



Existence and multiplicity of weak solutions for eigenvalue Robin problem with weighted $p(\cdot)$ -Laplacian

Ismail Aydin¹ · Cihan Unal²

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Abstract

By applying Mountain Pass Lemma, Ekeland's variational principle and Fountain Theorem, we prove the existence and multiplicity of solutions for the following Robin problem

$$\begin{cases} -\operatorname{div} (a(x) |\nabla u|^{p(x)-2} \nabla u) = \lambda b(x) |u|^{q(x)-2} u, & x \in \Omega \\ a(x) |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u = 0, & x \in \partial\Omega, \end{cases}$$

under some appropriate conditions in the space $W_{a,b}^{1,p(\cdot)}(\Omega)$.

Keywords Weak solution · $p(\cdot)$ -Laplacian · Mountain Pass Lemma · Fountain Theorem · Ekeland variational principle

Mathematics Subject Classification 35J35 · 46E35 · 35J60 · 35J70

1 Introduction

Let a and b be weight functions. In this paper the Robin problem

$$\begin{cases} -\operatorname{div} (a(x) |\nabla u|^{p(x)-2} \nabla u) = \lambda b(x) |u|^{q(x)-2} u, & x \in \Omega \\ a(x) |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

✉ Cihan Unal
cihanunal88@gmail.com

Ismail Aydin
iaydin@sinop.edu.tr

¹ Department of Mathematics, Faculty of Arts and Sciences, Sinop University, Sinop, Turkey

² Republic of Turkey, Assessment, Selection and Placement Center, Ankara, Turkey

is investigated with respect to some suitable assumptions, where $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative of u with respect to $\partial\Omega$, p, q are continuous functions on $\overline{\Omega}$, i.e. $p, q \in C(\overline{\Omega})$ with $\inf_{x \in \overline{\Omega}} p(x) > 1$, $\beta \in L^\infty(\partial\Omega)$ such that $\beta^- = \inf_{x \in \partial\Omega} \beta(x) > 0$, and $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded smooth domain.

In recent years, variable exponent function spaces, especially Sobolev spaces, have been very useful for exploring the existence of solutions of partial differential equations involving $p(x)$ -Laplacian. There are also several applications in this area, such as electrorheological fluids, image processing, elastic mechanics, fluid dynamics and calculus of variations [17,27,28,34]. In the last years many $p(x)$ -Laplacian problems have been investigated by several authors, see [2,3,5,9,13,26,29].

In 2009, Deng [10] studied the following Robin problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda f(x, u), & x \in \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u = 0, & x \in \partial\Omega, \end{cases} \quad (2)$$

under some conditions on f , and obtained that there exists $\lambda^* > 0$ such that the problem (2) has at least two positive solutions if $\lambda \in (0, \lambda^*)$, has at least one positive solution if $\lambda = \lambda^* < +\infty$ and has no solution if $\lambda > \lambda^*$. Kefi [19] investigated the Robin problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda V(x) |u|^{q(x)-2} u, & x \in \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u = 0, & x \in \partial\Omega, \end{cases} \quad (3)$$

where $V \in L^{s(\cdot)}(\partial\Omega)$ and $V > 0$ with $1 < q(x) < p(x) < s(x) < N$ for all $x \in \overline{\Omega}$. Allaoui [1] obtained two different situations of the existence of solutions for the problem (3) using variational methods. Chung [6] have more general results from Deng [10] and Kefi [19] under some suitable assumptions for q and V .

Inspired by the articles mentioned above, we obtain some new compact embedding theorems in $W_{a,b}^{1,p(\cdot)}(\Omega)$, and show the existence of several different weak solutions of the problem (1) in these spaces. Therefore, we improve and generalize the results presented in Allaoui [1], Chung [6], Deng [10] and Kefi [19].

This work has three parts. In Sect. 2, we mention several properties of $W_{a,b}^{1,p(\cdot)}(\Omega)$ and energy functional J_λ . Moreover, we define a more general norm in $W_{a,b}^{1,p(\cdot)}(\Omega)$ compared to the norm given by Deng [10] using compact embedding theorems. In Sect. 3, we investigate and discuss the major conclusions of the paper.

2 Notation and preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Then, the set is defined by

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \inf_{x \in \overline{\Omega}} p(x) > 1 \right\},$$

For any $p \in C_+(\overline{\Omega})$, we indicate

$$p^- = \inf_{x \in \Omega} p(x) \text{ and } p^+ = \sup_{x \in \Omega} p(x).$$

Let $p \in C_+(\overline{\Omega})$ and $1 < p^- \leq p^+ < \infty$. The space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \mid u : \Omega \rightarrow \mathbb{R} \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}$$

with the (Luxemburg) norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \mathcal{Q}_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

where

$$\mathcal{Q}_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx.$$

Moreover, the space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a reflexive Banach space, see [22].

The weighted Lebesgue space $L_a^{p(\cdot)}(\Omega)$ is defined by

$$L_a^{p(\cdot)}(\Omega) = \left\{ u \mid u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\Omega} |u(x)|^{p(x)} a(x) dx < \infty \right\}$$

such that $\|u\|_{p(\cdot),a} = \left\| u a^{\frac{1}{p(\cdot)}} \right\|_{p(\cdot)} < \infty$ for $u \in L_a^{p(\cdot)}(\Omega)$, where a is a weight function from Ω to $(0, \infty)$.

Remark 1 The space $(L_a^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot),a})$ is a Banach space.

Proof Let (u_j) be a Cauchy sequence in $L_a^{p(\cdot)}(\Omega)$. Then, $(u_j a^{\frac{1}{p(\cdot)}})$ is a Cauchy sequence in the Banach space $L^{p(\cdot)}(\Omega)$ due to

$$\|u_{j_1} - u_{j_2}\|_{p(\cdot),a} = \left\| (u_{j_1} - u_{j_2}) a^{\frac{1}{p(\cdot)}} \right\|_{p(\cdot)} \rightarrow 0,$$

so it converges to some u in $L^{p(\cdot)}(\Omega)$. This follows that (u_j) converges to $u a^{-\frac{1}{p(\cdot)}}$ in $L_a^{p(\cdot)}(\Omega)$. $\square$

We can define the space $L_a^{p(\cdot)}(\partial\Omega)$ similarly by

$$L_a^{p(\cdot)}(\partial\Omega) = \left\{ u \mid u : \partial\Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} a(x) d\sigma < +\infty \right\}$$

with the norm

$$\|u\|_{p(\cdot),a,\partial\Omega} = \inf \left\{ \tau > 0 : \int_{\partial\Omega} \left| \frac{u(x)}{\tau} \right|^{p(x)} a(x) d\sigma \leq 1 \right\}$$

for $u \in L_a^{p(\cdot)}(\partial\Omega)$, where $d\sigma$ is the measure on the boundary of Ω . Then $(L_a^{p(\cdot)}(\partial\Omega), \|\cdot\|_{p(\cdot),a,\partial\Omega})$ is a Banach space. If $a \in L^\infty(\Omega)$, then $L_a^{p(\cdot)} = L^{p(\cdot)}$, see [8,18].

Proposition 1 (See [2,13,20,24,25]) *For all $u, v \in L_a^{p(\cdot)}(\Omega)$, we have*

- (i) $\|u\|_{p(\cdot),a} < 1$ (resp. $= 1, > 1$) if and only if $\mathcal{Q}_{p(\cdot),a}(u) < 1$ (resp. $= 1, > 1$),
- (ii) $\|u\|_{p(\cdot),a}^{p^-} \leq \mathcal{Q}_{p(\cdot),a}(u) \leq \|u\|_{p(\cdot),a}^{p^+}$ with $\|u\|_{p(\cdot),a} > 1$,
- (iii) $\|u\|_{p(\cdot),a}^{p^+} \leq \mathcal{Q}_{p(\cdot),a}(u) \leq \|u\|_{p(\cdot),a}^{p^-}$ with $\|u\|_{p(\cdot),a} < 1$
- (iv) $\min \left\{ \|u\|_{p(\cdot),a}^{p^-}, \|u\|_{p(\cdot),a}^{p^+} \right\} \leq \mathcal{Q}_{p(\cdot),a}(u) \leq \max \left\{ \|u\|_{p(\cdot),a}^{p^-}, \|u\|_{p(\cdot),a}^{p^+} \right\}$,
- (v) $\min \left\{ \mathcal{Q}_{p(\cdot),a}(u)^{\frac{1}{p^-}}, \mathcal{Q}_{p(\cdot),a}(u)^{\frac{1}{p^+}} \right\} \leq \|u\|_{p(\cdot),a} \leq \max \left\{ \mathcal{Q}_{p(\cdot),a}(u)^{\frac{1}{p^-}}, \mathcal{Q}_{p(\cdot),a}(u)^{\frac{1}{p^+}} \right\}$,
- (vi) $\mathcal{Q}_{p(\cdot),a}(u - v) \rightarrow 0$ if and only if $\|u - v\|_{p(\cdot),a} \rightarrow 0$.

Definition 1 Let $a^{-\frac{1}{p(\cdot)-1}}, b^{-\frac{1}{p(\cdot)-1}} \in L_{loc}^1(\Omega)$. The double weighted variable exponent Sobolev space $W_{a,b}^{1,p(\cdot)}(\Omega)$ is defined by

$$W_{a,b}^{1,p(\cdot)}(\Omega) = \left\{ u \in L_b^{p(\cdot)}(\Omega) : |\nabla u| \in L_a^{p(\cdot)}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{a,b}^{1,p(\cdot)} = \|\nabla u\|_{p(\cdot),a} + \|u\|_{p(\cdot),b}.$$

If we take $a = b = 1$, then we obtain $W_{a,b}^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega)$.

Note that the spaces $L_a^{p(\cdot)}(\Omega)$ and $W_{a,b}^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces. In addition, the dual spaces of $L_a^{p(\cdot)}(\Omega)$ and $W_{a,b}^{1,p(\cdot)}(\Omega)$ are $L_{a^*}^{p'(\cdot)}(\Omega)$ and $W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)$, respectively, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, $a^* = a^{1-p'(\cdot)} = a^{-\frac{1}{p(\cdot)-1}}$ and $b^* = b^{1-p'(\cdot)} = b^{-\frac{1}{p(\cdot)-1}}$, see [4,15,23,25,33].

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf_{x \in A} p(x)$ and $p^+(A) = \sup_{x \in A} p(x)$. Define

$$p^\partial(x) = (p(x))^\partial = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases}$$

and

$$p_{r(x)}^\partial(x) = \frac{r(x) - 1}{r(x)} p^\partial(x)$$

for any $x \in \partial\Omega$ and $r \in C(\partial\Omega, \mathbb{R})$ with $r^- = \inf_{x \in \partial\Omega} r(x) > 1$.

Theorem 1 (See [8]) Assume that the set $\partial\Omega$ possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. If $q \in C(\partial\Omega)$ and the inequality $1 \leq q(x) < p_{r(x)}^\partial(x)$ is valid for all $x \in \partial\Omega$, then there is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L_a^{q(\cdot)}(\partial\Omega)$ for $a \in L^{r(\cdot)}(\partial\Omega)$, $r \in C(\partial\Omega)$ with $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. In particular, there is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$, where $1 \leq q(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

It is easy to see that $p_{r(x)}^\partial(x) < p^\partial(x)$ and $p(x) < p^\partial(x)$. So we have the following Corollary under conditions in Theorem 1.

Corollary 1 (See [8])

- (i) There is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\partial\Omega)$, where $1 \leq p(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.
- (ii) There is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L_a^{p(\cdot)}(\partial\Omega)$, where $1 \leq p(x) < p_{r(x)}^\partial(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.

Theorem 2 (See [5]) Let $a^{-\alpha(\cdot)} \in L^1(\Omega)$ with $\alpha(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$. Then we have the compact embedding

$$W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,p_*(\cdot)}(\Omega)$$

where $p_*(x) = \frac{\alpha(x)p(x)}{\alpha(x)+1}$.

Corollary 2 If the inequality $p(x) < p_{*,r(x)}^\partial(x) < p_*^\partial(x)$ is valid for all $x \in \partial\Omega$, then there exists a compact embedding between $W_{a,b}^{1,p(\cdot)}(\Omega)$ and $L_a^{p(\cdot)}(\partial\Omega)$.

Theorem 3 (See [13]) Assume that the set $\partial\Omega$ possesses the cone property and $p \in C(\overline{\Omega})$. Suppose that $b \in L^{r(\cdot)}(\Omega)$, $b(x) > 0$ for $x \in \Omega$, $r \in C(\overline{\Omega})$ and $r^- > 1$. If $q \in C(\overline{\Omega})$ and

$$1 \leq q(x) < \frac{r(x) - 1}{r(x)} p^\gamma(x)$$

for all $x \in \overline{\Omega}$, then there is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L_b^{q(\cdot)}(\Omega)$, where

$$p^\gamma(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

Corollary 3 If the inequality $1 \leq q(x) < \frac{r(x)-1}{r(x)} (p_*)^\gamma(x)$ is true for all $x \in \overline{\Omega}$, then there exists a compact embedding between $W_{a,b}^{1,p(\cdot)}(\Omega)$ and $L_b^{q(\cdot)}(\Omega)$.

The following Theorem plays an important role for the existence of weak solutions of the problem (1).

Theorem 4 *Let $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$. Then the norms $\|u\|_{\partial}$ and $\|u\|_{a,b}^{1,p(\cdot)}$ are equivalent on $W_{a,b}^{1,p(\cdot)}(\Omega)$, where*

$$\|u\|_{\partial} = \|\nabla u\|_{p(\cdot),a} + \|u\|_{p(\cdot),a,\partial\Omega}.$$

Proof Using the embedding $W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow L_a^{p(\cdot)}(\partial\Omega)$ in Corollary 2, there exists a $C_1 > 0$ such that

$$\|u\|_{p(\cdot),a,\partial\Omega} \leq (C_1 - 1) \|u\|_{a,b}^{1,p(\cdot)}$$

for all $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$. Thus, we get

$$\|u\|_{\partial} \leq C_1 \|u\|_{a,b}^{1,p(\cdot)}. \quad (4)$$

Now, we will show that the inequality

$$\|u\|_{p(\cdot),b} \leq C_2 \|u\|_{\partial}$$

is satisfied for any $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ and $C_2 > 0$. Assume that there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $W_{a,b}^{1,p(\cdot)}(\Omega)$ such that

$$\|u_n\|_{p(\cdot),b} > n \|u_n\|_{\partial}$$

for each $n \in \mathbb{N}$. Take $\|u_n\|_{p(\cdot),b} = 1$. Then we obtain

$$\|u_n\|_{\partial} \rightarrow 0 \quad (n \rightarrow \infty)$$

due to the boundedness of $\mathbb{N}$. Thus, we see that

$$\|\nabla u_n\|_{p(\cdot),a} \rightarrow 0$$

and

$$\|u_n\|_{p(\cdot),a,\partial\Omega} \rightarrow 0$$

as $n \rightarrow \infty$. This follows that the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $W_{a,b}^{1,p(\cdot)}(\Omega)$. Since the space $W_{a,b}^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space, there exists a subsequence $(u_n)_{n \in \mathbb{N}}$, such that

$$u_n \rightharpoonup v \text{ (weakly)}$$

as $n \rightarrow \infty$ in $W_{a,b}^{1,p(\cdot)}(\Omega)$. If we use the compact embedding in Corollary 3, then we get

$$u_n \rightarrow v \text{ (strongly)} \quad (5)$$

as $n \rightarrow \infty$ in $L_b^{p(\cdot)}(\Omega)$. Moreover, since strong convergence requires weak convergence, we have $u_n \rightharpoonup v$ as $n \rightarrow \infty$ in $L_b^{p(\cdot)}(\Omega)$. Therefore, we can get that

$$\nabla u_n \rightharpoonup \nabla v$$

as $n \rightarrow \infty$ in $\left(L_a^{p(\cdot)}(\Omega)\right)^N$. By Theorem 5.12-2 in [7], we see that

$$\liminf_{n \rightarrow \infty} \|\nabla u_n\|_{p(\cdot),a} \geq \|\nabla v\|_{p(\cdot),a},$$

i.e. $\|\nabla v\|_{p(\cdot),a} = 0$ and $v \equiv c$ (constant). Again using the compact embedding $W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow L_a^{p(\cdot)}(\partial\Omega)$, the convergence $u_n \rightarrow v$ (strongly), and so $u_n \rightharpoonup v$ (weakly) as $n \rightarrow \infty$ in $L_a^{p(\cdot)}(\partial\Omega)$ and Theorem 5.12-2 in [7], we have

$$\liminf_{n \rightarrow \infty} \|u_n\|_{p(\cdot),a,\partial\Omega} \geq \|v\|_{p(\cdot),a,\partial\Omega},$$

that is, $\|v\|_{p(\cdot),a,\partial\Omega} \equiv 0$ and $v \equiv 0$. Due to $\|u_n\|_{p(\cdot),b} = 1$ and (5), this is a contradiction. This completes the proof.

Let $\beta \in L^\infty(\partial\Omega)$ such that $\beta^- = \inf_{x \in \partial\Omega} \beta(x) > 0$. Then, the norm $\|u\|_{\beta(x)}$ is defined by

$$\|u\|_{\beta(x)} = \inf \left\{ \tau > 0 : \int_{\Omega} a(x) \left| \frac{\nabla u(x)}{\tau} \right|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \left| \frac{u(x)}{\tau} \right|^{p(x)} d\sigma \leq 1 \right\}$$

for any $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$. Moreover, $\|\cdot\|_{\beta(x)}$ and $\|\cdot\|_{a,b}^{1,p(\cdot)}$ are equivalent on $W_{a,b}^{1,p(\cdot)}(\Omega)$ by Theorem 4.

Proposition 2 (See [16,24,25]) *Let $I_{\beta(x)}(u) = \int_{\Omega} a(x) |\nabla u(x)|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)} d\sigma$ with $\beta^- > 0$. For any $u, u_k \in W_{a,b}^{1,p(\cdot)}(\Omega)$ ($k = 1, 2, \dots$), we have*

- (i) $\|u\|_{\beta(x)}^{p^-} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^+}$ with $\|u\|_{\beta(x)} \geq 1$,
- (ii) $\|u\|_{\beta(x)}^{p^+} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^-}$ with $\|u\|_{\beta(x)} \leq 1$,
- (iii) $\min \left\{ \|u\|_{\beta(x)}^{p^-}, \|u\|_{\beta(x)}^{p^+} \right\} \leq I_{\beta(x)}(u) \leq \max \left\{ \|u\|_{\beta(x)}^{p^-}, \|u\|_{\beta(x)}^{p^+} \right\}$,
- (iv) $\|u - u_k\|_{\beta(x)} \rightarrow 0$ if and only if $I_{\beta(x)}(u - u_k) \rightarrow 0$ as $k \rightarrow \infty$,
- (v) $\|u_k\|_{\beta(x)} \rightarrow \infty$ if and only if $I_{\beta(x)}(u_k) \rightarrow \infty$ as $k \rightarrow \infty$.

The following Proposition can be proved similarly using the methods in [6,14,16, 25].

Proposition 3 *Let us define the functional $L_{\beta(x)} : W_{a,b}^{1,p(\cdot)}(\Omega) \rightarrow \mathbb{R}$ by*

$$L_{\beta(x)}(u) = \int_{\Omega} \frac{a(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} d\sigma$$

for all $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$. Then we obtain $L_{\beta(x)} \in C^1\left(W_{a,b}^{1,p(\cdot)}(\Omega), \mathbb{R}\right)$ and

$$\begin{aligned} L'_{\beta(x)}(u)(v) = \langle L'_{\beta(x)}(u), v \rangle &= \int_{\Omega} a(x) |\nabla u|^{p(x)-2} \nabla u \nabla v dx \\ &+ \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)-2} u v d\sigma \end{aligned}$$

for any $u, v \in W_{a,b}^{1,p(\cdot)}(\Omega)$. In addition, we have the following properties;

- (i) $L'_{\beta(x)} : W_{a,b}^{1,p(\cdot)}(\Omega) \longrightarrow W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)$ is continuous, bounded and strictly monotone operator,
- (ii) $L'_{\beta(x)} : W_{a,b}^{1,p(\cdot)}(\Omega) \longrightarrow W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)$ is a mapping of type (S_+) , i.e., if $u_n \rightharpoonup u$ in $W_{a,b}^{1,p(\cdot)}(\Omega)$ and $\limsup_{n \rightarrow \infty} L'_{\beta(x)}(u_n)(u_n - u) \leq 0$, then $u_n \longrightarrow u$ in $W_{a,b}^{1,p(\cdot)}(\Omega)$,
- (iii) $L'_{\beta(x)} : W_{a,b}^{1,p(\cdot)}(\Omega) \longrightarrow W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)$ is a homeomorphism.

Definition 2 If the function $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ satisfies the condition

$$\begin{aligned} \int_{\Omega} a(x) |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)-2} u v d\sigma \\ - \lambda \int_{\Omega} b(x) |u|^{q(x)-2} u v dx = 0 \end{aligned}$$

for all $v \in W_{a,b}^{1,p(\cdot)}(\Omega)$, then u is called a weak solution of the problem (1). We point out that if $\lambda \in \mathbb{R}$ is an eigenvalue of the problem (1), then the corresponding $u \in W_{a,b}^{1,p(\cdot)}(\Omega) - \{0\}$ is a weak solution of (1).

Zhikov and Surnachev [35] prove that a sufficient condition for the density of smooth functions in the weighted Sobolev space with variable exponent is obtained. Moreover, it is known that the space $C_0^\infty(\Omega)$ is a subspace of $W_{a,b}^{1,p(\cdot)}(\Omega)$, and $W_{a,b}^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega)$ for $0 < a_1 \leq a(x) \leq a_2$, $0 < b_1 \leq b(x) \leq b_2$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Thus we obtain that the spaces $C_0^\infty(\Omega)$ and $C^\infty(\Omega) \cap W_{a,b}^{1,p(\cdot)}(\Omega)$ is dense in $W_{a,b}^{1,p(\cdot)}(\Omega)$ since log-H ölder continuity of the exponent is sufficient for the density of smooth functions, see [11, Theorem 9.1.8], [12,30,31].

The class $A_{p(\cdot)}$ consists of those weights w such that

$$\|w\|_{A_{p(\cdot)}} = \sup_{B \in \mathcal{B}} |B|^{-p_B} \|w\|_{L^1(B)} \left\| \frac{1}{w} \right\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(B)} < \infty$$

where $\mathcal{B}$ denotes the set of all balls in Ω , $p_B = \left(\frac{1}{|\mathcal{B}|} \int_B \frac{1}{p(x)} dx \right)^{-1}$ and $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

If $a \in A_{p(\cdot)}$, $0 < b_1 \leq b(x) \leq b_2$ and $p(\cdot) \in P^{\log}(\mathbb{R}^n)$, i.e. $p(\cdot)$ satisfy log-Hölder continuity condition, then $C_0^\infty(\Omega)$ is dense in $W_{a,b}^{1,p(\cdot)}(\Omega)$, see [2, 21]. Therefore, the weak solution of the problem (1) is well defined. Throughout the paper, we assume that $a \in A_{p(\cdot)}$, $0 < b_1 \leq b(x) \leq b_2$ and $p(\cdot) \in P^{\log}(\Omega)$.

To find out a weak solution to (1), let us introduce the functional $J_\lambda : W_{a,b}^{1,p(\cdot)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_\lambda(u) = \int_\Omega \frac{a(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} d\sigma - \lambda \int_\Omega \frac{b(x)}{q(x)} |u|^{q(x)} dx$$

for any $\lambda > 0$. It is easy to see that J_λ is well defined from $W_{a,b}^{1,p(\cdot)}(\Omega)$ to $\mathbb{R}$ and $J_\lambda \in C^1(W_{a,b}^{1,p(\cdot)}(\Omega), \mathbb{R})$ with the derivative given by

$$\begin{aligned} J'_\lambda(u)(v) &= \int_\Omega a(x) |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)-2} u v d\sigma \\ &\quad - \lambda \int_\Omega b(x) |u|^{q(x)-2} u v dx \end{aligned}$$

for all $u, v \in W_{a,b}^{1,p(\cdot)}(\Omega)$. The operator $J'_\lambda(u)$ belongs to the dual space $W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)$. We also have $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ is a weak solution of the problem (1) if and only if u is a critical point of J_λ .

3 Main results

In this section, we suppose that $p(x) < p_{*,r(x)}^\partial(x) < p_*^\partial(x)$ for all $x \in \partial\Omega$, and $1 \leq q(x) < \frac{r(x)-1}{r(x)} (p_*)^\gamma(x)$ for all $x \in \overline{\Omega}$.

Lemma 1 *If there is a constant $\eta > 0$ such that $p^+ < \eta < q^-$, then the operator J_λ satisfies the (PS) condition.*

Proof Let $(u_n)_{n \in \mathbb{N}} \subset W_{a,b}^{1,p(\cdot)}(\Omega)$ be such that

$$\sup |J_\lambda(u_n)| \leq M, \quad J'_\lambda(u_n) \rightarrow 0 \text{ in } W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega) \text{ as } n \rightarrow \infty, \quad (6)$$

i.e., $(u_n)_{n \in \mathbb{N}}$ is a (PS) sequence. First, it will be shown that the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $W_{a,b}^{1,p(\cdot)}(\Omega)$. If the sequence $(u_n)_{n \in \mathbb{N}}$ is not bounded, then we may write $\|u_n\|_{\beta(x)} > 1$ for all $n \in \mathbb{N}$. It follows that we have

$$\begin{aligned} M + \|u_n\|_{\beta(x)} &\geq J_\lambda(u_n) - \frac{1}{\eta} \langle J'_\lambda(u_n), u_n \rangle \\ &= \int_\Omega \frac{a(x)}{p(x)} |\nabla u_n|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n(x)|^{p(x)} d\sigma - \lambda \int_\Omega \frac{b(x)}{q(x)} |u_n|^{q(x)} dx \\ &\quad - \frac{1}{\eta} \int_\Omega a(x) |\nabla u_n|^{p(x)} dx - \frac{1}{\eta} \int_{\partial\Omega} \beta(x) |u_n(x)|^{p(x)} d\sigma + \frac{\lambda}{\eta} \int_\Omega b(x) |u_n|^{q(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\eta} \right) I_{\beta(x)}(u_n) + \lambda \left(\frac{1}{\eta} - \frac{1}{q^-} \right) \int_\Omega b(x) |u_n|^{q(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\eta} \right) \|u_n\|_{\beta(x)}^{p^-}. \end{aligned}$$

Dividing the last inequality by $\|u_n\|_{\beta(x)}^{p^-}$, passing to the limit as $n \rightarrow \infty$ we get $\eta \leq p^+$, which is a contradiction. Thus, (u_n) is bounded in $W_{a,b}^{1,p(\cdot)}(\Omega)$. Since $W_{a,b}^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space, there exists $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_{a,b}^{1,p(\cdot)}(\Omega)$. By Corollary 3, the embedding $W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow L_b^{q(\cdot)}(\Omega)$ is compact. Thus, we have $u_n \rightarrow u$ in $L_b^{q(\cdot)}(\Omega)$ and $\sup_n \|u_n\|_{\beta(x)} < \infty$. Moreover, using the Hölder inequality, we obtain

$$\begin{aligned} \left| \int_\Omega b(x) |u_n|^{q(x)-2} u_n (u_n - u) dx \right| &\leq \int_\Omega b(x)^{\frac{1}{q(x)} + \frac{1}{r(x)}} |u_n|^{q(x)-1} |u_n - u| dx \\ &\leq C \left\| |u_n|^{q(x)-1} b(x)^{\frac{1}{r(x)}} \right\|_{r(\cdot)} \cdot \left\| |u_n - u| b(x)^{\frac{1}{q(x)}} \right\|_{q(\cdot)} \\ &= C \|u_n\|_{q(\cdot),b} \cdot \|u_n - u\|_{q(\cdot),b} \\ &\leq C^* \|u_n\|_{\beta(x)} \cdot \|u_n - u\|_{q(\cdot),b} \rightarrow 0 \end{aligned}$$

where $\frac{1}{q(\cdot)} + \frac{1}{r(\cdot)} = 1$. So

$$\lim_{n \rightarrow \infty} \int_\Omega b(x) |u_n|^{q(x)-2} u_n (u_n - u) dx = 0. \quad (7)$$

By (6) and the boundedness of $\{u_n - u\}$ in $W_{a,b}^{1,p(\cdot)}(\Omega)$, we have

$$J'_\lambda(u_n)(u_n - u) \rightarrow 0. \quad (8)$$

If we combine (7) and (8), we prove that

$$\lim_{n \rightarrow \infty} \left(\int_{\Omega} a(x) |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx + \int_{\partial\Omega} \beta(x) |u_n(x)|^{p(x)-2} u_n (u_n - u) d\sigma \right) = 0$$

or

$$\lim_{n \rightarrow \infty} L'_{\beta(x)}(u_n) (u_n - u) = 0. \quad (9)$$

By (9) and Proposition 3, it is obtained that the sequence $\{u_n\}$ converges strongly to u in $W_{a,b}^{1,p(\cdot)}(\Omega)$. The proof is completed. $\square$

Theorem 5 *Let $p^+ < \eta < q^-$ for some $\eta > 0$. Then, there is a nontrivial weak solution of the problem (1) for $p^+ < \eta < q^-$.*

Proof By Lemma 1, the functional J_λ satisfies (PS) condition on $W_{a,b}^{1,p(\cdot)}(\Omega)$. Using the assumption $p^+ < \eta < q^-$ and the embedding $W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow L_b^{q(\cdot)}(\Omega)$ we have

$$\begin{aligned} J_\lambda(u) &= \int_{\Omega} \frac{a(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} d\sigma - \lambda \int_{\Omega} \frac{b(x)}{q(x)} |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \left(\int_{\Omega} a(x) |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)} d\sigma \right) - \frac{\lambda}{q^-} \int_{\Omega} b(x) |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} I_{\beta(x)}(u) - \frac{\lambda}{q^-} \max \left\{ \|u\|_{q(\cdot),b}^{q^-}, \|u\|_{q(\cdot),b}^{q^+} \right\} \\ &\geq \frac{1}{p^+} I_{\beta(x)}(u) - \frac{\lambda}{q^-} \max \left\{ \|u\|_{\beta(x)}^{q^-}, \|u\|_{\beta(x)}^{q^+} \right\} \\ &\geq \frac{1}{p^+} \|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^-} \|u\|_{\beta(x)}^{q^-} \\ &\geq \left(\frac{1}{p^+} - \frac{\lambda}{q^-} \right) \|u\|_{\beta(x)}^{p^+} \end{aligned}$$

for $\|u\|_{\beta(x)} \leq 1$ and $p^+ \lambda < q^-$. Thus, we get $J_\lambda(u) > 0$ for $\|u\|_{\beta(x)} = \rho$ sufficiently small. In addition, we have

$$\begin{aligned} J_\lambda(tv) &= \int_{\Omega} \frac{a(x)t^{p(x)}}{p(x)} |\nabla v|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)t^{p(x)}}{p(x)} |v|^{p(x)} d\sigma - \lambda \int_{\Omega} \frac{b(x)t^{q(x)}}{q(x)} |v|^{q(x)} dx \\ &\leq t^{p^+} I_{\beta(x)}(v) - \lambda t^{q^-} \int_{\Omega} b(x) |v|^{q(x)} dx \\ &\leq t^{p^+} I_{\beta(x)}(v) - \lambda t^{\eta^-} \int_{\Omega} b(x) |v|^{q(x)} dx \longrightarrow -\infty \end{aligned}$$

as $t \rightarrow \infty$ for $v \in W_{a,b}^{1,p(\cdot)}(\Omega) - \{0\}$. Note also that $J_\lambda(0) = 0$. So, it is showed that there is a nontrivial weak solution of the problem (1) by the Mountain Pass Theorem (see [32]).

To apply Ekeland's variational principle to the problem (1), we need the following lemmas.

Lemma 2 *Let $1 < q^- < p^- < q^+ < p^+$. Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho, \gamma > 0$ such that $J_\lambda(u) \geq \gamma > 0$ for any $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ with $\|u\|_{\beta(x)} = \rho$.*

Proof Due to the continuous embedding $W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow L_b^{q(\cdot)}(\Omega)$, there exists a positive constant $C_2 > 0$ such that

$$\|u\|_{q(\cdot),b} \leq C_2 \|u\|_{\beta(x)} \quad (10)$$

for any $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$. We take $\rho \in (0, 1)$ such that $\rho < \frac{1}{C_2}$. Then the relation (10) implies $\|u\|_{q(\cdot),b} < 1$ and

$$\int_{\Omega} |u|^{q(x)} b(x) dx \leq \|u\|_{q(\cdot),b}^{q^-} \quad (11)$$

for all $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ with $\|u\|_{\beta(x)} = \rho$. Combining (10) and (11), we obtain

$$\int_{\Omega} |u|^{q(x)} b(x) dx \leq C_2^{q^-} \|u\|_{\beta(x)}^{q^-} \quad (12)$$

for all $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ with $\|u\|_{\beta(x)} = \rho$. Due to the inequality (12), we deduce that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+} I_{\beta(x)}(u) - \frac{\lambda}{q^-} \int_{\Omega} b(x) |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^-} C_2^{q^-} \|u\|_{\beta(x)}^{q^-} \\ &= \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{q^-} C_2^{q^-} \rho^{q^-} \\ &= \rho^{q^-} \left(\frac{1}{p^+} \rho^{p^+ - q^-} - \frac{\lambda}{q^-} C_2^{q^-} \right). \end{aligned}$$

Taking

$$\lambda^* = \frac{\rho^{p^+ - q^-}}{2p^+} \cdot \frac{q^-}{C_2^{q^-}}, \quad (13)$$

then there exists $\gamma = \frac{\rho^{p^+}}{2p^+} > 0$ such that $J_\lambda(u) \geq \gamma > 0$ for all $\lambda \in (0, \lambda^*)$ and $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$ with $\|u\|_{\beta(x)} = \rho$.

Lemma 3 *There is a function ϕ in $W_{a,b}^{1,p(\cdot)}(\Omega)$ satisfying $\phi \geq 0$, $\phi \neq 0$ and $J_\lambda(t\phi) < 0$ when t is small enough for $1 < q^- < p^- < q^+ < p^+$.*

Proof Using the assumption $q^- < p^-$ we have $q^- + \varepsilon_0 < p^-$ for a $\varepsilon_0 > 0$. Since $q \in C(\overline{\Omega})$, we can find an open subset $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \varepsilon_0$ for any $x \in \Omega_0$. So, we infer that

$$q(x) \leq q^- + \varepsilon_0 < p^- \quad (14)$$

for any $x \in \Omega_0$. Moreover, there is a function $\phi \in C_0^\infty(\Omega)$ satisfying $\overline{\Omega_0} \subset \text{supp } \phi$, $\phi(x) = 1$ for all $x \in \overline{\Omega_0}$ and $0 \leq \phi \leq 1$ in Ω . Using (14) we have

$$\begin{aligned} J_\lambda(t\phi) &= \int_\Omega \frac{a(x)t^{p(x)}}{p(x)} |\nabla \phi|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)t^{p(x)}}{p(x)} |\phi|^{p(x)} d\sigma \\ &\quad - \lambda \int_\Omega \frac{b(x)t^{q(x)}}{q(x)} |\phi|^{q(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} I_{\beta(x)}(u) - \frac{\lambda}{q^+} \int_\Omega t^{q(x)} |\phi|^{q(x)} b(x) dx \\ &\leq \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+} \int_{\Omega_0} t^{q(x)} |\phi|^{q(x)} b(x) dx \\ &\leq \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^- + \varepsilon_0}}{q^+} \int_{\Omega_0} |\phi|^{q(x)} b(x) dx \end{aligned}$$

for any $t \in (0, 1)$ and $\|\phi\|_{\beta(x)} = I_{\beta(x)}(u) = 1$. Then, for any $t < \frac{1}{\delta^{p^- - q^- - \varepsilon_0}}$ with $0 < \delta < \min \left\{ 1, \frac{\lambda p^-}{q^+} \int_{\Omega_0} |\phi|^{q(x)} b(x) dx \right\}$, we obtain $J_\lambda(t\phi) < 0$. $\square$

Theorem 6 *Let $1 < q^- < p^- < q^+ < p^+$. Then, there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for the problem (1).*

Proof Let $\lambda \in (0, \lambda^*)$, where λ^* is given by (13). By Lemma 2 it follows that on the boundary of the ball centered at the origin and of radius ρ in $W_{a,b}^{1,p(\cdot)}(\Omega)$, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_\lambda > 0.$$

On the other hand, by Lemma 3, there exists $\phi \in W_{a,b}^{1,p(\cdot)}(\Omega)$ such that $J_\lambda(t\phi) < 0$ for $t > 0$ small enough. Moreover, we get

$$J_\lambda(u) \geq \frac{1}{p^+} \|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^-} C_2^{q^-} \|u\|_{\beta(x)}^{q^-}$$

for any $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$. This follows that

$$-\infty < \underline{c} = \inf_{B_\rho(0)} J_\lambda < 0.$$

Let us choose $\varepsilon > 0$. Therefore, we have

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda.$$

If we apply the Ekeland's variational principle to the functional $J_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$, then we find $u_\varepsilon \in B_\rho(0)$ such that

$$\begin{aligned} J_\lambda(u_\varepsilon) &< \inf_{B_\rho(0)} J_\lambda + \varepsilon, \\ J_\lambda(u_\varepsilon) &< J_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{\beta(x)}, \quad u \neq u_\varepsilon. \end{aligned}$$

Since

$$J_\lambda(u_\varepsilon) \leq \inf_{B_\rho(0)} J_\lambda + \varepsilon \leq \inf_{B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we can deduce that $u_\varepsilon \in B_\rho(0)$. Set a function $\psi : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ such that $\psi(u) = J_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{\beta(x)}$. It is easy to see that u_ε is a minimum point of ψ . Thus

$$\frac{\psi(u_\varepsilon + tv) - \psi(u_\varepsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_\rho(0)$. From the above relation we obtain

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\|_{\beta(x)} \geq 0.$$

By letting $t \rightarrow 0$, we get

$$\langle J'_\lambda(u_\varepsilon), v \rangle + \varepsilon \|v\|_{\beta(x)} \geq 0. \quad (15)$$

If we substitute $-v$ in (15) for v , we acquire $|\langle J'_\lambda(u_\varepsilon), v \rangle| \leq \varepsilon \|v\|_{\beta(x)}$. It is known that $J'_\lambda(u_\varepsilon) \in W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)$. If we use the definition on operator norm for $W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)$, we have $\|J'_\lambda(u_\varepsilon)\|_{W_{a^*,b^*}^{-1,p'(\cdot)}(\Omega)} \leq \varepsilon$. Thus we conclude that there is a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \rightarrow \underline{c} \text{ and } J'_\lambda(w_n) \rightarrow 0. \quad (16)$$

Since $\{w_n\} \subset B_\rho(0)$ and $\|w_n\|_{\beta(x)} < \rho$, $\{w_n\}$ is bounded in $W_{a,b}^{1,p(\cdot)}(\Omega)$. This follows that $\{w_n\}$ converges strongly to w in $W_{a,b}^{1,p(\cdot)}(\Omega)$ by Lemma 1. So, by (16),

$$J_\lambda(w) = \underline{c} < 0 \text{ and } J'_\lambda(w) = 0.$$

We conclude that w is a non-trivial weak solution for the problem (1) and thus any $\lambda \in (0, \lambda^*)$ is an eigenvalue of the problem (1). $\square$

For the multiplicity solutions of the problem (1), we will use some well-known results.

Lemma 4 (See [14]) *Let X be a reflexive and separable Banach space, then there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that*

$$X = \overline{\text{span}\{e_j \mid j = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{e_j^* \mid j = 1, 2, \dots\}}$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between X and X^* .

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^k X_j$, $Z_k = \bigoplus_{j=k}^\infty X_j$.

Lemma 5 *If we define the set*

$$\alpha_k = \sup \left\{ \|u\|_{q(\cdot),b} : \|u\|_{\beta(x)} = 1, u \in Z_k \right\},$$

then $\lim_{k \rightarrow \infty} \alpha_k = 0$.

Proof Using the continuous embedding $W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow L_b^{q(\cdot)}(\Omega)$ and the method in (Lemma 4.9, [14]), then we get $\lim_{k \rightarrow \infty} \alpha_k = 0$. $\square$

Using Lemma 5 we obtain the following theorem.

Theorem 7 *Let $p^+ < q^-$. Then there are infinitely many pairs of solutions of the problem (1), i.e., the functional J_λ has a sequence of critical points $\{u_n\}$ such that $J_\lambda(u_n) \rightarrow +\infty$.*

Proof It is well known that J_λ is an even functional and fulfills the (PS) condition. We will prove that if k is large enough, then there exist $\eta_k > \gamma_k > 0$ such that

$$\begin{aligned} (A_1) \quad b_k &:= \inf \left\{ J_\lambda(u) \mid u \in Z_k, \|u\|_{\beta(x)} = \gamma_k \right\} \rightarrow \infty \text{ as } k \rightarrow \infty \\ (A_2) \quad a_k &:= \max \left\{ J_\lambda(u) \mid u \in Y_k, \|u\|_{\beta(x)} = \eta_k \right\} \leq 0. \end{aligned}$$

(A₁) For any $u \in Z_k$ such that $\|u\|_{\beta(x)} = \gamma_k > 1$, we obtain

$$\begin{aligned}
 J_\lambda(u) &\geq \frac{1}{p^+} I_{\beta(x)}(u) - \frac{\lambda}{q^-} \int_{\Omega} b(x) |u|^{q(x)} dx \\
 &\geq \frac{1}{p^+} I_{\beta(x)}(u) - \frac{\lambda}{q^-} \max \left\{ \|u\|_{q(\cdot),b}^{q^-}, \|u\|_{q(\cdot),b}^{q^+} \right\} \\
 &\geq \frac{1}{p^+} \|u\|_{\beta(x)}^{p^-} - \frac{\lambda}{q^-} \max \left\{ \|u\|_{q(\cdot),b}^{q^-}, \|u\|_{q(\cdot),b}^{q^+} \right\} \\
 &\geq \begin{cases} \frac{1}{p^+} \|u\|_{\beta(x)}^{p^-} - \lambda, & \|u\|_{q(\cdot),b} \leq 1 \\ \frac{1}{p^+} \|u\|_{\beta(x)}^{p^-} - \lambda \alpha_k^{q^+} \|u\|_{\beta(x)}^{q^+}, & \|u\|_{q(\cdot),b} > 1 \end{cases} \\
 &\geq \frac{1}{p^+} \|u\|_{\beta(x)}^{p^-} - \lambda \alpha_k^{q^+} \|u\|_{\beta(x)}^{q^+} \\
 &= \frac{1}{p^+} \gamma_k^{p^-} - \lambda \alpha_k^{q^+} \gamma_k^{q^+}
 \end{aligned}$$

If we choose $\gamma_k = \left(\lambda q^+ \alpha_k^{q^+} \right)^{\frac{1}{p^- - q^+}}$, then we get

$$\begin{aligned}
 J_\lambda(u) &\geq \frac{1}{p^+} \left(\lambda q^+ \alpha_k^{q^+} \right)^{\frac{p^-}{p^- - q^+}} - \lambda \alpha_k^{q^+} \left(\lambda q^+ \alpha_k^{q^+} \right)^{\frac{q^+}{p^- - q^+}} \\
 &= \left(\frac{1}{p^+} - \frac{1}{q^+} \right) \left(\lambda q^+ \alpha_k^{q^+} \right)^{\frac{p^-}{p^- - q^+}} \longrightarrow \infty
 \end{aligned}$$

as $k \longrightarrow \infty$ because $p^+ < q^+$ and $\alpha_k \longrightarrow 0$. So (A₁) holds.

(A₂) Let $u \in Y_k$ be such that $\|u\|_{\beta(x)} = \eta_k > \gamma_k > 1$. Then, we obtain

$$\begin{aligned}
 J_\lambda(u) &\leq \frac{1}{p^-} \|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^+} \int_{\Omega} |u|^{q(x)} b(x) dx \\
 &\leq \frac{1}{p^-} \|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^+} \min \left\{ \|u\|_{q(\cdot),b}^{q^-}, \|u\|_{q(\cdot),b}^{q^+} \right\}.
 \end{aligned}$$

Since the space Y_k has finite dimension, the norms $\|\cdot\|_{\beta(x)}$ and $\|\cdot\|_{q(\cdot),b}$ are equivalent. Finally,

$$J_\lambda(u) \rightarrow -\infty \text{ as } \|u\|_{\beta(x)} \rightarrow +\infty, u \in Y_k$$

due to $p^+ < q^-$ by the Fountain Theorem (see [32, Theorem 3.6]). $\square$

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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