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ORIGINAL PAPER

Approximation theorems via power series statistical convergence and applications for sequences of monotone and sublinear operators

Sevda Yıldız[1] · Kamil Demirci[1] · Fadime Dirik[1]

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Abstract
In this paper, we address the problem of extending Korovkin-type approximation theorems to sequences of monotone and sublinear operators via the concept of power series statistical convergence (statistical convergence with respect to power series methods), which is incompatible with statistical convergence and in general a non-matrix method. It is important to emphasise that any positive linear operator is monotone sublinear, but the opposite is not correct. We establish several Korovkin-type theorems under these generalized settings and demonstrate their applicability with concrete examples.

Keywords Power series method · Statistical convergence · Monotone and sublinear operators · Choquet integral

Mathematics Subject Classification 40G10 · 41A36 · 47H05

1 Introduction

The field of approximation theory has undergone significant developments with the incorporation of statistical convergence methods [21, 28]. These methods facilitate a more expansive investigation of convergence phenomena, extending beyond the scope of traditional approaches. Such methods are particularly valuable in instances where traditional pointwise or uniform convergence may not fully capture the behaviour of operator sequences, particularly on unbounded intervals [5, 9, 12, 26] . In this context, Korovkin-type approximation theorems provide a robust framework for analysing the convergence properties of sequences of linear positive operators [22, 23]. Recently, however, the scope of these theorems has

Sevda Yıldız, Kamil Demirci and Fadime Dirik have contributed equally to this work.

✉ Sevda Yıldız
sevdaorhan@sinop.edu.tr

Kamil Demirci
kamild@sinop.edu.tr

Fadime Dirik
fdirik@sinop.edu.tr

1 Department of Mathematics, Sinop University, 57000 Sinop, Turkey

been extended to more general operator structures, including monotone and sublinear operators. This class of operators is much broader and more applicable, because while every positive linear operator is monotone and sublinear, the converse is not true. These operators are also characterised by their structural flexibility and applicability to approximation problems in various function spaces. Following the pioneering works of Gal and Niculescu [17, 19], the use of these operators in Korovkin-type approximation theory has seen a notable increase, driving further research into their convergence properties and applications (see, for example [14, 16, 20]). The concept of power series statistical convergence [34, 35], a novel method that is incompatible with classical statistical convergence and represents a non-matrix approach, allows for the analysis of operator sequences in settings where classical convergence fails, thereby offering a more versatile tool for approximation theory. Researchers have investigated its properties in various settings, including its relationship to classical convergence methods [30], its applications in summability theory [6, 8], and its extensions to more general frameworks [3, 7, 11]. It is noteworthy that several studies have focused on power series statistical convergence, thereby revealing its rich structural properties and potential for broader applications (see also, [27, 31–33]). By focusing on almost everywhere convergence and convergence in measure, this approach aligns with modern extensions of Korovkin-type theorems.

This paper is concerned with the development of Korovkin-type approximation theorems utilising the power series statistical convergence almost everywhere, in measure for sequences of operators that are monotone and sublinear. By extending statistical convergence concepts to these classes of operators, we establish a more comprehensive approach to approximation theory, with the objective of providing both qualitative and quantitative insights. Furthermore, we address the rate of convergence in terms of the modulus of continuity, which is incorporated to precisely characterize it.

2 Preliminaries

This section serves to recall some fundamental definitions and concepts related to our study, particularly those associated with Korovkin-type approximation theory, statistical convergence, power series methods, P -Statistical Convergence and finally, monotone and sublinear operators.

2.1 Statistical convergence

Statistical convergence was initially developed in the context of real sequences. It generalises the notion of convergence by considering the density of indices at which a sequence deviates from its limit, thereby making it a powerful tool for applications in approximation theory, functional analysis, and probability theory. Since the pivotal paper by Gadjiev and Orhan [13], statistical convergence has gained significant traction in approximation theory, opening up new avenues beyond classical notions of convergence.

Let E be a subset of $\mathbb{N}$, the set of natural numbers, then *the natural density of* E, denoted by $\delta(E)$, is given by:

$$\delta(E) := \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|$$

whenever the limit exists, where $|.|$ denotes the cardinality of the set [25]. The sequence $x = \{x_k\}$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) =$

0, i.e. for every $\varepsilon > 0$,

$$\lim_{n} \frac{1}{n} |\{k \leq n \ : |x_k - L| \geq \varepsilon \ \}| = 0 \ ([21, 28]).$$

We note that every convergent sequence is statistically convergent, but not conversely.

2.2 Power series methods

Power series methods have emerged as a versatile tool in the study of approximation and summability, particularly when traditional approaches prove inadequate. By representing sequences or series of operators as expansions in terms of power series, valuable insights can be gained into their behaviour. In the context of Korovkin-type theorems, power series methods offer a powerful means of investigating convergence properties that may elude standard linear methods, thereby broadening our understanding of approximation processes.

Let $\{p_k\}$ be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{k=0}^{\infty} p_k t^k$$

has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k x_k$$

exists then we say that $x = \{x_k\}$ is convergent in the sense of power series method [24, 29]. Note that the method is regular iff $\lim_{0<t\to R^-} \dfrac{p_k t^k}{p(t)} = 0$ for every k (see,e.g. [2]).

Remark 1 It follows immediately from the definition that in the case of $R = 1$, if $p_k = 1$ and $p_k = \frac{1}{k+1}$, the power series methods coincide with the Abel summability method and logarithmic summability method, respectively. In the case of $R = \infty$ and $p_k = \frac{1}{k!}$, the power series method coincides with Borel summability method.

Here and in the sequel, power series method is always assumed to be regular.

2.3 *P*-statistical convergence

Another noteworthy approach to convergence is P-statistical convergence, which was recently introduced by Ünver and Orhan [35] (see also [34]). This method represents a distinct form of statistical convergence derived through power series methods. The authors provided compelling examples that illustrate the differences between statistical convergence and P-statistical convergence. Among the various power series methods, the Abel and Borel methods stand out as particularly effective alternatives to standard convergence.

Let $E \subset \mathbb{N}_0$. If the limit

$$\delta_P(E) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k\in E} p_k t^k$$

exists, then $\delta_P(E)$ is called the P-density of E. Note that, from the definition of a power series method and P-density it is obvious that $0 \leq \delta_P(E) \leq 1$ whenever it exists.

Let $x=\{x_k\}$ be a sequence. Then x is said to be statistically convergent with respect to power series method (P-statistically convergent) to L if for any $\varepsilon>0$

$$\lim_{0<t\to R^-}\frac{1}{p\left(t\right)}\sum_{k\in E_{\varepsilon}}p_k t^k=0$$

where $E_{\varepsilon}=\{k\in\mathbb{N}_0:|x_k-L|\geq\varepsilon\}$, that is $\delta_P\left(E_{\varepsilon}\right)=0$ for any $\varepsilon>0$. This is denoted by $st_P-\lim x_k=L$.

2.4 Sublinear and monotone operators

In this subsection, we revisit some properties related to sublinear and monotone operators, which are fundamental to our study.

Let M be a metric measure space, denoted by the triple (M,d,μ) where M is a space equipped with the metric d and the measure μ defined on the sigma field of Borel subsets of M.

If we consider the vector lattice $VL(M)$ of all real-valued functions defined on M, endowed with the pointwise ordering, then important vector sublattices of $VL(M)$ are $C(M)$, which is the space of functions of all continuous and bounded, and $AC(M)$, which is the space of functions of all bounded and almost everywhere continuous, and $L^q(M)$ $(1\leq q<\infty)$, which is the space of functions of all Borel measurable and Lebesgue integrable with modulo equality a.e..

On $C(M)$, we consider the uniform norm $\|f\|=\sup_{x\in M}|f(x)|$, while on $L^q(M)$, $1\leq q<\infty$, we consider the usual q-norm $\|f\|_q=\left(\int_M|f(x)|^q\,dx\right)^{1/q}$.

Let M and N be two metric spaces and E and F two respectively ordered vector subspaces (or the positive cones) of $VL(M)$ and $VL(N)$ that contain the unity. An operator $S:E\to F$ is called *a weakly nonlinear operator* (respectively *a weakly nonlinear functional* when $F=\mathbb{R}$) if it satisfies the following three conditions:

1. *(Sublinearity)* S is positively homogeneous and subadditive, that is,

$$S\left(\alpha f\right)=\alpha S\left(f\right)\text{ and }S\left(f+g\right)\leq S\left(f\right)+S\left(g\right)$$

for all f,g in E and $\alpha\geq 0$,
2. *(Monotonicity)* $f\leq g$ in E implies $S\left(f\right)\leq S\left(g\right)$,
3. *(Translatability)* $S(f+\alpha.1)=S(f)+\alpha S(1)$ for all $f\in E$ and $\alpha\geq 0$.

In this paper, we are also interested in the operator which satisfies the following condition: *(Subunital property)* $S(1)\leq 1$.

If E and F are closed vector sublattices of the Banach lattices $C(M)$ and $C(N)$, respectively, then every monotone and subadditive operator (functional when $F=\mathbb{R}$) $S:E\to F$ satisfies the inequality

$$|S\left(f\right)-S\left(g\right)|\leq S\left(|f-g|\right)\text{ for all }f,g. \tag{1}$$

If an operator (functional when $F=\mathbb{R}$) S is monotone and positively homogeneous, then we necessarily have $S(0)=0$.

In this paper, we explore the convergence behavior of sequences of operators that are both monotone and sublinear.

2.5 Choquet integral

This subsection focuses on the Choquet integral, revisiting its essential definitions and properties.

Suppose that (M, Σ) is a measurable space with $M \neq \varnothing$ and Σ is a σ-algebra of subsets of M. The function $m : \Sigma \to [0, +\infty]$ is said to be a capacity (or monotone set function) iff $m(\varnothing) = 0$ and $m(E) \leq m(F)$ for all $E, F \in \Sigma$ with $E \subset F$ (monotonicity). A capacity m is submodular if

$$m(E \cup F) + m(E \cap F) \leq m(E) + m(F), \text{ for all } E, F \in \Sigma.$$

If $m(M) = 1$, then a capacity m is said to be normailzed. The capacities provide a non additive generalization of probability measures, that is, of capacities m having the property of σ-additivity,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

for every sequence $\{E_k\}$ of disjoint sets with $\bigcup_{k=1}^{\infty} E_k \in \Sigma$.

Let m be normailzed capacity on Σ. If $f : M \to \mathbb{R}$ is Σ-measurable, then for any $E \in \Sigma$, the Choquet integral is given by

$$(C)\int_E f dm = \int_0^{+\infty} m\left(F_\beta(f) \cap E\right) d\beta + \int_{-\infty}^{0} \left[m\left(F_\beta(f) \cap E\right) - m(E)\right] d\beta,$$

where $F_\beta(f) = \{u \in M : f(u) \geq \beta\}$. If $(C)\int_E f dm \in \mathbb{R}$, then f is called Choquet integrable on E. Note that if $f \geq 0$, then $\int_{-\infty}^{0} . = 0$. Also, if m is σ -additive measure, then the Choquet integral coincides with Lebesgue integral. We now conclude by mentioning the following assumptions concerning Choquet integrals:

- if $f \geq 0$, then $(C)\int_M f dm \geq 0$,
- $f \leq g$ implies $(C)\int_M f dm \leq (C)\int_M g dm$,
- for all $c \geq 0$, we have $(C)\int_M cf dm = c(C)\int_M f dm$,
- $(C)\int_M 1 dm = m(M)$,
- if m is submodular, then $(C)\int_M (f + g) dm \leq (C)\int_M f dm + (C)\int_M g dm$.

For some other notions about capacity and Choquet integral, we refer the readers to [1, 4, 10, 36, 37] (see also [16, 18], as well as the references therein).

We assume throughout the paper that the test functions are

$$f_0(x) = 1, \ f_1(x) = x, \ f_2(x) = -x, \ f_3(x) = x^2.$$

3 Approximation via *P*-statistical convergence almost everywhere

In this section, we establish and prove Korovkin-type approximation theorems using the concept of power series statistical convergence almost everywhere. This approach extends the classical framework of statistical convergence by incorporating power series methods.

Theorem 1 *Let M be a locally compact subset of $\mathbb{R}$ and we denote a vector sublattice of $VL(M)$ by E that contains the test functions: f_i, $i = 0, 1, 2, 3$. Suppose that $\{S_k\}$ is a sequence of monotone and sublinear operators from E into itself. If*

$$st_P - \lim S_k(f_i) = f_i \text{ a.e.}, i = 0, 1, 2, 3, \tag{2}$$

then for all nonnegative functions f in $E \cap AC(M)$, we have

$$st_P - \lim S_k(f) = f \text{ a.e.}. \tag{3}$$

Moreover, if S_k is translatable, then for all $f \in E \cap AC(M)$, we have

$$st_P - \lim S_k(f) = f a.e..$$

Clarifying a point, it is essential to mention that if M is included in $[0, \infty)$, then the test functions, f_i, $i = 0, 1, 2, 3$, are reduced to $1, -x, x^2$.

Proof Let $f \in E \cap AC(M)$ and suppose that y is a point of continuity of f with

$$st_P - \lim S_k(f_i)(y) = f_i(y) \text{ a.e.}, i = 0, 1, 2, 3.$$

Since y is a point of continuity of f, we can write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x \in M$ satisfying $|x - y| < \delta$. If $|x - y| \geq \delta$, then $|f(x) - f(y)| < \frac{2\|f\|}{\delta^2}|x - y|^2$, hence, for any $x \in M$ we have

$$|f(x) - f(y)| < \varepsilon + \frac{2\|f\|}{\delta^2}|x - y|^2 = \varepsilon + \frac{2\|f\|}{\delta^2}\left\{x^2 + 2x(C - y) + 2C(-x) + y^2\right\} \tag{4}$$

where $C = \max\{y, 0\}$. Thanks to inequality (1) and (4), monotonicity and sublinearity of the operators S_k, we conclude in case $f \geq 0$ that

$$\begin{aligned}
&|S_k(f) - f(y)| \\
&\quad= |S_k(f - f(y) + f(y)) - f(y)| \\
&\quad\leq |S_k(f) - S_k(f(y).1) + f(y)S_k(1) - f(y)| \\
&\quad\leq S_k(|f - f(y)|) + f(y)|S_k(1) - 1| \\
&\quad\leq S_k\left(\varepsilon + \frac{2\|f\|}{\delta^2}\left\{x^2 + 2x(C - y) + 2C(-x) + y^2\right\}\right) + f(y)|S_k(f_0) - f_0(y)| \\
&\quad\leq \varepsilon + \varepsilon|S_k(f_0) - f_0(y)| + \frac{2\|f\|}{\delta^2}\left\{S_k(f_3) + 2(C - y)S_k(f_1) + 2CS_k(f_2) + y^2 S_k(f_0)\right\} \\
&\qquad+ f(y)|S_k(f_0) - f_0(y)| \\
&\quad\leq \varepsilon + K\Big\{S_k(f_3) + 2(C - y)S_k(f_1) + 2CS_k(f_2) + y^2 S_k(f_0) \\
&\qquad+ |S_k(f_0) - f_0(y)|\Big\}
\end{aligned}$$

where $K = \max\left\{\frac{2\|f\|}{\delta^2}, \varepsilon + f(y)\right\}$. By our choice of y and in view of the above inequality, we obtain, for every $\varepsilon < \varepsilon'$,

$$
\begin{aligned}
&\left\{k \in \mathbb{N}_0 : |S_k(f) - f(y)| \geq \varepsilon'\right\} \\
&\quad \subset \left\{k \in \mathbb{N}_0 : K\left\{S_k(f_3) + 2(C-y)S_k(f_1) + 2CS_k(f_2) + y^2 S_k(f_0)\right.\right. \\
&\qquad \left.\left. + |S_k(f_0) - f_0(y)|\right\} \geq \varepsilon' - \varepsilon\right\} \\
&\quad = \{k \in \mathbb{N}_0 : (S_k(f_3) - f_3(y)) + 2(C-y)(S_k(f_1) - f_1(y)) \\
&\qquad \left. +2C(S_k(f_2) - f_2(y)) + |S_k(f_0) - f_0(y)| \geq \frac{\varepsilon' - \varepsilon}{K}\right\} \\
&\quad \subset \left\{k \in \mathbb{N}_0 : |S_k(f_3) - f_3(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\} \\
&\quad \cup \left\{k \in \mathbb{N}_0 : |S_k(f_1) - f_1(y)| \geq \frac{\varepsilon' - \varepsilon}{8K(C-y)}\right\} \\
&\quad \cup \left\{k \in \mathbb{N}_0 : |S_k(f_2) - f_2(y)| \geq \frac{\varepsilon' - \varepsilon}{8KC}\right\} \\
&\quad \cup \left\{k \in \mathbb{N}_0 : |S_k(f_0) - f_0(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\delta_P\left(\left\{k \in \mathbb{N}_0 : |S_k(f) - f(y)| \geq \varepsilon'\right\}\right) \\
&\quad \leq \delta_P\left(\left\{k \in \mathbb{N}_0 : |S_k(f_3) - f_3(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\}\right) \\
&\qquad +\delta_P\left(\left\{k \in \mathbb{N}_0 : |S_k(f_1) - f_1(y)| \geq \frac{\varepsilon' - \varepsilon}{8K(C-y)}\right\}\right) \\
&\qquad +\delta_P\left(\left\{k \in \mathbb{N}_0 : |S_k(f_2) - f_2(y)| \geq \frac{\varepsilon' - \varepsilon}{8KC}\right\}\right) \\
&\qquad +\delta_P\left(\left\{k \in \mathbb{N}_0 : |S_k(f_0) - f_0(y)| \geq \frac{\varepsilon' - \varepsilon}{4K}\right\}\right)
\end{aligned}
$$

which gives $\delta_P\left(\left\{k \in \mathbb{N}_0 : |S_k(f) - f(y)| \geq \varepsilon'\right\}\right) = 0$ with hypotheses, that is, $st_P - \lim S_k(f) = f(y)$. Now suppose in addition that S_k is translatable for all $k \in \mathbb{N}$. Since $f + \|f\| \geq 0$, in view of (3), we get

$$st_P - \lim S_k(f + \|f\|) = f + \|f\| \text{ a.e.}$$

and since S_k is also translatable, we can write

$$S_k(f + \|f\|) = S_k(f) + \|f\| S_k(1) = S_k(f) + \|f\| S_k(f_0).$$

Thanks to our hypotheses (2), we get, for all $f \in E \cap AC(M)$, $st_P - \lim S_k(f) = f$ a.e.. Notice now that if M is included in $[0, \infty)$, then one can rewrite the inequality (4) as

$$|f(x) - f(y)| < \varepsilon + \frac{2\|f\|}{\delta^2}\left\{x^2 + 2C(-x) + y^2\right\}.$$

Hence, we get

$$
\begin{aligned}
|S_k(f) - f(y)| &\leq S_k(|f - f(y)|) + f(y)|S_k(f_0) - f_0(y)| \\
&\leq \varepsilon S_k(f_0) + \frac{2\|f\|}{\delta^2}\left\{S_k(f_3) + 2CS_k(f_2) + y^2 S_k(f_0)\right\} \\
&\quad + f(y)|S_k(f_0) - f_0(y)|
\end{aligned}
$$

where $f \geq 0$. So the proofs continue as above and we get the desired result. □

Remark 2 If we replace the space $AC(M)$ with $C(M)$, then Theorem 1 still works.

Corollary 1 *Let $\mathcal{R}([a,b])$ stands for the vector lattice of all Riemann integrable functions defined on one dimensional compact interval $[a,b]$. Suppose that $\{S_k\}$ is a sequence of monotone and sublinear operators from $\mathcal{R}([a,b])$ into $\mathcal{R}([a,b])$. If*

$$st_P - \lim S_k(f_i) = f_i \text{ a.e.}, i = 0, 1, 2, 3,$$

then for all nonnegative functions f in $\mathcal{R}([a,b])$, we have

$$st_P - \lim S_k(f) = f \text{ a.e.}.$$

Moreover, if S_k is translatable, then for all $f \in \mathcal{R}([a,b])$, we have $st_P - \lim S_k(f) = f$ a.e..

4 Approximation via *P*-statistical convergence in measure

This section focuses on approximation theorem of Korovkin type by power series statistical convergence in measure.

Theorem 2 *Suppose that M be a compact subset of $\mathbb{R}$ endowed with positive Borel measure μ and let $\{S_k\}$ is a sequence of monotone, subunital and sublinear operators from $C(M)$ into itself. If*

$$st_P - \lim \mu(\{y \in M : |S_k(f_i)(y) - f_i(y)| \geq \varepsilon\}) = 0, \ i = 0, 1, 2, 3, \tag{5}$$

then for all nonnegative functions f in $C(M)$, we have

$$st_P - \lim \mu(\{y \in M : |S_k(f)(y) - f(y)| \geq \varepsilon\}) = 0. \tag{6}$$

Moreover, if S_k is translatable, then for all $f \in C(M)$, we have

$$st_P - \lim \mu(\{y \in M : |S_k(f)(y) - f(y)| \geq \varepsilon\}) = 0.$$

Proof Let f be nonnegative function in $C(M)$ and $\varepsilon > 0$. Thanks to the uniform continuity of f, by classical method, it can be written that for every $\varepsilon' < \varepsilon$ with $\varepsilon' > 0$, there is $\delta > 0$, such that

$$|f(x) - f(y)| < \varepsilon' + \frac{2\|f\|}{\delta^2}|x-y|^2 \text{ for all } x, y \in M.$$

Then as in Theorem 1, we have

$$\begin{aligned}|S_k(f) - f(y)| &\leq \varepsilon' + \frac{2\|f\|}{\delta^2} S_k\left((x-y)^2\right) + f(y)|S_k(f_0) - f_0(y)| \\ &\leq \varepsilon' + \frac{2\|f\|}{\delta^2}\left\{S_k(f_3)(y) + 2(C-y)S_k(f_1)(y) + 2CS_k(f_2)(y) + y^2\right\} \\ &\quad + f(y)|S_k(f_0) - f_0(y)|\end{aligned}$$

where $C = \max\{y, 0\}$. In view of the above inequality, we obtain

$$
\begin{aligned}
&\{y \in M : |S_k(f)(y) - f(y)| \geq \varepsilon\} \\
&\subset \left\{ y \in M : \varepsilon' + \frac{2\|f\|}{\delta^2} \left\{ S_k(f_3)(y) + 2(C-y) S_k(f_1)(y) + 2C S_k(f_2)(y) + y^2 \right\} \right. \\
&\quad + f(y) |S_k(f_0) - f_0(y)| \geq \varepsilon \} \\
&= \{ y \in M : (S_k(f_3) - f_3(y)) + 2(C-y)(S_k(f_1) - f_1(y)) \\
&\quad \left. + 2C(S_k(f_2) - f_2(y)) + |S_k(f_0) - f_0(y)| \geq \frac{\varepsilon - \varepsilon'}{K} \right\} \\
&\subset \left\{ y \in M : |S_k(f_3) - f_3(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \\
&\cup \left\{ y \in M : |S_k(f_1) - f_1(y)| \geq \frac{\varepsilon - \varepsilon'}{8K(C-y)} \right\} \\
&\cup \left\{ y \in M : |S_k(f_2) - f_2(y)| \geq \frac{\varepsilon - \varepsilon'}{8KC} \right\} \\
&\cup \left\{ y \in M : |S_k(f_0) - f_0(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\}
\end{aligned}
$$

where $K = \max\left\{ \frac{2\|f\|}{\delta^2}, f(y) \right\}$. Hence, we get

$$
\begin{aligned}
&\mu(\{y \in M : |S_k(f) - f(y)| \geq \varepsilon\}) \\
&\leq \mu\left(\left\{ y \in M : |S_k(f_3) - f_3(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \right) \\
&\quad + \mu\left(\left\{ y \in M : |S_k(f_1) - f_1(y)| \geq \frac{\varepsilon - \varepsilon'}{8K(C-y)} \right\} \right) \\
&\quad + \mu\left(\left\{ y \in M : |S_k(f_2) - f_2(y)| \geq \frac{\varepsilon - \varepsilon'}{8KC} \right\} \right) \\
&\quad + \mu\left(\left\{ y \in M : |S_k(f_0) - f_0(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \right).
\end{aligned}
$$

Now, for every $\eta > 0$,

$$
\begin{aligned}
&\{k \in \mathbb{N}_0 : \mu(\{y \in M : |S_k(f) - f(y)| \geq \varepsilon\}) \geq \eta\} \\
&\subset \left\{ k \in \mathbb{N}_0 : \mu\left(\left\{ y \in M : |S_k(f_3) - f_3(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \right) \geq \frac{\eta}{4} \right\} \\
&\cup \left\{ k \in \mathbb{N}_0 : \mu\left(\left\{ y \in M : |S_k(f_1) - f_1(y)| \geq \frac{\varepsilon - \varepsilon'}{8K(C-y)} \right\} \right) \geq \frac{\eta}{4} \right\} \\
&\cup \left\{ k \in \mathbb{N}_0 : \mu\left(\left\{ y \in M : |S_k(f_2) - f_2(y)| \geq \frac{\varepsilon - \varepsilon'}{8KC} \right\} \right) \geq \frac{\eta}{4} \right\} \\
&\cup \left\{ k \in \mathbb{N}_0 : \mu\left(\left\{ y \in M : |S_k(f_0) - f_0(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \right) \geq \frac{\eta}{4} \right\}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\delta_P(\{k \in \mathbb{N}_0 : \mu(\{y \in M : |S_k(f) - f(y)| \geq \varepsilon\}) \geq \eta\}) \\
&\leq \delta_P\left(\left\{ k \in \mathbb{N}_0 : \mu\left(\left\{ y \in M : |S_k(f_3) - f_3(y)| \geq \frac{\varepsilon - \varepsilon'}{4K} \right\} \right) \geq \frac{\eta}{4} \right\} \right)
\end{aligned}
$$

$$+\delta_P\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(f_1)-f_1(y)|\geq\frac{\varepsilon-\varepsilon'}{8K(C-y)}\right\}\right)\geq\frac{\eta}{4}\right\}\right)$$
$$+\delta_P\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(f_2)-f_2(y)|\geq\frac{\varepsilon-\varepsilon'}{8KC}\right\}\right)\geq\frac{\eta}{4}\right\}\right)$$
$$+\delta_P\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{y\in M:|S_k(f_0)-f_0(y)|\geq\frac{\varepsilon-\varepsilon'}{4K}\right\}\right)\geq\frac{\eta}{4}\right\}\right).$$

In view of our hypotheses (5), we obtain $\delta_P(\{k\in\mathbb{N}_0:\mu(\{y\in M:|S_k(f)-f(y)|\geq\varepsilon\})\geq\eta\})=0$, i.e.,

$$st_P-\lim\mu\left(\{y\in M:|S_k(f)-f(y)|\geq\varepsilon\}\right)=0.$$

For the second part, suppose now that S_k is also translatable for all $k\in\mathbb{N}$. Since $f+\|f\|\geq 0$, in view of (6), we get

$$st_P-\lim\mu\left(\{y\in M:|S_k(f+\|f\|)-(f+\|f\|)(y)|\geq\varepsilon\}\right)=0$$

and as in Theorem 1, we have

$$S_k(f+\|f\|)=S_k(f)+\|f\|\,S_k(f_0).$$

Thus, for all $f\in C(M)$, we get $st_P-\lim\mu\left(y\in M:|S_k(f)-f(y)|\geq\varepsilon\right)=0.$ □

Remark 3 If M is considered to be locally compact, then Theorem 2 remains true under the following local convergence condition

$$st_P-\lim\mu\left(\{y\in E:|S_k(f)-f(y)|\geq\varepsilon\}\right)=0$$

for every compact subset E of M.

5 Applications to approximation theorems

In this part, we apply the theoretical framework developed in the previous sections to concrete problems in approximation theory. We provide detailed examples to demonstrate the effectiveness of our approach.

Example 1 First, let us consider monotone, sublinear and translatable operators S_k, the possibilistic Kantorovich operators [15], given by

$$S_k(f)(y)=\sum_{n=0}^{k}\binom{k}{n}y^n(1-y)^{k-n}\sup\left\{f(y):y\in\left[\frac{n}{k+1},\frac{n+1}{k+1}\right]\right\}.$$

As it is known, $S_k(f_0)(y)=f_0(y)$, $S_k(f_1)(y)=\frac{k}{k+1}f_1(y)+\frac{1}{k+1}$, $S_k(f_2)(y)=\frac{k}{k+1}f_2(y)$ and $S_k(f_3)(y)=\left(\frac{k}{k+1}\right)^2f_3(y)+\frac{2k}{(k+1)^2}f_1(y)+\frac{1}{(k+1)^2}$ (see, also [20]). Using these operators, we introduce the following

$$T_k(f)(y)=(1+x_k)\,S_k(f)(y),\ \text{ for all } f\in E\cap AC(M),$$

where $\{x_k\}$ is P-statistically convergent to 0 and, neither statistically nor classically convergent to 0. It is easy to see that each T_k is a monotone, sublinear and translatable operator. Applying now Theorem 1, it follows that

$$st_P-\lim T_k(f)=f(y)\ \text{ at each point } y \text{ of continuity of } f.$$

Example 2 Let us consider the Bernstein-Kantorovich-Choquet (BKC) polynomial operators $K_{k,m}^{(1)} : C([0,1]) \to C([0,1])$ by

$$K_{k,m}^{(1)}(f)(y) = \sum_{n=0}^{k} \frac{(C)\int_{\frac{n}{k+1}}^{\frac{n+1}{k+1}} f(u)\,dm(u)}{m\left(\left[\frac{n}{k+1}, \frac{n+1}{k+1}\right]\right)} \binom{k}{n} y^n (1-y)^{k-n}$$

where $m := \sqrt{\lambda}$ with the Lebesque measure λ. It is known from [19] that the operators monotone, translatable and sublinear. Because of the results given in [17], we can say that $K_{k,m}^{(1)}(f_0) \to f_0$, $K_{k,m}^{(1)}(f_1) \to f_1$, $K_{k,m}^{(1)}(f_2) \to f_2$ and $K_{k,m}^{(1)}(f_3) \to f_3$ uniformly on $[0,1]$. Now, define the sequence

$$K_{k,m}(f)(y) = \sum_{n=0}^{k} \frac{(C)\int_{\frac{n}{k+1}}^{\frac{n+1}{k+1}} f(x_k u)\,dm(u)}{m\left(\left[\frac{n}{k+1}, \frac{n+1}{k+1}\right]\right)} \binom{k}{n} y^n (1-y)^{k-n}$$

where $0 < x_k \leq 1$ with $st_P - \lim x_k = 1$, neither statistically nor classically convergent. Observe that $K_{k,m}(f_0) = f_0$, $K_{k,m}(f_1) = x_k K_{k,m}^{(1)}(f_1)$, $K_{k,m}(f_2) = x_k K_{k,m}^{(1)}(f_2)$ and $K_{k,m}(f_3) = x_k^2 K_{k,m}^{(1)}(f_3)$. Therefore, $K_{k,m}$ satisfies the conditions of Theorem 2, then for all $f \in C([0,1])$, we have

$$st_P - \lim \mu\left(\left\{y \in [0,1] : \left|K_{k,m}(f)(y) - f(y)\right| \geq \varepsilon\right\}\right) = 0.$$

Example 3 Let us consider the Bernstein-Chlodovsky-Kantorovich-Choquet (BCKC) polynomial operators $C_{k,m}^{(1)} : \mathcal{R}([0,\alpha_k]) \to \mathcal{R}([0,\alpha_k])$ by

$$C_{k,m}^{(1)}(f)(y) = \sum_{n=0}^{k} \frac{(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} f(u)\,dm(u)}{m\left(\left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right]\right)} \binom{k}{n} \left(\frac{y}{\alpha_{k+1}}\right)^n \left(1 - \frac{y}{\alpha_{k+1}}\right)^{k-n}$$

where $m := \sqrt{\lambda}$ with the Lebesque measure λ and $\{\alpha_n\}$ is a positive sequence with $\lim_n \alpha_n = \infty$, $\lim_n \frac{\alpha_n}{n} = 0$ [38]. It is important to point out that if $m := \lambda$, then the operators become Bernstein-Chlodovsky-Kantorovich operators and if $m := \delta_k$ (the Dirac measure), then the operators become Bernstein-Chlodovsky operators. Now, thanks to Choquet integral's sublinearity, monotonity and comonotone additivity with a submodular capacity m, we have these properties for $C_{k,m}^{(1)}$.

Clearly, $C_{k,m}^{(1)}(f_0)(y) = f_0(y)$. For the convergence of the $C_{k,m}^{(1)}(f_1)(y)$, we will first calculate the integrals $(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} u\,dm(u)$. We get

$$(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} u\,dm(u) = \int_0^\infty m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1}\right] : u \geq \beta\right\}\right) d\beta$$

$$= \int_{0}^{\frac{n\alpha_{k+1}}{k+1}} m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\,\alpha_{k+1}}{k+1}\right] : u \geq \beta\right\}\right) d\beta$$

$$+ \int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\,\alpha_{k+1}}{k+1}\right] : u \geq \beta\right\}\right) d\beta$$

$$= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right) + \int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} \sqrt{\frac{(n+1)\,\alpha_{k+1}}{k+1} - \beta} d\beta$$

$$= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{n\alpha_{k+1}}{k+1}\right) + \int_{0}^{\frac{\alpha_{k+1}}{k+1}} \sqrt{\eta} d\eta = \frac{3n+2}{3}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{3}{2}}.$$

Using this equality in $C_{k,m}^{(1)}(f_1)(y)$, we have

$$C_{k,m}^{(1)}(f_1)(y) = \frac{k}{k+1}y + \frac{2\alpha_{k+1}}{3\,(k+1)}.$$

For the convergence of the $C_{k,m}^{(1)}(f_2)(y)$, we will first calculate the integrals $(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} (-u)\, dm(u)$. We get

$$(C) \int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} (-u)\, dm(u)$$

$$= \int_{-\infty}^{0} \left\{m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\,\alpha_{k+1}}{k+1}\right] : -u \geq \beta\right\}\right) - \sqrt{\frac{\alpha_{k+1}}{k+1}}\right\} d\beta$$

$$= \int_{-\frac{n\alpha_{k+1}}{k+1}}^{0} \left\{m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\,\alpha_{k+1}}{k+1}\right] : u \leq -\beta\right\}\right) - \sqrt{\frac{\alpha_{k+1}}{k+1}}\right\} d\beta$$

$$+ \int_{-\frac{(n+1)\alpha_{k+1}}{k+1}}^{-\frac{n\alpha_{k+1}}{k+1}} \left\{m\left(\left\{u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\,\alpha_{k+1}}{k+1}\right] : u \leq -\beta\right\}\right) - \sqrt{\frac{\alpha_{k+1}}{k+1}}\right\} d\beta$$

$$= \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{-n\alpha_{k+1}}{k+1}\right) + \int_{-\frac{(n+1)\alpha_{k+1}}{k+1}}^{-\frac{n\alpha_{k+1}}{k+1}} \sqrt{-\beta_1 - \frac{n\alpha_{k+1}}{k+1}} d\beta_1 - \sqrt{\frac{\alpha_{k+1}}{k+1}}\left(\frac{\alpha_{k+1}}{k+1}\right)$$

$$= \sqrt{\frac{\alpha_{k+1}}{k+1}} \left(\frac{-n\alpha_{k+1}}{k+1} \right) + \int\limits_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} \sqrt{\beta - \frac{n\alpha_{k+1}}{k+1}} d\beta - \sqrt{\frac{\alpha_{k+1}}{k+1}} \left(\frac{\alpha_{k+1}}{k+1} \right)$$

$$= \sqrt{\frac{\alpha_{k+1}}{k+1}} \left(\frac{n\alpha_{k+1}}{k+1} \right) + \int\limits_{0}^{\frac{\alpha_{k+1}}{k+1}} \sqrt{\eta} d\eta - \sqrt{\frac{\alpha_{k+1}}{k+1}} \left(\frac{\alpha_{k+1}}{k+1} \right) = \frac{-3n+1}{3} \left(\frac{\alpha_{k+1}}{k+1} \right)^{\frac{3}{2}}$$

which gives

$$C_{k,m}^{(1)} (f_2) (y) = \frac{k}{k+1} (-y) + \frac{\alpha_{k+1}}{3 (k+1)}.$$

Finally, for the convergence of the $C_{k,m}^{(1)} (f_3) (y)$, we will first calculate the integrals $(C) \int\limits_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} u^2 dm (u)$ and we get

$$\begin{aligned}
&(C) \int\limits_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}} u^2 dm (u) \\
&= \int\limits_{0}^{\infty} m \left(\left\{ u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1} \right] : u \geq \sqrt{\beta} \right\} \right) d\beta \\
&= \int\limits_{0}^{\left(\frac{n\alpha_{k+1}}{k+1}\right)^2} m \left(\left\{ u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1} \right] : u \geq \sqrt{\beta} \right\} \right) d\beta \\
&\quad + \int\limits_{\left(\frac{n\alpha_{k+1}}{k+1}\right)^2}^{\left(\frac{(n+1)\alpha_{k+1}}{k+1}\right)^2} m \left(\left\{ u \in \left[\frac{n\alpha_{k+1}}{k+1}, \frac{(n+1)\alpha_{k+1}}{k+1} \right] : u \geq \sqrt{\beta} \right\} \right) d\beta \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}} \left(\frac{n\alpha_{k+1}}{k+1} \right)^2 + \int\limits_{\left(\frac{n\alpha_{k+1}}{k+1}\right)^2}^{\left(\frac{(n+1)\alpha_{k+1}}{k+1}\right)^2} \sqrt{\frac{(n+1)\alpha_{k+1}}{k+1} - \sqrt{\beta} d\beta} \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}} \left(\frac{n\alpha_{k+1}}{k+1} \right)^2 + 2 \int\limits_{0}^{\frac{\alpha_{k+1}}{k+1}} \sqrt{\eta} \left(\frac{(n+1)\alpha_{k+1}}{k+1} - \eta \right) d\eta \\
&= \sqrt{\frac{\alpha_{k+1}}{k+1}} \left(\frac{n\alpha_{k+1}}{k+1} \right)^2 + \frac{4(n+1)}{3} \left(\frac{\alpha_{k+1}}{k+1} \right)^{\frac{5}{2}} - \frac{4}{5} \left(\frac{\alpha_{k+1}}{k+1} \right)^{\frac{5}{2}}
\end{aligned}$$

which gives

$$C_{k,m}^{(1)}\left(f_3\right)(y)=\frac{k\left(k-1\right)}{\left(k+1\right)^2}y^2+\frac{7}{3}\frac{k\alpha_{k+1}}{\left(k+1\right)^2}y+\frac{4}{3}\left(\frac{\alpha_{k+1}}{k+1}\right)^2-\frac{4}{5}\left(\frac{\alpha_{k+1}}{k+1}\right)^{\frac{5}{2}}.$$

Using now $C_{k,m}^{(1)}$, we define the following

$$C_{k,m}\left(f\right)(y)=\sum_{n=0}^{k}\frac{(C)\int_{\frac{n\alpha_{k+1}}{k+1}}^{\frac{(n+1)\alpha_{k+1}}{k+1}}f\left(x_k u\right)dm\left(u\right)}{m\left(\left[\frac{n\alpha_{k+1}}{k+1},\frac{(n+1)\alpha_{k+1}}{k+1}\right]\right)}\binom{k}{n}\left(\frac{y}{\alpha_{k+1}}\right)^n\left(1-\frac{y}{\alpha_{k+1}}\right)^{k-n}$$

where $0<x_k\leq 1$ with $st_P-\lim x_k=1$, neither statistically nor classically convergent. It is easy to see that each $C_{k,m}$ is a monotone, sublinear and translatable operator and observe that $C_{k,m}\left(f_0\right)=f_0$, $C_{k,m}\left(f_1\right)=x_k C_{k,m}^{(1)}\left(f_1\right)$, $C_{k,m}\left(f_2\right)=x_k C_{k,m}^{(1)}\left(f_2\right)$ and $C_{k,m}\left(f_3\right)=x_k^2 C_{k,m}^{(1)}\left(f_3\right)$. Consequently: Applying Corollary 1, for all $f\in\mathcal{R}\left([0,\alpha_k]\right)$, we have $st_P-\lim C_{k,m}\left(f\right)=f$ a.e..

6 Rate of convergence

In this section, we present some estimates of rates of convergence for our Korovkin-type theorems.

We consider the following usual modulus of continuity:

$$\omega\left(f,\delta\right):=\sup_{x,y\in M:\ d(x,y)\leq\delta}\left|f\left(x\right)-f\left(y\right)\right|$$

where $f\in C(M)$ and $\delta>0$. It is readily seen that, $\omega\left(f,\delta\right)$ is an increasing function of δ, $\left|f\left(x\right)-f\left(y\right)\right|\leq\omega\left(f,d\left(x,y\right)\right)$ for each $x,y\in M$, $\omega\left(f,\delta\right)\leq 2\left\|f\right\|$ for every $\delta>0$, and $\omega\left(f,\gamma\delta\right)\leq\left(1+\gamma\right)\omega\left(f,\delta\right)$ for every $\gamma,\delta>0$.

Theorem 3 *Let M, E and $\{S_k\}$ be as in Theorem 1. If*

(i) $st_P-\lim S_k\left(f_0\right)=f_0$ a.e.,

(ii) $st_P-\lim\omega\left(f,\delta_S\right)=0$ a.e., where $\delta_S:=\sqrt{S_k\left((.-y)^2\right)}$.

Then for all nonnegative functions f in $E\cap AC(M)$, we have

$$st_P-\lim S_k\left(f\right)=f\ a.e..$$

Moreover, if S_k is translatable, then for all $f\in E\cap AC(M)$, we have $st_P-\lim S_k\left(f\right)=f$ a.e..

Proof Let $f\in E\cap AC(M)$ and suppose that y is a point of continuity of f with

$$st_P-\lim S_k\left(f_0\right)(y)=f_0\left(y\right)\ \text{a.e. and}\ st_P-\lim\omega\left(f,\delta_S\right)=0\ \text{a.e..}$$

With the help of monotonicity and sublinearity of the operators S_k and property of modulus of continuity ω, we get

$$\left|S_k\left(f\right)-f\left(y\right)\right|\leq S_k\left(\left|f-f\left(y\right)\right|\right)+f\left(y\right)\left|S_k\left(f_0\right)-f_0\right|$$

$$\leq S_k\left(\left(1+\frac{(x-y)^2}{\delta^2}\right)\omega(f,\delta)\right)+f(y)\left|S_k(f_0)-f_0(y)\right|$$
$$\leq \omega(f,\delta)\,S_k(f_0)+\frac{\omega(f,\delta)}{\delta^2}S_k\left((x-y)^2\right)+f(y)\left|S_k(f_0)-f_0(y)\right|$$

If we choose $\delta := \delta_S := \sqrt{S_k\left((x-y)^2\right)}$, this yield that

$$\left|S_k(f)-f(y)\right| \leq \omega(f,\delta_S)\left(S_k(f_0)+1\right)+f(y)\left|S_k(f_0)-f_0(y)\right|.$$

By our choice of y, we get

$$st_P-\lim S_k(f)=f(y).$$

Now suppose in addition that S_k is translatable for all $k\in\mathbb{N}$. Similar to the proof of Theorem 1, we can easily get, for all $f\in E\cap AC(M)$, $st_P-\lim S_k(f)=f$ a.e.. □

Theorem 4 *Suppose that M, $\{S_k\}$ be as in Theorem 2. If*
(i) $st_P-\lim\mu\left(\{y\in M:\left|S_k(f_0)(y)-f_0(y)\right|\geq\varepsilon\}\right)=0$,
(ii) $st_P-\lim\mu\left(\{y\in M:\omega(f,\delta_S)\geq\varepsilon\}\right)=0$ a.e., where $\delta_S:=\sqrt{S_k\left((.-y)^2\right)}$,
then for all nonnegative functions f in $C(M)$, we have

$$st_P-\lim\mu\left(\{y\in M:\left|S_k(f)(y)-f(y)\right|\geq\varepsilon\}\right)=0.$$

Moreover, if S_k is translatable, then for all $f\in C(M)$, we have

$$st_P-\lim\mu\left(\{y\in M:\left|S_k(f)(y)-f(y)\right|\geq\varepsilon\}\right)=0.$$

Proof Let f be nonnegative function in $C(M)$ and $\varepsilon>0$. As in Theorem 3, we have

$$\left|S_k(f)-f(y)\right| \leq \omega(f,\delta_S)\left(S_k(f_0)+1\right)+f(y)\left|S_k(f_0)-f_0(y)\right|.$$

In view of the above inequality, we obtain

$$\begin{aligned}&\{y\in M:\left|S_k(f)(y)-f(y)\right|\geq\varepsilon\}\\&\quad\subset\{y\in M:\omega(f,\delta_S)\left(S_k(f_0)+1\right)+f(y)\left|S_k(f_0)-f_0(y)\right|\geq\varepsilon\}.\end{aligned}$$

Now considering the above inclusion, the hypotheses (i), (ii), the proof is completed at once. Also, suppose now in addition that S_k is translatable for all $k\in\mathbb{N}$. Similar to the proof of Theorem 3, we can easily get, for all $f\in C(M)$,

$$st_P-\lim\mu\left(\{y\in M:\left|S_k(f)(y)-f(y)\right|\geq\varepsilon\}\right)=0.$$

□

7 Conclusion

In the present paper, the Korovkin-type approximation theorems have been extended to sequences of monotone and sublinear operators by introducing the concept of power series statistical convergence almost everywhere and in measure. The convergence method used in this paper is incompatible with statistical convergence and is a non-matrix method. Another important point is that while every positive linear operator is monotone and sublinear, the converse is not true, highlighting the broader applicability of our results. The theoretical framework is applied to concrete problems in the approximation theory. The effectiveness of our approach is illustrated through detailed examples. Furthermore, the investigation of the rates of convergences in terms of the modulus of continuity was undertaken.

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References

1. Adams, D.R.: Choquet integrals in potential theory. Publ. Math. **42**, 3–66 (1998)
2. Boos, J.: Classical and Modern Methods in Summability. Oxford University Press, Oxford (2000)
3. Cai, Q.B., Gorka, S., Raj, K.: Deferred Riesz statistical convergence via power series method. J. Appl. Math. Comput. **71**(1), 1141–1158 (2025)
4. Choquet, G.: Theory of capacities. Ann. l' Inst. Fourier **5**, 131–295 (1954)
5. Cinar, S., Yildiz, S., Demirci, K.: Korovkin type approximation via triangular A-statistical convergence on an infinite interval. Turk. J. Math. **45**(2), 929–942 (2021)
6. Çınar, S., Yıldız, S.: P-statistical summation process of sequences of convolution operators. Indian J. Pure Appl. Math. **53**(3), 648–659 (2022)
7. Demirci, K., Dirik, F., Yıldız, S.: Approximation via equi-statistical convergence in the sense of power series method. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **116**(2), 65 (2022)
8. Demirci, K., Yıldız, S.: Approximation on modular spaces via P-statistical relative A-summation process. Sinop Üniversitesi Fen Bilimleri Dergisi **7**(2), 152–172 (2022)
9. Demirci, K., Karakuş, S.: A-statistical Korovkin-type approximation theorem for of two variables on an infinite interval. Acta Math. Univ. Comen. **81**(2), 151–157 (2012)
10. Denneberg, D.: Non-additive Measure and Integral. Kluwer Academic Publisher, Dordrecht (1994)
11. Dinar, R., Yurdakadim, T.: Approximation properties of convolution operators via statistical convergence based on a power series. Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat. **74**(1), 92–102 (2025)
12. Dutta, S., Ghosh, R.: Korovkin type approximation theorem on an infinite interval in A^I-statistical sense. Acta Math. Univ. Comen. **89**(1), 131–142 (2019)
13. Gadjiev, A.D., Orhan, C.: Some approximation theorems via statistical convergence. Rocky Mt. J. Math. **32**, 129–138 (2002)
14. Gal, S.G.: Approximation by Choquet integral operators. Ann. Mat. (1923-) **195**, 881–896 (2016)
15. Gal, S.G.: Approximation by polynomial possibilistic integral operators. Ann. Acad. Rom. Sci. Ser. Math. Appl. **12**, 132–141 (2020)
16. Gal, S.G., Iancu, I.T.: Korovkin-type theorems for statistically convergent sequences of monotone and sublinear operators. Bull. Malays. Math. Sci. Soc. **46**(2), 79 (2023)
17. Gal, S.G., Niculescu, C.P.: A nonlinear extension of Korovkin's theorem. Mediterr. J. Math. **17**, 1–14 (2020)
18. Gal, S.G., Niculescu, C.P.: A note on the Choquet type operators. Aequ. Math. **95**, 433–447 (2021)
19. Gal, S.G., Niculescu, C.P.: Nonlinear versions of Korovkin's abstract theorems. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas **116**(2), 68 (2022)
20. Iancu, I.T.: Statistical Korovkin-type theorem for monotone and sublinear operators. Stud. Univ. Babes-Bolyai Math. **68**(2) (2023)
21. Fast, H.: Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)

22. Korovkin, P.P.: On convergence of linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR **90**, 961–964 (1953). (**in Russian**)
23. Korovkin, P.P.: Linear Operators and Approximation Theory, Fitzmatgiz, Moscow (1959); English transl.: Hindustan Publ. Corp., Delhi (1960) **(in Russian)**
24. Kratz, W., Stadtmüller, U.: Tauberian theorems for J_p-summability. J. Math. Anal. Appl. **139**, 362–371 (1989)
25. Niven, I., Zuckerman, H.S.: An Introduction to the Theory of Numbers, 4th edn. Wiley, New York (1980)
26. Orhan, S., Acar, T., Dirik, F.: Korovkin type theorems in weighted L_p spaces via statistical A-summability. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **62**(2), 537–546 (2016)
27. Söylemez, D.: A Korovkin type approximation theorem for Balazs type Bleimann, Butzer and Hahn operators via power series statistical convergence. Math. Slov. **72**(1), 153–164 (2022)
28. Steinhaus, H.: Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math. **2**, 73–74 (1951)
29. Stadtmüller, U., Tali, A.: On certain families of generalized Nörlund methods and power series methods. J. Math. Anal. Appl. **238**, 44–66 (1999)
30. Sümbül, C., Belen, C., Yıldırım, M.: On statistical limit points with respect to power series methods and modulus functions. Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat. **72**(2), 438–448 (2023)
31. Bayram, N.Ş., Yıldız, S.: Approximation by statistical convergence with respect to power series methods. Hacettepe J. Math. Stat. 1–13 (2022)
32. Uluçay, H., Ünver, M., Söylemez, D.: Some Korovkin type approximation applications of power series methods. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **117**(1), 24 (2023)
33. Ünver, M.: Abel transforms of positive linear operators. In: 11th International Conference of Numerical Analysis and Applied Mathematics 2013: ICNAAM 2013, vol. 1558, pp. 1148-1151 (2013)
34. Ünver, M.: Abel summability in topological spaces. Monatshefte für Mathematik **178**, 633–643 (2015)
35. Ünver, M., Orhan, C.: Statistical convergence with respect to power series methods and applications to approximation theory. Numer. Funct. Anal. Optim. **40**(5), 535–547 (2019)
36. Wang, Z., Klir, G.J.: Generalized Measure Theory. Springer, New York (2009)
37. Wang, Z., Yan, J.-A.: A Selective Overview of Applications of Choquet Integrals. Advanced Lectures in Mathematics, pp. 484–515. Higher Educational Press, Beijing (2007)
38. Yıldız, S., Demirci, K., Dirik, F.: Approximation results via power series method for sequences of monotone and sublinear operators. RM **80**(2), 1–19 (2025)