



Some properties of thinness and fine topology with relative capacity

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Abstract

In this paper, we introduce a thinness in sense to a type of relative capacity for weighted variable exponent Sobolev space. Moreover, we reveal some properties of this thinness and consider the relationship with finely open and finely closed sets. We discuss fine topology and compare this topology with Euclidean one. Finally, we give some information about importance of the fine topology in the potential theory.

Keywords Fine topology · Thinness · Relative capacity · Weighted variable exponent Sobolev spaces

Mathematics Subject Classification 31C40 · 46E35 · 32U20 · 43A15

1 Introduction

The history of potential theory begins in 17th century. Its development can be traced to such greats as Newton, Euler, Laplace, Lagrange, Fourier, Green, Gauss, Poisson, Dirichlet, Riemann, Weierstrass, Poincaré. We refer to the book by Kellogg [1] for references to some of the old works.

The Sobolev spaces $W^{k,p}(\Omega)$ are usually defined for open sets Ω . This makes sometimes difficulties to classical method for nonopen sets. The authors in [2] and [3] present different approach is to investigate Sobolev spaces on finely open sets. This is just a part of fine potential theory in $\mathbb{R}^d$.

Kováčik and Rákosník [4] introduced the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^d)$ and the Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^d)$. They present some basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^d)$ and the Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^d)$ such as reflexivity and Hölder inequalities were obtained. For a historical journey, we refer [4–7] and [8].

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The variational capacity has been used extensively in nonlinear potential theory on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ is open and $K \subset \Omega$ is compact. Then the relative variational p -capacity is defined by

$$\text{cap}_p(K, \Omega) = \inf_f \int_{\Omega} |\nabla f(x)|^p dx,$$

where the infimum is taken over smooth and zero boundary valued functions f in Ω such that $f \geq 1$ in K . The set of admissible functions f can be replaced by the continuous first order Sobolev functions with $f \geq 1$ in K . The p -capacity is a Choquet capacity relative to Ω . For more details and historical background, see [9]. Also, Harjulehto et al. [10] defined a relative capacity with variable exponent. They studied properties of the capacity and compare it with the Sobolev capacity. In [11], the authors expanded this relative capacity to weighted variable exponent. Moreover, they investigate properties of this capacity and give some relationship between defined capacity in [10] and Sobolev capacity. Besides to these studies, the Riesz capacity which is an another representative for capacity theory has been considered by [12].

In [13] and [14], the authors have explored some properties of the $p(\cdot)$ -Dirichlet energy integral

$$\int_{\Omega} |\nabla f(x)|^{p(x)} dx$$

over a bounded domain $\Omega \subset \mathbb{R}^d$. They have discussed the existence and regularity of energy integral minimizers. As an alternative method the minimizers in one dimensional case have been studied by the authors in [15]. Moreover, Harjulehto et al. [16] considered the Dirichlet energy integral, with boundary values given in the Sobolev sense, has a minimizer provided the variable exponent satisfies a certain jump condition.

The fine topology was introduced by Cartan [17] in 1946. Classical fine topology has found many applications such as its connections to the theory of analytic functions and probability. For classical treatment we can refer [18–21] and [22]. Also, Meyers [23] first generalized the fine topology to nonlinear theories. For the historical background and an excellent scientific survey we refer [9] and references therein.

In this study, we present $(p(\cdot), \vartheta)$ -thin sets in sense to $(p(\cdot), \vartheta)$ -relative capacity and consider the basic and advanced properties. We discuss some results about $(p(\cdot), \vartheta)$ -relative capacity in $(p(\cdot), \vartheta)$ -thin sets. Moreover, we generalize several properties of fine topology and find new results by Wiener type integral.

2 Notation and preliminaries

In this paper, we will work on $\mathbb{R}^d$ with Lebesgue measure dx . The measure μ is doubling if there is a fixed constant $c_d \geq 1$, called the doubling constant of μ such that

$$\mu(B(x_0, 2r)) \leq c_d \mu(B(x_0, r))$$

for every ball $B(x_0, r)$ in $\mathbb{R}^d$. Also, the elements of the space $C_0^\infty(\mathbb{R}^d)$ are the infinitely differentiable functions with compact support. We denote the family of all measurable

functions $p(\cdot) : \mathbb{R}^d \rightarrow [1, \infty)$ (called the variable exponent on $\mathbb{R}^d$) by the symbol $\mathcal{P}(\mathbb{R}^d)$. In this paper, the function $p(\cdot)$ always denotes a variable exponent. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, put

$$p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^d} p(x) \quad p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} p(x).$$

A measurable and locally integrable function $\vartheta : \mathbb{R}^d \rightarrow (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\varrho_{p(\cdot), \vartheta}(f) = \int_{\mathbb{R}^d} |f(x)|^{p(x)} \vartheta(x) dx.$$

The weighted variable exponent Lebesgue spaces $L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ consist of all measurable functions f on $\mathbb{R}^d$ endowed with the Luxemburg norm

$$\|f\|_{p(\cdot), \vartheta} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \vartheta(x) dx \leq 1 \right\}.$$

When $\vartheta(x) = 1$, the space $L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ is the variable exponent Lebesgue space. The space $L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ is a Banach space with respect to $\|\cdot\|_{p(\cdot), \vartheta}$. Also, some basic properties of this space were investigated in [24–26]. We set the weighted variable exponent Sobolev spaces $W^{k, p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ by

$$W^{k, p(\cdot)}_{\vartheta}(\mathbb{R}^d) = \left\{ f \in L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d) : D^{\alpha} f \in L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d), 0 \leq |\alpha| \leq k \right\}$$

with the norm

$$\|f\|_{k, p(\cdot), \vartheta} = \sum_{0 \leq |\alpha| \leq k} \|D^{\alpha} f\|_{p(\cdot), \vartheta}$$

where $\alpha \in \mathbb{N}_0^d$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and $D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}$. It is already known that $W^{k, p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ is a Banach space.

Now, let $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, $k \in \mathbb{N}$ and $\vartheta^{-\frac{1}{p(\cdot)-1}} \in L^1_{loc}(\mathbb{R}^d)$. Thus, the embedding $L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d) \hookrightarrow L^1_{loc}(\mathbb{R}^d)$ holds and then the weighted variable exponent Sobolev spaces $W^{k, p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ is well-defined by [[25], Proposition 2.1].

In particular, the space $W^{1, p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ is defined by

$$W^{1, p(\cdot)}_{\vartheta}(\mathbb{R}^d) = \left\{ f \in L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d) : |\nabla f| \in L^{p(\cdot)}_{\vartheta}(\mathbb{R}^d) \right\}.$$

The function $\rho_{1, p(\cdot), \vartheta} : W^{1, p(\cdot)}_{\vartheta}(\mathbb{R}^d) \rightarrow [0, \infty)$ is shown as $\rho_{1, p(\cdot), \vartheta}(f) = \rho_{p(\cdot), \vartheta}(f) + \rho_{p(\cdot), \vartheta}(|\nabla f|)$. Also, the norm $\|f\|_{1, p(\cdot), \vartheta} = \|f\|_{p(\cdot), \vartheta} + \|\nabla f\|_{p(\cdot), \vartheta}$ makes the space $W^{1, p(\cdot)}_{\vartheta}(\mathbb{R}^d)$ a Banach space. The local weighted variable exponent Sobolev space $W^{1, p(\cdot)}_{\vartheta, loc}(\mathbb{R}^d)$ is defined in the classical way. More information on the classic theory of variable exponent spaces can be found in [4, 27].

Let $\Omega \subset \mathbb{R}^d$ is bounded and ϑ is a weight function. It is known that a function $f \in C^\infty_0(\Omega)$ satisfies Poincaré inequality in $L^1_{\vartheta}(\Omega)$ if and only if the inequality

$$\int_{\Omega} |f(x)|\vartheta(x)dx \leq c(\text{diam } \Omega) \int_{\Omega} |\nabla f(x)|\vartheta(x)dx$$

where the constant c satisfies the condition III in [9, page 7].

Unal and Aydin [11] defined an alternative capacity -called relative $(p(\cdot), \vartheta)$ -capacity-for Sobolev capacity in sense to [10]. For this, they recall that

$$C_0(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} : f \text{ is continuous and } \text{supp} f \subset \Omega \text{ is compact}\},$$

where $\text{supp} f$ is the support of f . Suppose that K is a compact subset of Ω . Also, they denote

$$R_{p(\cdot), \vartheta}(K, \Omega) = \left\{ f \in W_{\vartheta}^{1,p(\cdot)}(\Omega) \cap C_0(\Omega) : f > 1 \text{ on } K \text{ and } f \geq 0 \right\}$$

and define

$$\text{cap}_{p(\cdot), \vartheta}^*(K, \Omega) = \inf_{f \in R_{p(\cdot), \vartheta}(K, \Omega)} \int_{\Omega} |\nabla f(x)|^{p(x)} \vartheta(x) dx.$$

In addition, if $U \subset \Omega$ is open, then

$$\text{cap}_{p(\cdot), \vartheta}(U, \Omega) = \sup_{\substack{K \subset U \\ \text{compact}}} \text{cap}_{p(\cdot), \vartheta}^*(K, \Omega),$$

and also for an arbitrary set $E \subset \Omega$ we define

$$\text{cap}_{p(\cdot), \vartheta}(E, \Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot), \vartheta}(U, \Omega).$$

They call $\text{cap}_{p(\cdot), \vartheta}(E, \Omega)$ the variational $(p(\cdot), \vartheta)$ -capacity of E relative to Ω , briefly the relative $(p(\cdot), \vartheta)$ -capacity. Also, the relative $(p(\cdot), \vartheta)$ -capacity has the following properties.

P1 . $\text{cap}_{p(\cdot), \vartheta}(\emptyset, \Omega) = 0$.

P2 . If $E_1 \subset E_2 \subset \Omega_2 \subset \Omega_1$, then $\text{cap}_{p(\cdot), \vartheta}(E_1, \Omega_1) \leq \text{cap}_{p(\cdot), \vartheta}(E_2, \Omega_2)$.

P3 . If E is a subset of Ω , then

$$\text{cap}_{p(\cdot), \vartheta}(E, \Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot), \vartheta}(U, \Omega).$$

P4 . If K_1 and K_2 are compact subsets of Ω , then

$$\begin{aligned} \text{cap}_{p(\cdot), \vartheta}(K_1 \cup K_2, \Omega) + \text{cap}_{p(\cdot), \vartheta}(K_1 \cap K_2, \Omega) &\leq \text{cap}_{p(\cdot), \vartheta}(K_1, \Omega) \\ &\quad + \text{cap}_{p(\cdot), \vartheta}(K_2, \Omega). \end{aligned}$$

P5 . Let K_n is a decreasing sequence of compact subsets of Ω for $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} \text{cap}_{p(\cdot), \vartheta}(K_n, \Omega) = \text{cap}_{p(\cdot), \vartheta}\left(\bigcap_{n=1}^{\infty} K_n, \Omega\right).$$

P6 . If E_n is an increasing sequence of subsets of Ω for $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} \text{cap}_{p(\cdot), \vartheta}(E_n, \Omega) = \text{cap}_{p(\cdot), \vartheta}\left(\bigcup_{n=1}^{\infty} E_n, \Omega\right).$$

P7 . If $E_n \subset \Omega$ for $n \in \mathbb{N}$, then

$$\text{cap}_{p(\cdot), \vartheta}\left(\bigcup_{n=1}^{\infty} E_n, \Omega\right) \leq \sum_{n=1}^{\infty} \text{cap}_{p(\cdot), \vartheta}(E_n, \Omega).$$

Theorem 1 ([11]) If $\text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) \geq 1$ and μ_{ϑ} is a doubling measure, then we obtain

$$C_1 \mu_{\vartheta}(B(x_0, r)) \leq \text{cap}_{p(\cdot), \vartheta}(B(x_0, r), B(x_0, 2r)) \leq C_2 \mu_{\vartheta}(B(x_0, r))$$

such that $C_1 = \frac{C}{r}$ and $C_2 = 2^{p^+} c_d \max\{r^{-p^-}, r^{-p^+}\}$.

Theorem 2 ([11]) If $A \subset B(x_0, r)$, $\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 4r)) \geq 1$ and $0 < r \leq s \leq 2r$, then

$$\frac{1}{C} \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r)) \leq \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2s)) \leq \text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 2r))$$

such that $C = 2^{p^+} + 2^{2p^++1} c c_1 \max\{r^{1-p^-}, r^{1-p^+}\}$.

We say that a property holds $(p(\cdot), \vartheta)$ -quasieverywhere if it satisfies except in a set of capacity zero. Recall also a function f is $(p(\cdot), \vartheta)$ -quasicontinuous in $\mathbb{R}^d$ if for each $\varepsilon > 0$ there exists a set A with the capacity of A is less than ε such that f restricted to $\mathbb{R}^d - A$ is continuous. If the capacity is an outer capacity, we can suppose that A is open. More detail can be found in [25].

Let $\Omega \subset \mathbb{R}^d$ be an open set. The space $W_{0, \vartheta}^{1, p(\cdot)}(\Omega)$ is denoted as the set of all measurable functions f if there exists a $(p(\cdot), \vartheta)$ -q.c. function $f^* \in W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^d)$ such that $f = f^*$ a.e. in Ω and $f^* = 0$ $(p(\cdot), \vartheta)$ -q.e. in $\mathbb{R}^d - \Omega$. In other words, $f \in W_{0, \vartheta}^{1, p(\cdot)}(\Omega)$, if there exist a $(p(\cdot), \vartheta)$ -q.c. function $f^* \in W_{\vartheta}^{1, p(\cdot)}(\mathbb{R}^d)$ such that the trace of f^* vanishes. More detail about the space can be seen by [9, 28, 29].

Moreover, $A \Subset B$ means that $\bar{A}$ is a compact subset of B . Throughout this paper, we assume that $1 < p^- \leq p(\cdot) \leq p^+ < \infty$ and $\vartheta^{-\frac{1}{p(\cdot)-1}} \in L_{loc}^1(\mathbb{R}^d)$. Also, we will denote

$$\mu_{\vartheta}(\Omega) = \int_{\Omega} \vartheta(x) dx.$$

3 The $(p(\cdot), \vartheta)$ -thinness and fine topology

Now, we present $(p(\cdot), \vartheta)$ -thinness and consider some properties of this thinness before considering the fine topology.

Definition 1 A set $E \subset \mathbb{R}^d$ is $(p(\cdot), \vartheta)$ -thin at $x \in \mathbb{R}^d$ if

$$\int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(E \cap B(x, r), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty. \quad (1)$$

Also, we say that E is $(p(\cdot), \vartheta)$ -thick at $x \in \mathbb{R}^d$ if E is not $(p(\cdot), \vartheta)$ -thin at $x \in \mathbb{R}^d$.

In [9], the authors have been considered (1) in the case of constant exponents. Moreover, in the definition of $(p(\cdot), \vartheta)$ -thinness we make a convention that the integral is 1 if $\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r)) = 0$. Also, the integral in (1) is usually called the Wiener type integral, briefly Wiener integral, as

$$W_{p(\cdot), \vartheta}(E, x) = \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(E \cap B(x, r), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r}.$$

In addition, we denote the Wiener sum $W_{p(\cdot), \vartheta}^{\text{sum}}(E, x)$ as

$$W_{p(\cdot), \vartheta}^{\text{sum}}(E, x) = \sum_{i=0}^{\infty} \left(\frac{\text{cap}_{p(\cdot), \vartheta}(E \cap B(x, 2^{-i}), B(x, 2^{1-i}))}{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}}.$$

Now we give a relationship between these two notions. The proof can be found in [11].

Theorem 3 Assume that the hypotheses of Theorem 1 and Theorem 2 hold. Then there exist positive constants C_1, C_2 such that

$$C_1 W_{p(\cdot), \vartheta}(E, x) \leq W_{p(\cdot), \vartheta}^{\text{sum}}(E, x) \leq C_2 W_{p(\cdot), \vartheta}(E, x)$$

for every $E \subset \mathbb{R}^d$ and $x_0 \notin E$. In particular, $W_{p(\cdot), \vartheta}(E, x_0)$ is finite if and only if $W_{p(\cdot), \vartheta}^{\text{sum}}(E, x_0)$ is finite.

The previous theorem tell us that the notions $W_{p(\cdot), \vartheta}$ and $W_{p(\cdot), \vartheta}^{\text{sum}}$ are equivalent under some conditions. In some cases, the Wiener sum $W_{p(\cdot), \vartheta}^{\text{sum}}$ is more practical than the Wiener integral $W_{p(\cdot), \vartheta}$.

Definition 2 A set $U \subset \mathbb{R}^d$ is called $(p(\cdot), \vartheta)$ -finely open if $\mathbb{R}^d - U$ is $(p(\cdot), \vartheta)$ -thin at $x \in U$. Equivalently, a set is $(p(\cdot), \vartheta)$ -finely closed if it includes all points where it is not $(p(\cdot), \vartheta)$ -thin. Moreover, the fine interior of A , briefly fine-int A , is the largest $(p(\cdot), \vartheta)$ -finely open set contained in A . In a similar way, the fine closure of F , briefly fine-clo F , is the smallest $(p(\cdot), \vartheta)$ -finely closed set containing F .

Theorem 4 The $(p(\cdot), \vartheta)$ -fine topology on $\mathbb{R}^d$ is generated by $(p(\cdot), \vartheta)$ -finely open sets.

Proof Firstly, we denote

$$\begin{aligned}\tau_F = & \left\{ E \subset \mathbb{R}^d : \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}((\mathbb{R}^d - E) \cap B(x, r), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty \right\} \cup \emptyset \\ = & \left\{ E \subset \mathbb{R}^d : \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(x, r) - E, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty \right\} \cup \emptyset.\end{aligned}$$

It is obvious that $\emptyset \in \tau_F$. Since $\text{cap}_{p(\cdot), \vartheta}(\emptyset, B(x, 2r)) = 0$, we have

$$\begin{aligned}& \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(x, r) - \mathbb{R}^d, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &= \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(\emptyset, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty.\end{aligned}$$

This follows that $\mathbb{R}^d \in \tau_F$. Now, we assert that finite intersections of $(p(\cdot), \vartheta)$ -finely open sets are $(p(\cdot), \vartheta)$ -finely open. Assume that $x \in \bigcap_{i=1}^n U_i$ where $U_1, U_2, \dots, U_n$ are $(p(\cdot), \vartheta)$ -finely open. Thus, if we consider the subadditivity of relative $(p(\cdot), \vartheta)$ -capacity and the cases of the exponent $p(\cdot)$ as $1 < p(\cdot) \leq 2$ and $p(\cdot) > 2$, then we get

$$\begin{aligned}& \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}\left(B(x, r) - \bigcap_{i=1}^n U_i, B(x, 2r)\right)}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &\leq \int_0^1 \left(\sum_{i=1}^n \frac{\text{cap}_{p(\cdot), \vartheta}(B(x, r) - U_i, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &\leq C \sum_{i=1}^n \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(x, r) - U_i, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty\end{aligned}$$

where $C > 0$ depends on n, p^-, p^+ . Therefore $\bigcap_{i=1}^n U_i$ is $(p(\cdot), \vartheta)$ -finely open. Finally, we need to show that arbitrary unions of $(p(\cdot), \vartheta)$ -finely open sets are $(p(\cdot), \vartheta)$ -finely open. Let $x \in \bigcup_{i \in I} U_i$ where $U_i, i \in I$, are $(p(\cdot), \vartheta)$ -finely open sets, and I is an index set. Thus, for every $i \in I$, we have

$$\int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(x, r) - U_i, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty. \quad (2)$$

Moreover, it is clear that $B(x, r) - \bigcup_{i \in I} U_i \subset B(x, r) - U_j$ or equivalently $\bigcap_{i \in I} (B(x, r) - U_i) \subset B(x, r) - U_j$ for $j \in I$. If we consider the properties of relative $(p(\cdot), \vartheta)$ -capacity and (2), then we get

$$\begin{aligned} & \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta} \left(B(x, r) - \bigcup_{i \in I} U_i, B(x, 2r) \right)}{\text{cap}_{p(\cdot), \vartheta} (B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &= \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta} \left(\bigcap_{i \in I} (B(x, r) - U_i), B(x, 2r) \right)}{\text{cap}_{p(\cdot), \vartheta} (B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &\leq \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta} (B(x, r) - U_j, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta} (B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty. \end{aligned}$$

Therefore, $\mathbb{R}^d - \bigcup_{i \in I} U_i$ is $(p(\cdot), \vartheta)$ -thin at x and as $x \in \bigcup_{i \in I} U_i$ was arbitrary, $\bigcup_{i \in I} U_i$ is $(p(\cdot), \vartheta)$ -finely open. $\square$

Corollary 5 Every open set is $(p(\cdot), \vartheta)$ -finely open.

Proof Assume that A is an open set in $\mathbb{R}^d$. For every $x \in A$, by the definition of openness, there exists a $t > 0$ such that $B(x, t) \subset A$. It is easy to see that $B(x, r) \subset B(x, t) \subset A$ for small enough $r > 0$. This follows that

$$\int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta} (B(x, r) - A, B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta} (B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty,$$

that is, $A \subset \mathbb{R}^d$ is $(p(\cdot), \vartheta)$ -finely open. $\square$

Remark 1 By the similar method in Corollary 5, it can be shown that every closed set is $(p(\cdot), \vartheta)$ -finely closed and that finite union of $(p(\cdot), \vartheta)$ -finely closed sets is $(p(\cdot), \vartheta)$ -finely closed again.

Corollary 6 The $(p(\cdot), \vartheta)$ -fine topology generated by the $(p(\cdot), \vartheta)$ -finely open sets is finer than Euclidean topology.

The opposite claim of Corollary 5 is not true in general. To see this, we give the Lebesgue spine

$$E = \left\{ (x, t) \in \mathbb{R}^2 \times \mathbb{R} : t > 0 \text{ and } |x| < e^{-\frac{1}{t}} \right\}$$

as a counter example, see [30, Example 13.4].

Now, we consider the more general case in sense to Corollary 5.

Theorem 7 Assume that $A \subset \mathbb{R}^d$ is an open or $(p(\cdot), \vartheta)$ -finely open set. Moreover, let, the relative $(p(\cdot), \vartheta)$ -capacity of E is zero. Then $A - E$ is $(p(\cdot), \vartheta)$ -finely open.

Proof By the Corollary 5, we can consider that $A \subset \mathbb{R}^d$ is an open set. Thus, for all $y \in A$,

$$\int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(y, r) - A, B(y, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(y, r), B(y, 2r))} \right)^{\frac{1}{p(y)-1}} \frac{dr}{r} < \infty. \quad (3)$$

Moreover, if we consider the properties of relative $(p(\cdot), \vartheta)$ -capacity, for all $x \in A - E$ and $r > 0$, we have

$$\begin{aligned} & \text{cap}_{p(\cdot), \vartheta}(B(x, r) - (A - E), B(x, 2r)) \\ &= \text{cap}_{p(\cdot), \vartheta}((B(x, r) - A) \cup (B(x, r) \cap E), B(x, 2r)) \\ &\leq \text{cap}_{p(\cdot), \vartheta}((B(x, r) - A), B(x, 2r)) \\ &\quad + \text{cap}_{p(\cdot), \vartheta}((B(x, r) \cap E), B(x, 2r)). \end{aligned} \quad (4)$$

Using the (3) and (4), we get

$$\begin{aligned} & \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(x, r) - (A - E), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &\leq \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}((B(x, r) - A), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &\quad + \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}((B(x, r) \cap E), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &< \infty. \end{aligned}$$

This completes the proof. $\square$

Now, we give that $(p(\cdot), \vartheta)$ -thinness is a local property.

Theorem 8 $A \subset \mathbb{R}^d$ is $(p(\cdot), \vartheta)$ -thin at $x \in \mathbb{R}^d$ if and only if for any $\delta > 0$, the set $A \cap B(x, \delta)$ is $(p(\cdot), \vartheta)$ -thin at $x \in \mathbb{R}^d$.

Proof Let $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Assume that $A \subset \mathbb{R}^d$ is $(p(\cdot), \vartheta)$ -thin at $x \in \mathbb{R}^d$. This follows that

$$\int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x, r), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty.$$

By the monotonicity of relative $(p(\cdot), \vartheta)$ -capacity, we have

$$\int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x, \delta) \cap B(x, r), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} < \infty \quad (5)$$

for any $\delta > 0$. This completes the necessary condition part of the proof. Now, we assume that for any $\delta > 0$, the set $A \cap B(x, \delta)$ is $(p(\cdot), \vartheta)$ -thin at $x \in \mathbb{R}^d$. Thus (5) is satisfied for all $\delta > 0$, in particular, for $0 < r \leq \delta \leq 1$. Let A is $(p(\cdot), \vartheta)$ -thick at $x \in \mathbb{R}^d$. Then we have

$$\begin{aligned} & \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x, \delta) \cap B(x, r), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} \\ &= \int_0^1 \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap B(x, r), B(x, 2r))}{\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r))} \right)^{\frac{1}{p(x)-1}} \frac{dr}{r} = \infty. \end{aligned}$$

This is a contradiction. That is the desired result. $\square$

Theorem 9 *Let the hypotheses of Theorem 3 hold. Moreover, assume that there is a point $x \in A$ such that $\mathbb{R}^d - A$ is $(p(\cdot), \vartheta)$ -thin at x . Then there exists $s > 0$ such that*

$$\text{cap}_{p(\cdot), \vartheta}(B(x, s) - A, B(x, 2s)) < \text{cap}_{p(\cdot), \vartheta}(B(x, s), B(x, 2s)).$$

Proof Since $\mathbb{R}^d - A$ is $(p(\cdot), \vartheta)$ -thin at $x \in A$, by the Theorem 3, we have

$$\begin{aligned} W_{p(\cdot), \vartheta}^{\text{sum}}(\mathbb{R}^d - A, x) &= \sum_{i=0}^{\infty} \left(\frac{\text{cap}_{p(\cdot), \vartheta}((\mathbb{R}^d - A) \cap B(x, 2^{-i}), B(x, 2^{1-i}))}{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} \\ &= \sum_{i=0}^{\infty} \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}) - A, B(x, 2^{1-i}))}{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} < \infty. \end{aligned}$$

This follows that

$$\liminf_{i \rightarrow \infty} \left(\frac{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}) - A, B(x, 2^{1-i}))}{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} = 0.$$

By the definition of limit, we get the desired result. $\square$

Remark 2 The proof of the previous theorem can be considered by using Wiener integral $W_{p(\cdot), \vartheta}(A, x)$ with similar method. Here, there is not necessary the condition that the hypotheses of Theorem 3 are hold.

Definition 3 Let U be a $(p(\cdot), \vartheta)$ -finely open set. A function $f : U \rightarrow \mathbb{R}$ is $(p(\cdot), \vartheta)$ -finely continuous at $x_0 \in U$ if $\left\{ x \in U : |f(x) - f(x_0)| \geq \varepsilon \right\}$ is $(p(\cdot), \vartheta)$ -thin at x_0 for each $\varepsilon > 0$.

Remark 3 Assume that U is a $(p(\cdot), \vartheta)$ -finely open set and $f : U \rightarrow \mathbb{R}$ is $(p(\cdot), \vartheta)$ -finely continuous at $x_0 \in U$. Then f is continuous function with respect to the $(p(\cdot), \vartheta)$ -fine topology on U . Indeed, if we consider the definition of finely continuous, then the set

$\{x \in U : |f(x) - f(x_0)| < \varepsilon\}$ is $(p(\cdot), \vartheta)$ -finely open. This follows that $f : U \rightarrow \mathbb{R}$ is continuous at $x_0 \in U$ in sense to the $(p(\cdot), \vartheta)$ -fine topology on U . The converse argument is still an open problem, see [28]. Moreover, this argument for the constant exponent was considered by [3, Theorem 2.136].

We note that a set A is a $(p(\cdot), \vartheta)$ -fine neighbourhood of a point x if and only if $x \in A$ and $\mathbb{R}^d - A$ is $(p(\cdot), \vartheta)$ -thin at x , see [9].

Theorem 10 *Let $A \subset B(x_0, r)$, $\text{cap}_{p(\cdot), \vartheta}(A, B(x_0, 4r)) \geq 1$ and $0 < r \leq s \leq 2r$. Assume that $A \subset \mathbb{R}^d$ is $(p(\cdot), \vartheta)$ -thin at $x \in A - A$. Then there exists an open neighbourhood U of A such that U is $(p(\cdot), \vartheta)$ -thin at x and $x \notin U$.*

Proof Using the same methods in the Theorem 2 and Theorem 1, it can be found for $r \leq s \leq 2r$ that

$$\text{cap}_{p(\cdot), \vartheta}(A \cap B(x, r), B(x, 2r)) \approx \text{cap}_{p(\cdot), \vartheta}(A \cap B(x, r), B(x, 2s)) \quad (6)$$

and

$$\text{cap}_{p(\cdot), \vartheta}(B(x, r), B(x, 2r)) \approx \text{cap}_{p(\cdot), \vartheta}(B(x, s), B(x, 2s)) \quad (7)$$

where the constants in $\approx$ depend on r , p^- , p^+ , constants of doubling measure and Poincaré inequality, see [11]. If we consider the Theorem 2 and the monotonicity of relative $(p(\cdot), \vartheta)$ -capacity, then we have

$$\begin{aligned} \text{cap}_{p(\cdot), \vartheta}(A \cap \overline{B(x, 2^{-i})}, B(x, 2^{1-i})) &\leq T \text{cap}_{p(\cdot), \vartheta}(A \cap \overline{B(x, 2^{-i})}, B(x, 2^{2-i})) \\ &\leq T \text{cap}_{p(\cdot), \vartheta}(A \cap B(x, 2^{1-i}), B(x, 2^{2-i})). \end{aligned}$$

By the definition of relative $(p(\cdot), \vartheta)$ -capacity, it can be taken open sets $U_i \supset A \cap \overline{B(x, 2^{-i})}$ such that

$$\begin{aligned} &\left(\frac{\text{cap}_{p(\cdot), \vartheta}(U_i, B(x, 2^{1-i}))}{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} \\ &\leq \left(\frac{\text{cap}_{p(\cdot), \vartheta}(A \cap \overline{B(x, 2^{-i})}, B(x, 2^{1-i}))}{\text{cap}_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} + \frac{1}{2^i}. \end{aligned} \quad (8)$$

Denote

$$U = (\mathbb{R}^d - \overline{B_1}) \cup (U_1 - \overline{B_2}) \cup ((U_1 \cap U_2) - \overline{B_3}) \cup ((U_1 \cap U_2 \cap U_3) - \overline{B_4}) \cup \dots$$

where $B_i = \overline{B(x, 2^{-i})}$. It is easy to see that U is open, $A \subset U$ holds and $x \notin U$. Since $B(x, 2^{-i}) \subset B(x, 2^{-i})$ holds for $i \in \mathbb{N}$, it is clear that $U \cap B(x, 2^{-i}) \subset U_i$. By (8), we have

$$\begin{aligned}
& \left(\frac{\text{cap}_{p(\cdot),\vartheta}(U \cap B(x, 2^{-i}), B(x, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} \\
& \leq \left(\frac{\text{cap}_{p(\cdot),\vartheta}(U_i, B(x, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} \\
& \leq \left(\frac{\text{cap}_{p(\cdot),\vartheta}(A \cap \overline{B(x, 2^{-i})}, B(x, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} + \frac{1}{2^i}.
\end{aligned}$$

Moreover, if we consider (6), then we get

$$\begin{aligned}
& \text{cap}_{p(\cdot),\vartheta}(A \cap \overline{B(x, 2^{-i})}, B(x, 2^{1-i})) \\
& \leq C_1 \text{cap}_{p(\cdot),\vartheta}(A \cap \overline{B(x, 2^{-i})}, B(x, 2^{2-i})) \\
& \leq C_2 \text{cap}_{p(\cdot),\vartheta}(A \cap B(x, 2^{1-i}), B(x, 2^{2-i})).
\end{aligned} \tag{9}$$

By (7), the inequality

$$\begin{aligned}
& \text{cap}_{p(\cdot),\vartheta}(B(x, 2^{-i}), B(x, 2^{1-i})) \\
& \geq \frac{1}{C_2} \text{cap}_{p(\cdot),\vartheta}(B(x, 2^{1-i}), B(x, 2^{2-i}))
\end{aligned} \tag{10}$$

holds. If we combine (9) and (10), then we have

$$\begin{aligned}
& \sum_{i=1}^{\infty} \left(\frac{\text{cap}_{p(\cdot),\vartheta}(U \cap B(x, 2^{-i}), B(x, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} \\
& \leq C \sum_{i=1}^{\infty} \left(\frac{\text{cap}_{p(\cdot),\vartheta}(A \cap B(x, 2^{1-i}), B(x, 2^{2-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x, 2^{1-i}), B(x, 2^{2-i}))} \right)^{\frac{1}{p(x)-1}} + 1.
\end{aligned}$$

Since A is $(p(\cdot), \vartheta)$ -thin at x , by considering the definition of Wiener sum $W_{p(\cdot),\vartheta}^{sum}$, we conclude

$$\sum_{i=1}^{\infty} \left(\frac{\text{cap}_{p(\cdot),\vartheta}(U \cap B(x, 2^{-i}), B(x, 2^{1-i}))}{\text{cap}_{p(\cdot),\vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} < \infty.$$

This follows that

$$\begin{aligned}
& W_{p(\cdot), \vartheta}^{sum}(U, x) \\
&= \sum_{i=0}^{\infty} \left(\frac{cap_{p(\cdot), \vartheta}(U \cap B(x, 2^{-i}), B(x, 2^{1-i}))}{cap_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} \\
&= \left(\frac{cap_{p(\cdot), \vartheta}(U \cap B(x, 1), B(x, 2))}{cap_{p(\cdot), \vartheta}(B(x, 1), B(x, 2))} \right)^{\frac{1}{p(x)-1}} \\
&\quad + \sum_{i=1}^{\infty} \left(\frac{cap_{p(\cdot), \vartheta}(U \cap B(x, 2^{-i}), B(x, 2^{1-i}))}{cap_{p(\cdot), \vartheta}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{\frac{1}{p(x)-1}} < \infty.
\end{aligned}$$

Hence U is $(p(\cdot), \vartheta)$ -thin at x . Thus the claim is follows from definition of open neighbourhood. $\square$

Now, we consider the usage area of $(p(\cdot), \vartheta)$ -fine topology in potential theory. We define $(p(\cdot), \vartheta)$ -Laplace equation as

$$-\Delta_{p(\cdot), \vartheta} = -div(\vartheta(x)|\nabla f|^{p(\cdot)-2}\nabla f) = 0 \quad (11)$$

for every $f \in W_{0, \vartheta}^{1, p(\cdot)}(\Omega)$.

Definition 4 ([29]) Let $\Omega \subset \mathbb{R}^d$ for $d \geq 2$, be an open set. A function $f \in W_{\vartheta, loc}^{1, p(\cdot)}(\Omega)$ is called a (weak) weighted solution (briefly $(p(\cdot), \vartheta)$ -solution) of (11) in Ω , if

$$\int_{\Omega} |\nabla f(x)|^{p(x)-2} \nabla f(x) \cdot \nabla g(x) \vartheta(x) dx = 0$$

whenever $g \in C_0^\infty(\Omega)$. Moreover, a function $f \in W_{\vartheta, loc}^{1, p(\cdot)}(\Omega)$ is a (weak) weighted supersolution (briefly $(p(\cdot), \vartheta)$ -supersolution) of (11) in Ω , if

$$\int_{\Omega} |\nabla f(x)|^{p(x)-2} \nabla f(x) \cdot \nabla g(x) \vartheta(x) dx \geq 0$$

whenever $g \in C_0^\infty(\Omega)$ is nonnegative. A function f is a weighted subsolution in Ω if $-f$ is a $(p(\cdot), \vartheta)$ -supersolution in Ω , and a weighted solution in Ω .

Definition 5 ([9, 28]) A function $f : \Omega \rightarrow (-\infty, \infty]$ is $(p(\cdot), \vartheta)$ -superharmonic in Ω if

- (i) f is lower semicontinuous,
- (ii) f is finite almost everywhere,
- (iii) Assume that $D \Subset \Omega$ is an open set. If g is a $(p(\cdot), \vartheta)$ -solution in D , which is continuous in $\bar{D}$, and satisfies $f \geq g$ on ∂D , then $f \geq g$ in D .

Note that every $(p(\cdot), \vartheta)$ -supersolution in Ω , which satisfies

$$f(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} f(y)$$

for all $x \in \Omega$, is $(p(\cdot), \vartheta)$ -superharmonic in Ω . On the other hand every locally bounded $(p(\cdot), \vartheta)$ -superharmonic function is a $(p(\cdot), \vartheta)$ -supersolution. The proof can be easily seen by using the similar method in [9, 31].

Let $S(\mathbb{R}^d)$ be the class of all $(p(\cdot), \vartheta)$ -superharmonic functions in $\mathbb{R}^d$. Since $(p(\cdot), \vartheta)$ -superharmonic functions are lower semicontinuous and since $S(\mathbb{R}^d)$ is closed under truncations, $(p(\cdot), \vartheta)$ -fine topology is the coarsest topology on $\mathbb{R}^d$ making all locally bounded $(p(\cdot), \vartheta)$ -superharmonic functions continuous, see [9].

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