monotone, sublinear and translatable operators T_k, the possibilistic Kantorovich operators [12], given by

$$T_k\left(f\right)\left(y\right)=\sum_{n=0}^{k}\binom{k}{n}y^n\left(1-y\right)^{k-n}\sup\left\{f\left(y\right):y\in\left[\frac{n}{k+1},\frac{n+1}{k+1}\right]\right\}.$$

As it is known, $T_k\left(f_0\right)\left(y\right)=f_0\left(y\right)$, $T_k\left(f_1\right)\left(y\right)=\frac{k}{k+1}f_1\left(y\right)+\frac{1}{k+1}$, $T_k\left(f_2\right)\left(y\right)=\frac{k}{k+1}f_2\left(y\right)$ and $T_k\left(f_3\right)\left(y\right)=\left(\frac{k}{k+1}\right)^2f_3\left(y\right)+\frac{2k}{(k+1)^2}f_1\left(y\right)+\frac{1}{(k+1)^2}$ (see, also [18]). Using these operators, we introduce the following

$$L_k\left(f\right)\left(y\right)=\left(1+\left(-1\right)^k\right)T_k\left(f\right)\left(y\right)$$

for all $f\in E\cap AC(M)$. It is easy to see that each L_k is a monotone, sublinear and translatable operator. Let the power series method is given with $p_k=1$,

$k = 0, 1, \ldots$. Then, $(-1)^k$ is convergent in the sense of power series method to 0 and also, observe that

$$\sup_{0<t<1} \frac{1}{p(t)} \sum_{k=0}^{\infty} |L_k(f_0)| p_k t^k = \sup_{0<t<1} \frac{1}{p(t)} \sum_{k=0}^{\infty} \left|1 + (-1)^k\right| p_k t^k = 2 < \infty.$$

Applying now Theorem 1, it follows that

$$\lim_{0<t\to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} L_k(f)(y) p_k t^k = f(y) \text{ at each point } y \text{ of continuity of } f.$$

Before next examples, let us recall some notations:

Suppose that (M, Σ) is a measurable space with $M \neq \varnothing$ and Σ is a σ-algebra of subsets of M. The function $m : \Sigma \to [0, +\infty]$ is said to be a capacity (or monotone set function) iff $m(\varnothing) = 0$ and $m(E) \leq m(F)$ for all $E, F \in \Sigma$ with $E \subset F$ (monotonicity). A capacity m is submodular if

$$m(E \cup F) + m(E \cap F) \leq m(E) + m(F), \text{ for all } E, F \in \Sigma.$$

If $m(M) = 1$, then a capacity m is said to be normailzed. The capacities provide a non additive generalization of probability measures, that is, of capacities m having the property of σ-additivity,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

for every sequence $\{E_k\}$ of disjoint sets with $\bigcup\limits_{k=1}^{\infty} E_k \in \Sigma$.

Let m be normailzed capacity on Σ. If $f : M \to \mathbb{R}$ is Σ-measurable, then for any $E \in \Sigma$, the Choquet integral is given by

$$(C)\int_E f dm = \int_0^{+\infty} m(F_\beta(f) \cap E) d\beta + \int_{-\infty}^{0} [m(F_\beta(f) \cap E) - m(E)] d\beta,$$

where $F_\beta(f) = \{u \in M : f(u) \geq \beta\}$. If $(C)\int\limits_E f dm \in \mathbb{R}$, then f is called Choquet integrable on E. Note that if $f \geq 0$, then $\int_{-\infty}^{0} . = 0$. Also, if m is σ -additive measure, then the Choquet integral coincides with Lebesgue integral. We now conclude by mentioning the following assumptions concerning Choquet integrals:

- if $f \geq 0$, then $(C)\int_M f dm \geq 0$,
- $f \leq g$ implies $(C)\int_M f dm \leq (C)\int\limits_M g dm$,
- for all $c \geq 0$, we have $(C)\int_M c f dm = c(C)\int\limits_M f dm$,
- $(C)\int_M 1 dm = m(M)$,
- if m is submodular, then $(C)\int_M (f + g) dm \leq (C)\int_M f dm + (C)\int_M g dm$.

For some other notions about capacity and Choquet integral, we refer the readers to [1,4,9,33,34] (see also [15] and [13], as well as the references therein).

Example 2. Let us consider the Bernstein-Kantorovich-Choquet (BKC) polynomial operators $K_{k,m}^{(1)} : C([0,1]) \to C([0,1])$ by

$$K_{k,m}^{(1)}(f)(y) = \sum_{n=0}^{k} \frac{(C)\int_{\frac{n}{k+1}}^{\frac{n+1}{k+1}} f(u)\, dm(u)}{m\left(\left[\frac{n}{k+1}, \frac{n+1}{k+1}\right]\right)} \binom{k}{n} y^n (1-y)^{k-n}$$

where $m := \sqrt{\lambda}$ with the Lebesque measure λ. It is known from [16] that the operators monotone, translatable and sublinear. Because of the results given in [14], we can say that $K_{k,m}^{(1)}(f_0) \to f_0$, $K_{k,m}^{(1)}(f_1) \to f_1$, $K_{k,m}^{(1)}(f_2) \to f_2$ and $K_{k,m}^{(1)}(f_3) \to f_3$ uniformly on $[0,1]$. Now, define the sequence

$$K_{k,m}(f)(y) = \left(1 + (-1)^k\right) K_{k,m}^{(1)}(f)(y).$$

Let the power series method is given with $p_k = 1$, $k = 0, 1, \ldots$. Then, $(-1)^k$ is convergent in the sense of power series method to 0 and also, observe that

$$\sup_{0<t<1} \frac{1}{p(t)} \sum_{k=0}^{\infty} |K_{k,m}(f_0)| p_k t^k < \infty.$$

Therefore, $K_{k,m}$ satisfies the conditions of Theorem 2, then for all $f \in C([0,1])$, we have $\lim_{0<t\to R^-} \mu\left(\left\{y \in [0,1] : \left|\frac{1}{p(t)} \sum_{k=0}^{\infty} K_{k,m}(f)(y) p_k t^k - f(y)\right| \geq \varepsilon\right\}\right) = 0$.

Now, we also prove a quantitative estimate of BKC for all nonnegative function $f \in C([0,1])$. Let $S_t\left(K_{k,m}^{(1)}(f)\right) := \frac{1}{p(t)} \sum_{k=0}^{\infty} K_{k,m}^{(1)}(f) p_k t^k$. Then, for arbitrarily fixed y, as in Theorem 3, we obtain

$$\begin{aligned}
&\left|S_t\left(K_{k,m}^{(1)}(f)\right) - f(y)\right| \\
&\quad \leq \frac{1}{p(t)} \sum_{k=0}^{\infty} K_{k,m}^{(1)}(|f - f(y)|) p_k t^k + f(y) \left|\frac{1}{p(t)} \sum_{k=0}^{\infty} K_{k,m}^{(1)}(1) p_k t^k - 1\right| \\
&\quad \leq \frac{1}{p(t)} \sum_{k=0}^{\infty} K_{k,m}^{(1)}(|f - f(y)|) p_k t^k \\
&\leq \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta^2} S_t\left(K_{k,m}^{(1)}\left((. - y)^2\right)\right)
\end{aligned}$$

and choosing $\delta := \sqrt{S_t\left(K_{k,m}^{(1)}\left((.-y)^2\right)\right)}$, we get

$$\begin{aligned}\left|S_t\left(K_{k,m}^{(1)}(f)\right)-f(y)\right| &\leq 2\omega\left(f,\sqrt{S_t\left(K_{k,m}^{(1)}\left((.-y)^2\right)\right)}\right)\\ &\leq 2\omega\left(f,\sqrt{S_t\left(K_{k,m}^{(1)}(.^2)\right)+2yS_t\left(K_{k,m}^{(1)}(-.)\right)+y^2}\right).\end{aligned}$$

Example 3. Let us define the Bernstein-Chlodovsky-Kantorovich-Choquet (BCKC) polynomial operators $C_{k,m}^{(1)}:\mathcal{R}([0,\alpha_k])\to\mathcal{R}([0,\alpha_k])$ by

$$C_{k,m}^{(1)}(f)(y)=\sum_{n=0}^{k}\frac{(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}}f(u)\,dm(u)}{m\left(\left[\frac{n\alpha_{k+1}}{k+1},\frac{(n+1)\alpha_{k+1}}{k+1}\right]\right)}\binom{k}{n}\left(\frac{y}{\alpha_{k+1}}\right)^n\left(1-\frac{y}{\alpha_{k+1}}\right)^{k-n}$$

where $m:=\sqrt{\lambda}$ with the Lebesque measure λ and $\{\alpha_n\}$ is a positive sequence with $\lim_n\alpha_n=\infty$, $\lim_n\frac{\alpha_n}{n}=0$. It is important to point out that if $m:=\lambda$, then the operators become Bernstein-Chlodovsky-Kantorovich operators and if $m:=\delta_k$ (the Dirac measure), then the operators become Bernstein-Chlodovsky operators. Now, thanks to Choquet integral's sublinearity, monotonity and comonotone additivity with a submodular capacity m, we have these properties for $C_{k,m}^{(1)}$.

Clearly, $C_{k,m}^{(1)}(f_0)(y)=f_0(y)$. For the convergence of the $C_{k,m}^{(1)}(f_1)(y)$, we will first calculate the integrals $(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}}udm(u)$. We get

$$\begin{aligned}(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}}udm(u) &= \int_0^\infty m\left(\left\{u\in\left[\frac{n\alpha_{k+1}}{k+1},\frac{(n+1)\alpha_{k+1}}{k+1}\right]:u\geq\beta\right\}\right)d\beta\\ &= \int_0^{\frac{n\alpha_{k+1}}{k+1}} m\left(\left\{u\in\left[\frac{n\alpha_{k+1}}{k+1},\frac{(n+1)\alpha_{k+1}}{k+1}\right]:u\geq\beta\right\}\right)d\beta\\ &\quad+\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} m\left(\left\{u\in\left[\frac{n\alpha_{k+1}}{k+1},\frac{(n+1)\alpha_{k+1}}{k+1}\right]:u\geq\beta\right\}\right)d\beta\\ &= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right)+\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}}\sqrt{\frac{(n+1)\alpha_{k+1}}{k+1}-\beta}d\beta\end{aligned}$$

$$= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right) + \int_{0}^{\frac{\alpha_{k+1}}{k+1}} \sqrt{\eta}\, d\eta = \frac{3n+2}{3}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{3}{2}}.$$

Using this equality in $C_{k,m}^{(1)}(f_1)(y)$, we have

$$C_{k,m}^{(1)}(f_1)(y) = \frac{k}{k+1}y + \frac{2\alpha_{k+1}}{3(k+1)}.$$

For the convergence of the $C_{k,m}^{(1)}(f_2)(y)$, we will first calculate the integrals $(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} (-u)\, dm(u)$. We get

$$\begin{aligned}
&(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} (-u)\, dm(u) \\
&= \int_{-\infty}^{0} \left\{ m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right] : -u \geq \beta\right\}\right) - \sqrt{\frac{\alpha_{k+1}}{k+1}} \right\} d\beta \\
&= \int_{-\frac{n\alpha_{k+1}}{k+1}}^{0} \left\{ m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right] : u \leq -\beta\right\}\right) - \sqrt{\frac{\alpha_{k+1}}{k+1}} \right\} d\beta \\
&\quad + \int_{-\frac{(n+1)\alpha_{k+1}}{k+1}}^{-\frac{n\alpha_{k+1}}{k+1}} \left\{ m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right] : u \leq -\beta\right\}\right) - \sqrt{\frac{\alpha_{k+1}}{k+1}} \right\} d\beta \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{-n\alpha_{k+1}}{k+1}\right) + \int_{-\frac{(n+1)\alpha_{k+1}}{k+1}}^{-\frac{n\alpha_{k+1}}{k+1}} \sqrt{-\beta_1 - \frac{n\alpha_{k+1}}{k+1}}\, d\beta_1 - \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{\alpha_{k+1}}{k+1}\right) \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{-n\alpha_{k+1}}{k+1}\right) + \int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} \sqrt{\beta - \frac{n\alpha_{k+1}}{k+1}}\, d\beta - \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{\alpha_{k+1}}{k+1}\right) \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right) + \int_{0}^{\frac{\alpha_{k+1}}{k+1}} \sqrt{\eta}\, d\eta - \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{\alpha_{k+1}}{k+1}\right) = \frac{-3n+1}{3}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{3}{2}}
\end{aligned}$$

which gives

$$C_{k,m}^{(1)}(f_2)(y) = \frac{k}{k+1}(-y) + \frac{\alpha_{k+1}}{3(k+1)}.$$

Finally, for the convergence of the $C_{k,m}^{(1)}(f_3)(y)$, we will first calculate the integrals $(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} u^2 dm(u)$ and we get

$$
\begin{aligned}
&(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} u^2 dm(u) \\
&= \int_0^{\infty} m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right] : u \geq \sqrt{\beta}\right\}\right) d\beta \\
&= \int_0^{\left(\frac{n\alpha_{k+1}}{k+1}\right)^2} m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right] : u \geq \sqrt{\beta}\right\}\right) d\beta \\
&\quad + \int_{\left(\frac{n\alpha_{k+1}}{k+1}\right)^2}^{\left(\frac{(n+1)\alpha_{k+1}}{k+1}\right)^2} m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right] : u \geq \sqrt{\beta}\right\}\right) d\beta \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right)^2 + \int_{\left(\frac{n\alpha_{k+1}}{k+1}\right)^2}^{\left(\frac{(n+1)\alpha_{k+1}}{k+1}\right)^2} \sqrt{\frac{(n+1)\alpha_{k+1}}{k+1} - \sqrt{\beta}} d\beta \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right)^2 + 2\int_0^{\frac{\alpha_{k+1}}{k+1}} \sqrt{\eta}\left(\frac{(n+1)\alpha_{k+1}}{k+1} - \eta\right) d\eta \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right)^2 + \frac{4(n+1)}{3}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{5}{2}} - \frac{4}{5}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{5}{2}}
\end{aligned}
$$

which gives

$$C_{k,m}^{(1)}(f_3)(y) = \frac{k(k-1)}{(k+1)^2} y^2 + \frac{7}{3}\frac{k\alpha_{k+1}}{(k+1)^2} y + \frac{4}{3}\left(\frac{\alpha_{k+1}}{k+1}\right)^2 - \frac{4}{5}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{5}{2}}.$$

Using now $C_{k,m}^{(1)}$, we define the following

$$C_{k,m}(f)(y) = \left(1 + (-1)^k\right) C_{k,m}^{(1)}(f)(y).$$

It is easy to see that each $C_{k,m}$ is a monotone, sublinear and translatable operator. Let the power series method is given with $p_k = 1$, $k = 0, 1, \ldots$. Then, $(-1)^k$ is convergent in the sense of power series method to 0 and also, observe that

$$\sup_{0<t<1} \frac{1}{p(t)} \sum_{k=0}^{\infty} \left| C_{k,m}(f_0) \right| p_k t^k < \infty.$$

Consequently: Applying Corollary 1, for all $f \in \mathcal{R}([0, \alpha_k])$, we have $\lim_{0<t\to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} C_{k,m}(f) p_k t^k = f$ a.e..

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