

ON SOME PROPERTIES OF RELATIVE CAPACITY AND THINNESS IN WEIGHTED VARIABLE EXPONENT SOBOLEV SPACES

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Abstract. We define the weighted relative $p(\cdot)$ -capacity and discuss its properties in the space $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$. Also, we investigate some properties of the weighted variable Sobolev capacity. It is shown that there is a relation between these two capacities. Moreover, we introduce the notion of thinness related to this newly defined relative capacity and prove an equivalence statement for this thinness.

1. Introduction

Kováčik and Rákosník [17] introduced the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$. The boundedness of the maximal operator was an open problem in $L^{p(\cdot)}(\mathbb{R}^n)$ for a long time. Diening [4] was the first to prove this over bounded domains if $p(\cdot)$ satisfies locally the log-Hölder continuous condition, that is,

$$|p(x) - p(y)| \leq \frac{C}{-\ln|x - y|}, \quad x, y \in \Omega, \quad |x - y| \leq \frac{1}{2},$$

where Ω is a bounded domain. We denote by $P^{\log}(\mathbb{R}^n)$ the class of variable exponents which satisfy the log-Hölder continuous condition. Diening later extended the result to unbounded domains by supposing, in addition, that the exponent $p(\cdot) = p$ is a constant function outside a large ball. After this study, many absorbing and crucial papers appeared in non-weighted

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and weighted variable exponent spaces. For a historical journey, we refer to [5], [9], [17], [20] and [21]. The Sobolev capacity for constant exponent spaces has found a great number of uses, see [8] and [19]. Moreover, the weighted Sobolev capacity was revealed by Kilpeläinen [15]. He investigated the role of capacity in the pointwise definition of functions in Sobolev spaces involving weights of Muckenhoupt's A_p -class. Harjulehto et al. [13] introduced variable Sobolev capacity in the spaces $W^{1,p(\cdot)}(\mathbb{R}^n)$. Also, Aydın [2] generalized some results of the variable Sobolev capacity to the weighted variable exponent case.

The variational capacity has been used extensively in nonlinear potential theory on $\mathbb{R}^n$. Assume $\Omega \subset \mathbb{R}^n$ is open and $K \subset \Omega$ is compact. Then the relative variational p -capacity is defined by

$$\text{cap}_p(K, \Omega) = \inf_f \int_{\Omega} |\nabla f(x)|^p dx,$$

where the infimum is taken over smooth and zero boundary valued functions f in Ω such that $f \geq 1$ in K . The set of admissible functions f can be replaced by the continuous first order Sobolev functions with $f \geq 1$ in K . The p -capacity is a Choquet capacity relative to Ω . For more details and historical background, see [14]. Also, Harjulehto et al. [12] defined a relative capacity. They studied properties of the capacity and compared it with the Sobolev capacity.

In [7], the authors have considered a relative capacity and relationship with the well-known half-plane capacity. It is known that the half-plane capacity is a particular case because of its applications in geometric function theory and stochastic processes. Also, they proved some properties of the defined relative capacity such as the behavior of this capacity under various forms of symmetrization and under some other geometric transformations. Moreover, they investigated some applications to bounded holomorphic functions of the unit disk.

Our purpose is to investigate some properties of the Sobolev capacity and, also, the relative $p(\cdot)$ -capacity as in Harjulehto et al. [12] in the weighted variable exponent case. Also, we give relationship between these defined two capacities. Moreover, we present a thinness related to this newly defined relative capacity and prove an equivalence statement for this thinness.

2. Notation and preliminaries

In this paper, we will work on $\mathbb{R}^n$ with Lebesgue measure dx . The measure μ is called doubling if there is a fixed constant $c_d \geq 1$, called the doubling constant of μ , such that

$$\mu(B(x_0, 2r)) \leq c_d \mu(B(x_0, r))$$

for every ball $B(x_0, r)$ in $\mathbb{R}^n$. Also, the elements of the space $C_0^\infty(\mathbb{R}^n)$ are the infinitely differentiable functions with compact support. We denote the family of all measurable functions $p(\cdot): \mathbb{R}^n \rightarrow [1, \infty)$ (called the variable exponent on $\mathbb{R}^n$) by the symbol $\mathcal{P}(\mathbb{R}^n)$. In this paper, the function $p(\cdot)$ always denotes a variable exponent. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, put

$$p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

A positive, measurable and locally integrable function $\vartheta: \mathbb{R}^n \rightarrow (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\rho_{p(\cdot), \vartheta}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} \vartheta(x) \, dx.$$

The weighted variable exponent Lebesgue spaces $L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)$ consist of all measurable functions f on $\mathbb{R}^n$ endowed with the Luxemburg norm

$$\|f\|_{p(\cdot), \vartheta} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \vartheta(x) \, dx \leq 1 \right\}.$$

When $\vartheta(x) = 1$, the space $L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)$ is the variable exponent Lebesgue space. The space $L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with respect to $\|\cdot\|_{p(\cdot), \vartheta}$. Also, some basic properties of this space were investigated in [1], [2], [16].

Assume $\Omega \subset \mathbb{R}^n$ is bounded and ϑ is a weight function. It is known that a function $f \in C_0^\infty(\Omega)$ satisfies Poincaré inequality in $L_{\vartheta}^1(\Omega)$ if and only if there is a constant $c > 0$ such that the inequality

$$\int_{\Omega} |f(x)| \vartheta(x) \, dx \leq c(\operatorname{diam} \Omega) \int_{\Omega} |\nabla f(x)| \vartheta(x) \, dx$$

holds [14].

In recent decades, the variable exponent Lebesgue spaces $L^{p(\cdot)}$ and the corresponding variable exponent Sobolev spaces $W^{k, p(\cdot)}$ have attracted more and more attention. Let $1 < p^- \leq p(\cdot) \leq p^+ < \infty$ and $k \in \mathbb{N}$. The variable exponent Sobolev spaces $W^{k, p(\cdot)}(\mathbb{R}^n)$ consist of all measurable functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that the distributional derivatives $D^\alpha f$ are in $L^{p(\cdot)}(\mathbb{R}^n)$ for all $0 \leq |\alpha| \leq k$ where $\alpha \in \mathbb{N}_0^n$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$. The spaces $W^{k, p(\cdot)}(\mathbb{R}^n)$ are a special class of so-called generalized Orlicz–Sobolev spaces with the norm

$$\|f\|_{k, p(\cdot)} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{p(\cdot)}.$$

We define the weighted variable exponent Sobolev spaces $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ by

$$W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n) = \{f \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n) : D^{\alpha} f \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n), 0 \leq |\alpha| \leq k\}$$

equipped with the norm

$$\|f\|_{k,p(\cdot),\vartheta} = \sum_{0 \leq |\alpha| \leq k} \|D^{\alpha} f\|_{p(\cdot),\vartheta}.$$

It is already known that $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ is a reflexive Banach space.

Now let $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, $k \in \mathbb{N}$ and $\vartheta^{-\frac{1}{p(\cdot)-1}} \in L_{\text{loc}}^1(\mathbb{R}^n)$. Thus we have $L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{\text{loc}}^1(\mathbb{R}^n)$ and then the weighted variable exponent Sobolev spaces $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ are well-defined by [2, Proposition 2.1].

In particular, the space $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$ is defined by

$$W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n) = \{f \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n) : |\nabla f| \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)\}.$$

The function $\rho_{1,p(\cdot),\vartheta} : W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n) \rightarrow [0, \infty)$ is shown to satisfy

$$\rho_{1,p(\cdot),\vartheta}(f) = \rho_{p(\cdot),\vartheta}(f) + \rho_{p(\cdot),\vartheta}(|\nabla f|).$$

Also, the norm $\|f\|_{1,p(\cdot),\vartheta} = \|f\|_{p(\cdot),\vartheta} + \|\nabla f\|_{p(\cdot),\vartheta}$ makes the space $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$ a Banach space. The local weighted variable exponent Sobolev space $W_{\vartheta,loc}^{1,p(\cdot)}(\mathbb{R}^n)$ is defined in the classical way. More information on the classic theory of variable exponent spaces can be found in [17].

As an alternative to the Sobolev $p(\cdot)$ -capacity, Harjulehto et al. [12] introduced the relative $p(\cdot)$ -capacity. Recall that

$$C_0(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f \text{ is continuous and } \text{supp } f \subset \Omega \text{ is compact}\},$$

where $\text{supp } f$ is the support of f . Suppose that K is a compact subset of Ω . We denote

$$R_{p(\cdot)}(K, \Omega) = \{f \in W^{1,p(\cdot)}(\Omega) \cap C_0(\Omega) : f \geq 1 \text{ on } K\}$$

and define

$$\text{cap}_{p(\cdot)}^*(K, \Omega) = \inf_{f \in R_{p(\cdot)}(K, \Omega)} \int_{\Omega} |\nabla f(x)|^{p(x)} dx = \inf_{f \in R_{p(\cdot)}(K, \Omega)} \rho_{p(\cdot)}(|\nabla f|).$$

Further, if $U \subset \Omega$ is open, then

$$\text{cap}_{p(\cdot)}(U, \Omega) = \sup_{\substack{K \subset U \\ \text{compact}}} \text{cap}_{p(\cdot)}^*(K, \Omega),$$

and for an arbitrary set $E \subset \Omega$

$$\text{cap}_{p(\cdot)}(E, \Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot)}(U, \Omega).$$

The number $\text{cap}_{p(\cdot)}(E, \Omega)$ is called the variational $p(\cdot)$ -capacity of E relative to Ω . It is usually called simply the relative $p(\cdot)$ -capacity of the pair or condenser (E, Ω) .

Throughout this paper, we assume that $p(\cdot) \in P^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p(\cdot) \leq p^+ < \infty$ and $\vartheta^{-\frac{1}{p(\cdot)-1}} \in L^1_{\text{loc}}(\mathbb{R}^n)$. We write that $a \approx b$ for two quantities if there exists positive constants c_1, c_2 such that $c_1 a \leq b \leq c_2 a$. Also, we will denote

$$\mu_{\vartheta}(\Omega) = \int_{\Omega} \vartheta(x) dx.$$

3. The Sobolev $(p(\cdot), \vartheta)$ -capacity and the relative $(p(\cdot), \vartheta)$ -capacity

A capacity for subsets of $\mathbb{R}^n$ was introduced in [2]. To define this capacity we denote

$$S_{p(\cdot), \vartheta}(E) = \{f \in W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^n) : f \geq 1 \text{ in open set containing } E\}.$$

The Sobolev $(p(\cdot), \vartheta)$ -capacity of E is defined by

$$C_{p(\cdot), \vartheta}(E) = \inf_{f \in S_{p(\cdot), \vartheta}(E)} \rho_{1, p(\cdot), \vartheta}(f).$$

In case $S_{p(\cdot), \vartheta}(E) = \emptyset$, we set $C_{p(\cdot), \vartheta}(E) = \infty$. If $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, then the set function $E \rightarrow C_{p(\cdot), \vartheta}(E)$ is an outer measure. If $f \in S_{p(\cdot), \vartheta}(E)$, then $\min\{1, f\} \in S_{p(\cdot), \vartheta}(E)$ and $\rho_{1, p(\cdot), \vartheta}(\min\{1, f\}) \leq \rho_{1, p(\cdot), \vartheta}(f)$. Thus it is enough to test the Sobolev $(p(\cdot), \vartheta)$ -capacity for $f \in S_{p(\cdot), \vartheta}(E)$ with $0 \leq f \leq 1$.

REMARK 1. In general, it is known that the space $C^\infty(\mathbb{R}^n) \cap W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^n)$ is not dense in $W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^n)$. But Zhikov and Surnachev have investigated a sufficient condition for this denseness. This condition was formulated in terms of the asymptotic behavior of the integrals of negative and positive powers of the weight, see [22]. In this paper, we will assume that this denseness holds.

THEOREM 2. Assume that $1 < p^- \leq p(\cdot) \leq p^+ < \infty$ and the space $C^\infty(\mathbb{R}^n) \cap W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^n)$ is dense in $W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^n)$. If K is compact, then

$$C_{p(\cdot), \vartheta}(K) = \inf_{f \in S_{p(\cdot), \vartheta}^\infty(K)} \rho_{1, p(\cdot), \vartheta}(f),$$

where $S_{p(\cdot), \vartheta}^\infty(K) = S_{p(\cdot), \vartheta}(K) \cap C^\infty(\mathbb{R}^n)$.

PROOF. Take any $f \in S_{p(\cdot), \vartheta}(K)$ with $0 \leq f \leq 1$. Since by the assumption $C^\infty(\mathbb{R}^n) \cap W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$ is dense in $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$, we can find a sequence $(\alpha_n)_{n \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n) \cap W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$ such that $\alpha_n \rightarrow f$ in $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$. Now, we take an open bounded neighborhood U of K such that $f = 1$ in U . Also, we take a function $\alpha \in C^\infty(\mathbb{R}^n)$, $0 \leq \alpha \leq 1$, be such that $\alpha = 1$ in $\mathbb{R}^n - U$ and $\alpha = 0$ in an open neighborhood of K . Then f or α is equal to 1 in $\mathbb{R}^n$. Now we define $\beta_n = 1 - (1 - \alpha_n)\alpha$. Thus we get

$$f - \beta_n = (f - \alpha_n)\alpha + (1 - \alpha)(f - 1) = (f - \alpha_n)\alpha.$$

Therefore $\beta_n \rightarrow f$ in $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$. Indeed, first, if we use the definitions of the functions, then we get

$$\rho_{p(\cdot), \vartheta}((f - \alpha_n)\alpha) = \int_{\mathbb{R}^n - U} |f(x) - \alpha_n(x)|^{p(x)} \vartheta(x) dx \leq \rho_{p(\cdot), \vartheta}(f - \alpha_n) \rightarrow 0.$$

Similarly, we have $\rho_{p(\cdot), \vartheta}(|\nabla(f - \alpha_n)|) \rightarrow 0$. Since $p^+ < \infty$, we find that

$$\begin{aligned} \|f - \beta_n\|_{1,p(\cdot), \vartheta} &= \|(f - \alpha_n)\alpha\|_{1,p(\cdot), \vartheta} \\ &= \|(f - \alpha_n)\alpha\|_{p(\cdot), \vartheta} + \|\nabla((f - \alpha_n)\alpha)\|_{p(\cdot), \vartheta} \rightarrow 0. \end{aligned}$$

Finally, since $\beta_n = 1 - (1 - \alpha_n)\alpha \in S_{p(\cdot), \vartheta}^\infty(K)$, it is clear that $S_{p(\cdot), \vartheta}^\infty(K)$ is dense in $S_{p(\cdot), \vartheta}(K)$. This completes the proof. $\square$

As in the proof of [6, Proposition 10.1.10], we can show the following theorem.

THEOREM 3. *Let $A \subset \mathbb{R}^n$ and $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, $1 < q^- \leq q(\cdot) \leq q^+ < \infty$ with $q(\cdot) \leq p(\cdot)$. If $C_{p(\cdot), \vartheta}(A) = 0$, then $C_{q(\cdot), \vartheta}(A) = 0$.*

Now, we will introduce the relative $(p(\cdot), \vartheta)$ -capacity.

DEFINITION 4. Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $K \subset \Omega$ be a compact subset. We denote

$$R_{p(\cdot), \vartheta}(K, \Omega) = \{f \in W_{\vartheta}^{1,p(\cdot)}(\Omega) \cap C_0(\Omega) : f > 1 \text{ on } K \text{ and } f \geq 0\},$$

and set

$$\begin{aligned} \text{cap}_{p(\cdot), \vartheta}^*(K, \Omega) &= \inf_{f \in R_{p(\cdot), \vartheta}(K, \Omega)} \rho_{p(\cdot), \vartheta}(|\nabla f|) \\ &= \inf_{f \in R_{p(\cdot), \vartheta}(K, \Omega)} \int_{\Omega} |\nabla f(x)|^{p(x)} \vartheta(x) dx. \end{aligned}$$

Moreover, if $U \subset \Omega$ is an open subset, then we define

$$\text{cap}_{p(\cdot),\vartheta}(U, \Omega) = \sup_{\substack{K \subset U \\ \text{compact}}} \text{cap}_{p(\cdot),\vartheta}^*(K, \Omega),$$

and also for an arbitrary set $A \subset \Omega$ we define

$$\text{cap}_{p(\cdot),\vartheta}(A, \Omega) = \inf_{\substack{A \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot),\vartheta}(U, \Omega).$$

We call $\text{cap}_{p(\cdot),\vartheta}(A, \Omega)$ the variational $(p(\cdot), \vartheta)$ -capacity of A with respect to Ω . We say simply $\text{cap}_{p(\cdot),\vartheta}(A, \Omega)$ the relative $(p(\cdot), \vartheta)$ -capacity. It is evident that the same number $\text{cap}_{p(\cdot),\vartheta}(A, \Omega)$ is obtained if the infimum in the definition is taken over $f \in R_{p(\cdot),\vartheta}(K, \Omega)$ with $0 \leq f \leq 1$; when suitable, we implicitly assume this extra condition.

By the same arguments as in [6, Propositions 10.2.2 and 10.2.3], we obtain the following Theorems 5 and 6, respectively.

THEOREM 5. *Let $K \subset \Omega$ be a compact subset. We denote*

$$R_{p(\cdot),\vartheta}^*(K, \Omega) = \{f \in W_{\vartheta}^{1,p(\cdot)}(\Omega) \cap C_0(\Omega) : f \geq 1 \text{ on } K\}.$$

Then we have

$$\text{cap}_{p(\cdot),\vartheta}^*(K, \Omega) = \inf_{f \in R_{p(\cdot),\vartheta}^*(K, \Omega)} \rho_{p(\cdot),\vartheta}(|\nabla f|).$$

THEOREM 6. *Let $p(\cdot) \in \mathcal{P}(\Omega)$ and ϑ be a weight function. Then we have*

$$\text{cap}_{p(\cdot),\vartheta}^*(K, \Omega) = \text{cap}_{p(\cdot),\vartheta}(K, \Omega)$$

for every compact set $K \subset \Omega$.

Therefore the relative $(p(\cdot), \vartheta)$ -capacity is well defined on compact sets. But, if $p^+ = \infty$, then the elements of $R_{p(\cdot),\vartheta}^*(K, \Omega)$ do not satisfy equality in general. Also, the relative $(p(\cdot), \vartheta)$ -capacity has the following properties.

(P1) $\text{cap}_{p(\cdot),\vartheta}(\emptyset, \Omega) = 0$.

(P2) If $A_1 \subset A_2 \subset \Omega_2 \subset \Omega_1$, then $\text{cap}_{p(\cdot),\vartheta}(A_1, \Omega_1) \leq \text{cap}_{p(\cdot),\vartheta}(A_2, \Omega_2)$.

(P3) If A is a subset of Ω , then

$$\text{cap}_{p(\cdot),\vartheta}(A, \Omega) = \inf_{\substack{A \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot),\vartheta}(U, \Omega).$$

(P4) If K_1 and K_2 are compact subsets of Ω , then

$$\text{cap}_{p(\cdot),\vartheta}(K_1 \cup K_2, \Omega) + \text{cap}_{p(\cdot),\vartheta}(K_1 \cap K_2, \Omega)$$

$$\leq \text{cap}_{p(\cdot),\vartheta}(K_1, \Omega) + \text{cap}_{p(\cdot),\vartheta}(K_2, \Omega).$$

(P5) Let K_n be a decreasing sequence of compact subsets of Ω for $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} \text{cap}_{p(\cdot),\vartheta}(K_n, \Omega) = \text{cap}_{p(\cdot),\vartheta}\left(\bigcap_{n=1}^{\infty} K_n, \Omega\right).$$

(P6) If A_n is an increasing sequence of subsets of Ω for $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega) = \text{cap}_{p(\cdot),\vartheta}\left(\bigcup_{n=1}^{\infty} A_n, \Omega\right).$$

(P7) If $A_n \subset \Omega$ for $n \in \mathbb{N}$, then

$$\text{cap}_{p(\cdot),\vartheta}\left(\bigcup_{n=1}^{\infty} A_n, \Omega\right) \leq \sum_{n=1}^{\infty} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega).$$

The proof of these properties is the same as in [6], [12], [14]. Hence the relative $(p(\cdot), \vartheta)$ -capacity is an outer measure. A set function which satisfies the capacity properties (P1), (P2), (P5) and (P6) is called a Choquet capacity, see [3]. Therefore we have the following result.

COROLLARY 7. *The set function $A \rightarrow \text{cap}_{p(\cdot),\vartheta}(A, \Omega)$, $A \subset \Omega$, is a Choquet capacity. In particular, all Borel sets $A \subset \Omega$ are capacitable, that is,*

$$\text{cap}_{p(\cdot),\vartheta}(A, \Omega) = \inf_{\substack{A \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot),\vartheta}(U, \Omega) = \sup_{\substack{K \subset A \\ K \text{ compact}}} \text{cap}_{p(\cdot),\vartheta}(K, \Omega).$$

Note that each Borel set is a Suslin set and the definition of Suslin sets can be reach in [10]. Also, it is not necessary that $p^+ < \infty$ for having all these properties.

THEOREM 8. *If $A_1 \subset \Omega_1 \subset A_2 \subset \Omega_2 \subset \dots \subset \Omega = \bigcup_{n=1}^{\infty} \Omega_n$, then*

$$\text{cap}_{p(\cdot),\vartheta}(A_1, \Omega) \leq \left(\sum_{n=1}^{\infty} (\text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n))^{\frac{1}{1-p^-}} \right)^{1-p^-}.$$

PROOF. First we can assume that $\text{cap}_{p(\cdot),\vartheta}(A_1, \Omega_1) < \infty$. Otherwise the proof is clear. Fix an integer m . Also, let $\varepsilon > 0$ and take an open set $U \subset \Omega_1$ such that $A_1 \subset U$ and

$$(1) \quad \text{cap}_{p(\cdot),\vartheta}(U, \Omega_1) \leq \text{cap}_{p(\cdot),\vartheta}(A_1, \Omega_1) + \varepsilon.$$

Let $K_1 \subset U$ be compact and let $f_1 \in R_{p(\cdot),\vartheta}(K_1, \Omega_1)$ be such that

$$\int_{\Omega_1} |\nabla f_1(x)|^{p(x)} \vartheta(x) dx \leq \text{cap}_{p(\cdot),\vartheta}(K_1, \Omega_1) + \varepsilon.$$

Also, we can choose $f_n \in W_{\vartheta}^{1,p(\cdot)}(\Omega) \cap C_0(\Omega)$, $n = 2, 3, \dots, m$, such that $f_n \in R_{p(\cdot),\vartheta}(K_n, \Omega_n)$, where $K_n = \text{supp } f_{n-1}$, and that

$$\int_{\Omega_n} |\nabla f_n(x)|^{p(x)} \vartheta(x) dx \leq \text{cap}_{p(\cdot),\vartheta}(K_n, \Omega_n) + \varepsilon$$

by induction. Let a_n be a sequence of nonnegative numbers with $\sum_{n=1}^m a_n = 1$ and define $g = \sum_{n=1}^m a_n f_n$. Since the space $W_{\vartheta}^{1,p(\cdot)}(\Omega) \cap C_0(\Omega)$ is a vector space, $g \in W_{\vartheta}^{1,p(\cdot)}(\Omega) \cap C_0(\Omega)$ and then $g \in R_{p(\cdot),\vartheta}(K_1, \Omega)$. It is easy to see that $K_n \subset \Omega_{n-1} \subset A_n$, $n \geq 2$. Using the definition of the relative $(p(\cdot), \vartheta)$ -capacity, we have

$$\begin{aligned} \text{cap}_{p(\cdot),\vartheta}(K_1, \Omega) &\leq \int_{\Omega_1 \cup \Omega_2 \cup \dots} \left| \sum_{n=1}^m a_n \nabla f_n(x) \right|^{p(x)} \vartheta(x) dx \\ &\leq \sum_{n=1}^m a_n^{p^-} \int_{\Omega_n} |\nabla f_n(x)|^{p(x)} \vartheta(x) dx \end{aligned}$$

where $\nabla f_n \neq 0$ are pairwise disjoint. This yields

$$\text{cap}_{p(\cdot),\vartheta}(K_1, \Omega) \leq \left(a_1^{p^-} \text{cap}_{p(\cdot),\vartheta}(K_1, \Omega_1) + \sum_{n=2}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(K_n, \Omega_n) \right) + \varepsilon^*$$

where $\varepsilon^* = \varepsilon \sum_{n=1}^m a_n^{p^-}$. Since $K_1 \subset U$, we get

$$\text{cap}_{p(\cdot),\vartheta}(K_1, \Omega_1) \leq \text{cap}_{p(\cdot),\vartheta}(U, \Omega_1).$$

Also, it follows by the definition of the relative $(p(\cdot), \vartheta)$ -capacity that

$$\text{cap}_{p(\cdot),\vartheta}(K_n, \Omega_n) \leq \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n)$$

and then

$$\sum_{n=2}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(K_n, \Omega_n) \leq \sum_{n=2}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n).$$

Hence

$$(2) \quad \text{cap}_{p(\cdot),\vartheta}(K_1, \Omega) \leq a_1^{p^-} \text{cap}_{p(\cdot),\vartheta}(U, \Omega_1) + \sum_{n=2}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n) + \varepsilon^*.$$

If we use (1) in (2), then we have

$$\begin{aligned} & \text{cap}_{p(\cdot),\vartheta}(K_1, \Omega) \\ & \leq \left(a_1^{p^-} \text{cap}_{p(\cdot),\vartheta}(A_1, \Omega_1) + \sum_{n=2}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n) + \varepsilon^* a_1^{p^-} \right) + \varepsilon^* \\ & = \sum_{n=1}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n) + \varepsilon^{**} \end{aligned}$$

where $\varepsilon^{**} = (1 + a_1^{p^-})\varepsilon^*$. Letting $\varepsilon^{**} \rightarrow 0$ we get

$$\text{cap}_{p(\cdot),\vartheta}(K_1, \Omega) \leq \sum_{n=1}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n).$$

Using the definition of infimum and the relative $(p(\cdot), \vartheta)$ -capacity, respectively, we obtain

$$(3) \quad \text{cap}_{p(\cdot),\vartheta}(A_1, \Omega) \leq \text{cap}_{p(\cdot),\vartheta}(U, \Omega) \leq \sum_{n=1}^m a_n^{p^-} \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n).$$

Since the equality

$$\sum_{n=1}^m \left[\text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n)^{\frac{1}{1-p^-}} \left(\sum_{k=1}^m \text{cap}_{p(\cdot),\vartheta}(A_k, \Omega_k)^{\frac{1}{1-p^-}} \right)^{-1} \right] = 1$$

holds, we can choose

$$a_n = \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n)^{\frac{1}{1-p^-}} \left(\sum_{k=1}^m \text{cap}_{p(\cdot),\vartheta}(A_k, \Omega_k)^{\frac{1}{1-p^-}} \right)^{-1}$$

for $n = 1, 2, \dots, m$. If $\text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n) > 0$ for every $n = 1, 2, \dots, m$, then we have

$$\begin{aligned} \text{cap}_{p(\cdot),\vartheta}(A_1, \Omega) & \leq \sum_{n=1}^m \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n)^{1+\frac{p^-}{1-p^-}} \left(\sum_{k=1}^m \text{cap}_{p(\cdot),\vartheta}(A_k, \Omega_k)^{\frac{1}{1-p^-}} \right)^{-p^-} \\ & = \left(\sum_{n=1}^m \text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n)^{\frac{1}{1-p^-}} \right)^{1-p^-}. \end{aligned}$$

When $\text{cap}_{p(\cdot),\vartheta}(A_n, \Omega_n) = 0$ for some n , then $\text{cap}_{p(\cdot),\vartheta}(A_1, \Omega) = 0$ as well by considering (3), and the proof is obvious. The claim follows by letting $m \rightarrow \infty$. $\square$

REMARK 9. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then the claim of [18, Proposition 2.4] holds even if $p(\cdot) = 1$. This yields $L_\vartheta^{p(\cdot)}(\Omega) \hookrightarrow L_\vartheta^1(\Omega)$.

THEOREM 10. If $\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) \geq 1$ and μ_ϑ is a doubling measure, then we obtain

$$C_1 \mu_\vartheta(B(x_0, r)) \leq \text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) \leq C_2 \mu_\vartheta(B(x_0, r))$$

such that $C_1 = \frac{C}{r}$ and $C_2 = 2^{p^+} c_d \max\{r^{-p^-}, r^{-p^+}\}$.

PROOF. Let $f \in C_0^\infty(B(x_0, 2r))$ be a function such that $f = 1$ in $B(x_0, r)$ and $|\nabla f| \leq \frac{2}{r}$. Since μ_ϑ is doubling, we get

$$\begin{aligned} (4) \quad & \text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) \\ & \leq \int_{B(x_0, 2r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx \leq 2^{p^+} c_d \max\{r^{-p^-}, r^{-p^+}\} \mu_\vartheta(B(x_0, r)). \end{aligned}$$

On the other hand, let $0 < s < r$ and take a function

$$f \in R_{p(\cdot), \vartheta}^*(B(x_0, s), B(x_0, 2r)).$$

Since $\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) \geq 1$, it is easy to see that

$$\rho_{L_\vartheta^{p(\cdot)}(B(x_0, 2r))}(|\nabla f|) \geq 1$$

and then we have

$$\|\nabla f\|_{L_\vartheta^{p(\cdot)}(B(x_0, 2r))} < \rho_{L_\vartheta^{p(\cdot)}(B(x_0, 2r))}(|\nabla f|),$$

see [18]. Hence if we use the Poincaré inequality in $L_\vartheta^1(B(x_0, 2r))$ and the embedding $L_\vartheta^{p(\cdot)}(B(x_0, 2r)) \hookrightarrow L_\vartheta^1(B(x_0, 2r))$, then we obtain

$$\begin{aligned} (5) \quad & \mu_\vartheta(B(x_0, s)) \leq cr \int_{B(x_0, 2r)} |\nabla f(x)| \vartheta(x) dx \\ & \leq cr c_1 \|\nabla f\|_{L_\vartheta^{p(\cdot)}(B(x_0, 2r))} \leq Cr \int_{B(x_0, 2r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx. \end{aligned}$$

If we take the infimum over $f \in R_{p(\cdot), \vartheta}^*(B(x_0, s), B(x_0, 2r))$ and letting $s \rightarrow r$ in inequality (5), then we get

$$(6) \quad \mu_\vartheta(B(x_0, r)) \leq Cr \text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)).$$

We conclude the proof by considering the inequalities (4) and (6). Hence it is clear that we can write $\mu_\vartheta(B(x_0, r)) \approx \text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r))$ under the hypotheses. $\square$

REMARK 11. Note that the equivalence in Theorem 10 is not true in general. But if we use the following trick in inequality (5)

$$\begin{aligned} \mu_{\vartheta}(B(x_0, s)) &\leq cr \int_{B(x_0, 2r)} |\nabla f(x)| \vartheta(x) dx \\ &\leq cr \int_{B(x_0, 2r)} \max \{1, |\nabla f(x)|\}^{p(x)} \vartheta(x) dx \\ &\leq cr \int_{B(x_0, 2r)} (1 + |\nabla f(x)|^{p(x)}) \vartheta(x) dx \leq cr (\mu_{\vartheta}(B(x_0, 2r)) + \rho_{p(\cdot), \vartheta}(|\nabla f|)), \end{aligned}$$

then this will allow us to obtain some estimates even in case

$$\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) < 1.$$

THEOREM 12. If $A \subset B(x_0, r)$, $\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 4r)) \geq 1$ and $0 < r \leq s \leq 2r$, then

$$\frac{1}{C} \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)) \leq \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2s)) \leq \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r))$$

such that $C = 2^{p^+} + 2^{2p^++1} c_{C1} \max \{r^{1-p^-}, r^{1-p^+}\}$.

PROOF. Since $B(x_0, 2r) \subset B(x_0, 2s)$, it is clear that

$$\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2s)) \leq \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)).$$

Thus, we need to have the first inequality in case $s = 2r$. Because of the fact that the relative $(p(\cdot), \vartheta)$ -capacity is a Choquet capacity, we can suppose that A is compact. Let $g \in C_0^\infty(B(x_0, 2r))$, $0 \leq g \leq 1$, be a cut-off function such that $g = 1$ in $B(x_0, r)$ and $|\nabla g| \leq \frac{2}{r}$. Also, let the function $f \in R_{p(\cdot), \vartheta}^*(A, B(x_0, 4r))$ be given. If we use the definition of $R_{p(\cdot), \vartheta}^*(A, B(x_0, 4r))$ and the function g and also the fact that the space $C_0^\infty(B(x_0, 2r))$ is dense in $W_{\vartheta}^{1, p(\cdot)}(B(x_0, 2r))$, then we get that

$$gf \in W_{\vartheta}^{1, p(\cdot)}(B(x_0, 2r)) \cap C_0(B(x_0, 2r))$$

such that $gf = 1$ on A . Thus $gf \in R_{p(\cdot), \vartheta}^*(A, B(x_0, 2r))$. Therefore, we have

$$\begin{aligned} \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)) &\leq 2^{p^+} \int_{B(x_0, 2r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx \\ &\quad + 2^{2p^+} \max \{r^{-p^-}, r^{-p^+}\} \int_{B(x_0, 2r)} |f(x)|^{p(x)} \vartheta(x) dx \end{aligned}$$

$$\begin{aligned} &\leq 2^{p^+} \int_{B(x_0, 4r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx \\ &+ 2^{2p^+} \max \{r^{-p^-}, r^{-p^+}\} \int_{B(x_0, 4r)} |f(x)|^{p(x)} \vartheta(x) dx. \end{aligned}$$

Since $\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 4r)) \geq 1$, we have

$$\|\nabla f\|_{L_{\vartheta}^{p(\cdot)}(B(x_0, 4r))} < \rho_{L_{\vartheta}^{p(\cdot)}(B(x_0, 4r))}(|\nabla f|),$$

see [18]. Hence if we use the Poincaré inequality in $L_{\vartheta}^1(B(x_0, 4r))$ and the embedding $L_{\vartheta}^{p(\cdot)}(B(x_0, 4r)) \hookrightarrow L_{\vartheta}^1(B(x_0, 4r))$, then we obtain

$$\begin{aligned} \int_{B(x_0, 4r)} |f(x)|^{p(x)} \vartheta(x) dx &\leq 2rc \int_{B(x_0, 4r)} |\nabla f(x)| \vartheta(x) dx \\ &\leq 2rc c_1 \|\nabla f\|_{L_{\vartheta}^{p(\cdot)}(B(x_0, 4r))} \leq 2rc c_1 \rho_{L_{\vartheta}^{p(\cdot)}(B(x_0, 4r))}(|\nabla f|). \end{aligned}$$

This yields

$$\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)) \leq C \int_{B(x_0, 4r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx$$

where $C = 2^{p^+} + 2^{2p^+ + 1} c c_1 \max\{r^{1-p^-}, r^{1-p^+}\}$. The proof is completed by taking the infimum over $f \in R_{p(\cdot), \vartheta}^*(A, B(x_0, 4r))$ in the last inequality. Hence it is clear that we can write

$$\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2s)) \approx \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r))$$

under the hypotheses. $\square$

REMARK 13. By the same arguments as in Theorem 10 the equivalence in Theorem 12 is not true in general. But if we use the same trick as in Remark 11, then we can find some estimates even in case $\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 4r)) < 1$.

THEOREM 14. Let $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, $1 < q^- \leq q(\cdot) \leq q^+ < \infty$ and $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. Assume that ϑ is a weight function such that $\vartheta(x) \geq 1$ for $x \in \mathbb{R}^n$. If $0 < r_1 < r_2 < \infty$ and $\text{cap}_{p(\cdot), \vartheta}(A(x_0; r_1, r_2), B(x_0, r_2)) \geq 1$, then

$$\omega_{n-1} \leq C \text{cap}_{p(\cdot), \vartheta}(B(x_0, r_1), B(x_0, r_2))$$

where $A(x_0; r_1, r_2)$ is the annulus $B(x_0, r_2) - B(x_0, r_1)$. Here

$$\begin{aligned} C = c_h \max \left\{ \left[\max \{r_2^{(1-n)q^+}, r_2^{(1-n)q^-}\} |A(x_0; r_1, r_2)| \right]^{\frac{1}{q^+}}, \right. \\ \left. \left[\max \{r_2^{(1-n)q^+}, r_2^{(1-n)q^-}\} |A(x_0; r_1, r_2)| \right]^{\frac{1}{q^-}} \right\}, \end{aligned}$$

where $|A(x_0; r_1, r_2)|$ is the Lebesgue measure of $A(x_0; r_1, r_2)$ and c_h is the constant in the Hölder inequality for variable exponent Lebesgue spaces.

PROOF. Let $f \in C_0^\infty(B(x_0, r_2))$ be a function such that $f = 1$ on $B(x_0, r_1)$. Then $f \in R_{p(\cdot), \vartheta}^*(B(x_0, r_1), B(x_0, r_2))$. By [11, Lemma 7.14], we get

$$f(y) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\nabla f(x)(y-x)}{|x-y|^n} dx.$$

Also, it is well known that the $(n-1)$ -dimensional measure of the unit sphere ω_{n-1} in $\mathbb{R}^n$ equals $n\omega_n$. Hence the following integral is obtained:

$$f(y) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla f(x)(y-x)}{|x-y|^n} dx$$

for all $y \in \mathbb{R}^n$. Since $\text{cap}_{p(\cdot), \vartheta}(A(x_0; r_1, r_2), B(x_0, r_2)) \geq 1$, it is easy to see that $\rho_{L_{\vartheta}^{p(\cdot)}(B(x_0, r_2))}(|\nabla f|) \geq 1$ and then we have $\|\nabla f\|_{L_{\vartheta}^{p(\cdot)}(B(x_0, r_2))} < \rho_{L_{\vartheta}^{p(\cdot)}(B(x_0, r_2))}(|\nabla f|)$, see [18]. Also, if we use the Hölder inequality for variable exponent Lebesgue spaces, then we find

$$\begin{aligned} \omega_{n-1} = \omega_{n-1} f(x_0) &= \int_{A(x_0; r_1, r_2)} \frac{\nabla f(x)(x_0-x)}{|x-x_0|^n} \vartheta(x)^{\frac{1}{p(x)}} \vartheta(x)^{-\frac{1}{p(x)}} dx \\ &\leq c_h \left\| |x-x_0|^{1-n} \vartheta(x)^{-\frac{1}{p(\cdot)}} \right\|_{L^{q(\cdot)}(A(x_0; r_1, r_2))} \rho_{L_{\vartheta}^{p(\cdot)}(B(x_0, r_2))}(|\nabla f|) \end{aligned}$$

for some $c_h > 0$ where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. Using the relationship between the Luxemburg norm and the modular (see [17]), we get

$$\begin{aligned} \omega_{n-1} &\leq c_h \max \left\{ \left(\rho_{L^{q(\cdot)}(A(x_0; r_1, r_2))}(|x-x_0|^{1-n} \vartheta(x)^{-\frac{1}{p(\cdot)}}) \right)^{\frac{1}{q^+}}, \right. \\ &\quad \left. \left(\rho_{L^{q(\cdot)}(A(x_0; r_1, r_2))}(|x-x_0|^{1-n} \vartheta(x)^{-\frac{1}{p(\cdot)}}) \right)^{\frac{1}{q^-}} \right\} \rho_{L_{\vartheta}^{p(\cdot)}(B(x_0, r_2))}(|\nabla f|) \\ &\leq c_h \max \left\{ \left[\max \left\{ r_2^{(1-n)q^+}, r_2^{(1-n)q^-} \right\} |A(x_0; r_1, r_2)| \right]^{\frac{1}{q^+}}, \right. \\ &\quad \left. \left[\max \left\{ r_2^{(1-n)q^+}, r_2^{(1-n)q^-} \right\} |A(x_0; r_1, r_2)| \right]^{\frac{1}{q^-}} \right\} \rho_{L_{\vartheta}^{p(\cdot)}(B(x_0, r_2))}(|\nabla f|) \end{aligned}$$

for some $c_h > 0$. Taking the infimum over $f \in R_{p(\cdot), \vartheta}^*(B(x_0, r_1), B(x_0, r_2))$ in the last inequality, we have the desired result by the continuity of the integral. $\square$

4. The relationship between capacities

Now we will give several inequalities between the capacities previously mentioned.

THEOREM 15. *If $\Omega \subset \mathbb{R}^n$ is bounded and $K \subset \Omega$ is compact, then*

$$C_{p(\cdot),\vartheta}(K) \leq C \max \left\{ \text{cap}_{p(\cdot),\vartheta}(K, \Omega)^{\frac{1}{p^-}}, \text{cap}_{p(\cdot),\vartheta}(K, \Omega) \right\}$$

where the constant C depends on the dimension n , the Poincaré inequality constant and $\text{diam}(\Omega)$.

PROOF. We can assume that $\text{cap}_{p(\cdot),\vartheta}(K, \Omega) < \infty$. Otherwise the proof is clear. Let $0 < \varepsilon < 1$ and $f \in R_{p(\cdot),\vartheta}^*(K, \Omega)$ be a function such that

$$(7) \quad \rho_{p(\cdot),\vartheta}(|\nabla f|) \leq \text{cap}_{p(\cdot),\vartheta}(K, \Omega) + \varepsilon.$$

Now let us extend f to be zero outside of Ω , that is

$$f(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n - \Omega, \end{cases}$$

and define $g = \min\{1, f\}$. If we consider the definitions of the relative $(p(\cdot), \vartheta)$ -capacity and the Sobolev capacity, then we get $g \in S_{p(\cdot),\vartheta}(K)$. Hence

$$C_{p(\cdot),\vartheta}(K) \leq \int_{\mathbb{R}^n} (|g(x)|^{p(x)} + |\nabla f(x)|^{p(x)})\vartheta(x) dx.$$

It follows by $0 \leq g \leq 1$ that

$$(8) \quad \int_{\Omega} |g(x)|^{p(x)}\vartheta(x) dx \leq \int_{\Omega} |g(x)|\vartheta(x) dx \leq \int_{\Omega} |f(x)|\vartheta(x) dx.$$

Also, if we use the Poincaré inequality in $L_{\vartheta}^1(\Omega)$ and Remark 9, then we have

$$(9) \quad \|f\vartheta\|_1 = \|f\|_{1,\vartheta} \leq c \text{diam}(\Omega) \|\nabla f\|_{1,\vartheta} \leq c \text{diam}(\Omega) c_1 \|\nabla f\|_{p(\cdot),\vartheta}.$$

By (8) and (9), we have

$$\begin{aligned} C_{p(\cdot),\vartheta}(K) &\leq \int_{\Omega} |f(x)|\vartheta(x) dx + \int_{\Omega} |\nabla f(x)|^{p(x)}\vartheta(x) dx \\ &\leq C^* [\|\nabla f\|_{p(\cdot),\vartheta} + \rho_{p(\cdot),\vartheta}(|\nabla f|)] \\ &\leq C^* \left(\max \left\{ \rho_{p(\cdot),\vartheta}(|\nabla f|)^{\frac{1}{p^-}}, \rho_{p(\cdot),\vartheta}(|\nabla f|)^{\frac{1}{p^+}} \right\} + \rho_{p(\cdot),\vartheta}(|\nabla f|) \right) \end{aligned}$$

where $C^* = \max\{1, c \operatorname{diam}(\Omega) c_1\}$. Considering the fact that $1 < p^- \leq p(\cdot) \leq p^+ < \infty$ and (7), it is easy to see that

$$\begin{aligned} & \max \left\{ \rho_{p(\cdot), \vartheta}(|\nabla f|)^{\frac{1}{p^+}}, \rho_{p(\cdot), \vartheta}(|\nabla f|)^{\frac{1}{p^-}} \right\} + \rho_{p(\cdot), \vartheta}(|\nabla f|) \\ & \leq 2 \max \left\{ \rho_{p(\cdot), \vartheta}(|\nabla f|)^{\frac{1}{p^+}}, \rho_{p(\cdot), \vartheta}(|\nabla f|)^{\frac{1}{p^-}} \right\} \\ & \leq 2 \max \left\{ \left(\operatorname{cap}_{p(\cdot), \vartheta}(K, \Omega) + \varepsilon \right)^{\frac{1}{p^+}}, \operatorname{cap}_{p(\cdot), \vartheta}(K, \Omega) + \varepsilon \right\} \\ & \leq 2 \max \left\{ \left(\operatorname{cap}_{p(\cdot), \vartheta}(K, \Omega) \right)^{\frac{1}{p^+}}, \operatorname{cap}_{p(\cdot), \vartheta}(K, \Omega) \right\} + \varepsilon^{\frac{1}{p^+}} + \varepsilon. \end{aligned}$$

Hence, we get

$$C_{p(\cdot), \vartheta}(K) \leq C \left[\max \left\{ \left(\operatorname{cap}_{p(\cdot), \vartheta}(K, \Omega) \right)^{\frac{1}{p^+}}, \operatorname{cap}_{p(\cdot), \vartheta}(K, \Omega) \right\} + \varepsilon^{\frac{1}{p^+}} + \varepsilon \right]$$

where $C = 2 \max\{1, c \operatorname{diam}(\Omega) c_1\}$. This yields the claim as ε tends to zero. $\square$

The proof of the following theorem is similar to [6, Theorem 10.3.2].

THEOREM 16. *If $\Omega \subset \mathbb{R}^n$ is bounded and $A \subset \Omega$, then*

$$C_{p(\cdot), \vartheta}(A) \leq C \max \left\{ \operatorname{cap}_{p(\cdot), \vartheta}(A, \Omega)^{\frac{1}{p^+}}, \operatorname{cap}_{p(\cdot), \vartheta}(A, \Omega) \right\}$$

where the constant C depends on the dimension n , the Poincaré inequality constant and $\operatorname{diam}(\Omega)$.

COROLLARY 17. *Let $A \subset \Omega$. If $\operatorname{cap}_{p(\cdot), \vartheta}(A, \Omega) = 0$, then $C_{p(\cdot), \vartheta}(A) = 0$.*

Note that the opposite implication of the previous corollary is not always true. We need to consider an additional hypothesis for this. By the same arguments as in [6, Proposition 10.3.4], we obtain the following statement.

THEOREM 18. *Let $A \subset \Omega$. Assume that the space $W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^n)$. If $C_{p(\cdot), \vartheta}(A) = 0$, then $\operatorname{cap}_{p(\cdot), \vartheta}(A, \Omega) = 0$.*

Now we give a relationship between the Sobolev $(p(\cdot), \vartheta)$ -capacity and the relative $(p(\cdot), \vartheta)$ -capacity.

THEOREM 19. *If $A \subset B(x_0, r)$ and $\operatorname{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)) \geq 1$, then*

$$\frac{1}{C_1} C_{p(\cdot), \vartheta}(A) \leq \operatorname{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)) \leq C_2 C_{p(\cdot), \vartheta}(A)$$

where $C_1 = 1 + cr(1 + |B(x_0, 2r)|)$ and $C_2 = 2^{2p^+} (1 + \max\{r^{-p^-}, r^{-p^+}\})$ and c is the Poincaré inequality constant.

PROOF. Suppose that $K \subset B(x_0, r)$ is compact. Let $g \in C_0^\infty(B(x_0, 2r))$, $0 \leq g \leq 1$, be a cut-off function such that $g = 1$ in $B(x_0, r)$ and $|\nabla g| \leq 2/r$. Also, let the function $f \in S_{p(\cdot), \vartheta}(K)$ be given. Thus we get $gf \in R_{p(\cdot), \vartheta}^*(A, B(x_0, 2r))$. Therefore

$$\begin{aligned} \text{cap}_{p(\cdot), \vartheta}(K, B(x_0, 2r)) &\leq 2^{p^+} \int_{B(x_0, 2r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx \\ &\quad + 2^{2p^+} \max\{r^{-p^-}, r^{-p^+}\} \int_{B(x_0, 2r)} |f(x)|^{p(x)} \vartheta(x) dx \\ &\leq 2^{2p^+} (1 + \max\{r^{-p^-}, r^{-p^+}\}) \int_{\mathbb{R}^n} (|f(x)|^{p(x)} + |\nabla f(x)|^{p(x)}) \vartheta(x) dx. \end{aligned}$$

If we take the infimum over $f \in S_{p(\cdot), \vartheta}(K)$ in the last inequality, then we have

$$\text{cap}_{p(\cdot), \vartheta}(K, B(x_0, 2r)) \leq C_2 C_{p(\cdot), \vartheta}(K)$$

where $C_2 = 2^{2p^+} (1 + \max\{r^{-p^-}, r^{-p^+}\})$.

Now, we take $f \in C_0^\infty(B(x_0, 2r))$, $0 \leq f \leq 1$, such that $f = 1$ in an open set containing K . Then $f \in R_{p(\cdot), \vartheta}^*(K, B(x_0, 2r))$. Since $\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)) \geq 1$, it is easy to see that $\rho_{L_\vartheta^{p(\cdot)}(B(x_0, 2r))}(|\nabla f|) \geq 1$ and then we have $\|\nabla f\|_{L_\vartheta^{p(\cdot)}(B(x_0, 2r))} < \rho_{L_\vartheta^{p(\cdot)}(B(x_0, 2r))}(|\nabla f|)$, see [18]. If we use the fact $0 \leq f \leq 1$, the Poincaré inequality in $L_\vartheta^1(B(x_0, 2r))$ and the embedding $L_\vartheta^{p(\cdot)}(B(x_0, 2r)) \hookrightarrow L_\vartheta^1(B(x_0, 2r))$, then we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|^{p(x)} \vartheta(x) dx &\leq cr \int_{B(x_0, 2r)} |\nabla f(x)| \vartheta(x) dx \\ &\leq cr c_1 \int_{B(x_0, 2r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx. \end{aligned}$$

It follows that

$$C_{p(\cdot), \vartheta}(K) \leq C_1 \int_{B(x_0, 2r)} |\nabla f(x)|^{p(x)} \vartheta(x) dx$$

where $C_1 = 1 + cr c_1$. This completes the proof for the compact sets if we take the infimum over $f \in R_{p(\cdot), \vartheta}^*(K, B(x_0, 2r))$ in the last inequality. If we consider the definition of the relative $(p(\cdot), \vartheta)$ -capacity and use the first part of the proof, then it is shown that the desired result holds for an arbitrary set $A \subset B(x_0, r)$. $\square$

5. $(p(\cdot), \vartheta)$ -thinness

DEFINITION 20. The set $A \subset \mathbb{R}^n$ is called $(p(\cdot), \vartheta)$ -thin at x_0 if

$$(10) \quad \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x_0, r), B(x_0, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r))} \right)^{\frac{1}{p(x_0)-1}} \frac{dr}{r} < \infty.$$

We say that A is $(p(\cdot), \vartheta)$ -thick at x_0 if A is not $(p(\cdot), \vartheta)$ -thin at x_0 .

The integral in inequality (10) is called a Wiener type integral, see [14]. From now on, we write that

$$W_{p(\cdot), \vartheta}(A, x_0) = \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x_0, r), B(x_0, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r))} \right)^{\frac{1}{p(x_0)-1}} \frac{dr}{r}$$

for convenience. Also, we denote the Wiener sum by

$$W_{p(\cdot), \vartheta}^{\text{sum}}(A, x_0) = \sum_{i=0}^{\infty} \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x_0, 2^{-i}), B(x_0, 2^{1-i}))}{\text{cap}_{p(\cdot), \vartheta}(B(x_0, 2^{-i}), B(x_0, 2^{1-i}))} \right)^{\frac{1}{p(x_0)-1}}.$$

The Wiener sum is more useful than the Wiener type integral in most cases. Now we give a relationship between these two notions.

THEOREM 21. Assume that the hypotheses of Theorems 10 and 12 are hold. Then there exist constants C_1, C_2 such that

$$C_1 W_{p(\cdot), \vartheta}(A, x_0) \leq W_{p(\cdot), \vartheta}^{\text{sum}}(A, x_0) \leq C_2 W_{p(\cdot), \vartheta}(A, x_0)$$

for every $A \subset \mathbb{R}^n$ and $x_0 \notin A$. In particular, $W_{p(\cdot), \vartheta}(A, x_0)$ is finite if and only if $W_{p(\cdot), \vartheta}^{\text{sum}}(A, x_0)$ is finite.

PROOF. Using the same methods as in Theorems 12 and 10, it is easy to see for $r \leq s \leq 2r$ that

$$\text{cap}_{p(\cdot), \vartheta}(A \cap B(x_0, r), B(x_0, 2r)) \approx \text{cap}_{p(\cdot), \vartheta}(A \cap B(x_0, r), B(x_0, 2s))$$

and

$$\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) \approx \text{cap}_{p(\cdot), \vartheta}(B(x_0, s), B(x_0, 2s))$$

where the constants in $\approx$ depend on r, p^-, p^+ , the constants of the doubling measure and the Poincaré inequality. Thus for $2^{-1-i} \leq r \leq 2^{-i}$ we have

$$\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x_0, r), B(x_0, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r))}$$

$$\begin{aligned}
&\leq C \frac{\text{cap}_{p(\cdot),\vartheta}(A \cap B(x_0, 2^{-i}), B(x_0, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x_0, 2^{-i}), B(x_0, 2^{1-i}))} \\
&\leq C \frac{\text{cap}_{p(\cdot),\vartheta}(A \cap B(x_0, 2r), B(x_0, 4r))}{\text{cap}_{p(\cdot),\vartheta}(B(x_0, 2r), B(x_0, 4r))}.
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
W_{p(\cdot),\vartheta}(A, x_0) &= \sum_{i=0}^{\infty} \int_{2^{-1-i}}^{2^{-i}} \left(\frac{\text{cap}_{p(\cdot),\vartheta}(A \cap B(x_0, r), B(x_0, 2r))}{\text{cap}_{p(\cdot),\vartheta}(B(x_0, r), B(x_0, 2r))} \right)^{\frac{1}{p(x_0)-1}} \frac{dr}{r} \\
&\leq C \sum_{i=0}^{\infty} \left(\frac{\text{cap}_{p(\cdot),\vartheta}(A \cap B(x_0, 2^{-i}), B(x_0, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x_0, 2^{-i}), B(x_0, 2^{1-i}))} \right)^{\frac{1}{p(x_0)-1}} = CW_{p(\cdot),\vartheta}^{\text{sum}}(A, x_0).
\end{aligned}$$

In a similar way we find

$$\begin{aligned}
W_{p(\cdot),\vartheta}^{\text{sum}}(A, x_0) &\leq \sum_{i=0}^{\infty} \int_{2^{-1-i}}^{2^{-i}} \left(\frac{\text{cap}_{p(\cdot),\vartheta}(A \cap B(x_0, 2^{-i}), B(x_0, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x_0, 2^{-i}), B(x_0, 2^{1-i}))} \right)^{\frac{1}{p(x_0)-1}} \frac{dr}{r} \\
&\leq C \int_0^1 \left(\frac{\text{cap}_{p(\cdot),\vartheta}(A \cap B(x_0, 2r), B(x_0, 4r))}{\text{cap}_{p(\cdot),\vartheta}(B(x_0, 2r), B(x_0, 4r))} \right)^{\frac{1}{p(x_0)-1}} \frac{dr}{r} \leq CW_{p(\cdot),\vartheta}(A, x_0).
\end{aligned}$$

This completes the proof. $\square$

Theorem 21 gives us an equivalent claim for $(p(\cdot), \vartheta)$ -thinness at x_0 .

THEOREM 22. *Assume that $A \subset \mathbb{R}^n$ and $x_0 \notin A$.*

(i) *If A is $(p(\cdot), \vartheta)$ -thin at x_0 , then there exists an open neighborhood U of A such that U is $(p(\cdot), \vartheta)$ -thin at x_0 .*

(ii) *If A is a Borel set and $(p(\cdot), \vartheta)$ -thick at x_0 , then there exists a compact set $K \subset A \cup \{x_0\}$ such that K is $(p(\cdot), \vartheta)$ -thick at x_0 .*

PROOF. Firstly we denote $B_i = B(x_0, 2^{1-i})$. Assume that V_1 and V_2 are $(p(\cdot), \vartheta)$ -thin at x_0 . By the subadditivity property of the relative $(p(\cdot), \vartheta)$ -capacity, it is clear that $V_1 \cup V_2$ is $(p(\cdot), \vartheta)$ -thin at x_0 . Since $x_0 \notin A$ and x_0 is the center of the balls B_i for each i , we may assume that $A \cap \partial B_i = \emptyset$. Moreover, let $U_0 = \mathbb{R}^n$ and for each $i = 1, 2, \dots$ take an open set $U_i \subset B_i \cap U_{i-1}$ such that $A_i = A \cap B_i \subset U_i$ and that

$$\left(\frac{\text{cap}_{p(\cdot),\vartheta}(U_i, B_{i-1})}{\text{cap}_{p(\cdot),\vartheta}(B_i, B_{i-1})} \right)^{\frac{1}{p(x_0)-1}} \leq \left(\frac{\text{cap}_{p(\cdot),\vartheta}(A_i, B_{i-1})}{\text{cap}_{p(\cdot),\vartheta}(B_i, B_{i-1})} \right)^{\frac{1}{p(x_0)-1}} + 2^{-i-1}.$$

Let us denote $U = \bigcup_{i=0}^{\infty} (U_i - \overline{B_{i+1}})$. Then we obtain that $A \subset U$, U is open, and

$$W_{p(\cdot), \vartheta}^{\text{sum}}(U, x_0) \leq \sum_{i=0}^{\infty} \left(\frac{\text{cap}_{p(\cdot), \vartheta}(U_i, B_{i-1})}{\text{cap}_{p(\cdot), \vartheta}(B_i, B_{i-1})} \right)^{\frac{1}{p(x_0)-1}} \leq W_{p(\cdot), \vartheta}^{\text{sum}}(A, x_0) + 1 < \infty.$$

This completes the proof of (i) because of the fact that U is the desired neighborhood of A .

Now we consider the proof of (ii). Again we denote $B_i = B(x_0, 2^{1-i})$. Since the sets $A \cap B_i$ are Borel, we have

$$\text{cap}_{p(\cdot), \vartheta}(A \cap B_i, B_{i-1}) = \sup_{\substack{K \subset A \cap B_i \\ \text{compact}}} \text{cap}_{p(\cdot), \vartheta}(K, B_{i-1})$$

for all $i \in \mathbb{N}$. For each i take a compact $K_i \subset A \cap B_i$ such that

$$\left(\frac{\text{cap}_{p(\cdot), \vartheta}(A_i, B_{i-1})}{\text{cap}_{p(\cdot), \vartheta}(B_i, B_{i-1})} \right)^{\frac{1}{p(x_0)-1}} \leq \left(\frac{\text{cap}_{p(\cdot), \vartheta}(K_i, B_{i-1})}{\text{cap}_{p(\cdot), \vartheta}(B_i, B_{i-1})} \right)^{\frac{1}{p(x_0)-1}} + 2^{-i}.$$

Hence $K = \bigcup_{i=0}^{\infty} K_i \cup \{x_0\}$ is the desired compact set. $\square$

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