



The Kolmogorov–Riesz theorem and some compactness criteria of bounded subsets in weighted variable exponent amalgam and Sobolev spaces

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Abstract

We study totally bounded subsets in weighted variable exponent amalgam and Sobolev spaces. Moreover, this paper includes several detailed generalized results of some compactness criteria in these spaces.

Keywords Weighted variable exponent amalgam and Sobolev spaces · Compactness · Totally bounded set

Mathematics Subject Classification 46E35 · 43A15 · 46E30

1 Introduction

Initially, the classical Riesz–Kolmogorov theorem states about the compactness of subsets in L^p $[0, 1]$ for $1 < p < \infty$, see [24]. This theorem has been generalized to some function spaces, such as Takahashi [35] for Orlicz spaces, Goes and Welland [11] for Köthe spaces, Musielak [30] for Musielak–Orlicz spaces, Rafeiro [32] for variable exponent Lebesgue spaces, Bandaliev [5] for weighted variable exponent Lebesgue spaces, Górká and Rafeiro [14] for more general framework, namely in the case of Banach function spaces (shortly, BF-space) and grand variable variable Lebesgue spaces, Górká and Macios [12, 13] for classical Lebesgue variable exponent Lebesgue in metric measure spaces. Hanche-Olsen et al. [20] offer an important development of the classical result by Kolmogorov–Riesz on compact subsets of Lebesgue spaces. Weil [36] considered compactness in L^p -spaces on locally compact groups. Moreover, the compactness problems for various spaces of differentiable functions on the Euclidean spaces have been studied by several authors. The classical criterion of Kolmogorov–Riesz for compactness of subsets of L^p ($1 \leq p < \infty$) has been extended by

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Feichtinger [9] to translation invariant Banach function spaces. More details can be seen [18] and [31].

The amalgam of L^p and l^q on the real line is the space $(L^p, l^q)(\mathbb{R})$ (briefly, (L^p, l^q)) consisting of functions which are locally in L^p and have l^q behavior at infinity. Wiener [37] studied several special cases of amalgam spaces including (L^1, l^2) , (L^2, l^∞) , (L^∞, l^1) and (L^1, l^∞) . Comprehensive information about amalgam spaces can be found in [10, 22] and [33]. Recently, there have been many interesting and important papers appeared in variable exponent amalgam spaces $(L^{p(\cdot)}, \ell^s)$ such as Aydın [1], Aydın [3], Aydın and Gurkanli [4], Gurkanli [16], Gurkanli and Aydın [17], Kokilashvili, Meskhi and Zaighum [23] and Kulak and Gurkanli [26], Meskhi and Zaighum [29]. In 2003, Pandey studied the compactness of bounded subsets in a Wiener amalgam space $W(B, Y)$ whose local and global components are solid Banach function spaces and satisfy conditions in [31, Definition 5.1].

In this study, we focus especially on the spaces $(L_w^{p(\cdot)}, \ell^q)$ on $\mathbb{R}$, and discuss totally bounded subsets in weighted variable exponent amalgam and Sobolev spaces. Moreover, it is well known that the spaces $L_w^{p(\cdot)}$ and $(L_w^{p(\cdot)}, \ell^q)$ are not translation invariant and the map $y \rightarrow L_y f$ is not continuous for $f \in (L_w^{p(\cdot)}, \ell^q)$. Also, the Young theorem $\|f * g\|_{p(\cdot), w} \leq \|f\|_{p(\cdot), w} \|g\|_1$ is not valid for $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, see [25]. Hence, the local component of the amalgam space $(L_w^{p(\cdot)}, \ell^q)$ does not satisfy the criteria in [31, Definition 5.1]. In this regard, we will propose some new conditions and criterions for compactness of bounded subsets in these spaces. In addition, Gurkanli [16] showed that $(L_w^{p(\cdot)}, \ell^q) = L_w^{p(\cdot)}$ under the some conditions, that is, the space $(L_w^{p(\cdot)}, \ell^q)$ is a extension of $L_w^{p(\cdot)}$. Finally, we will obtain that our main theorem provides a generalization of the corresponding results, such as Bandaliyev [5], Bandaliyev and Górká [6], Górká and Macios [12, 13], Górká and Rafeiro [14], Hanche-Olsen and Holden [18, 19], Sudakov [34] and Rafeiro [32].

2 Notations and preliminaries

In this section, we give some essential definitions, theorems and compactness criterions about totally bounded subsets in weighted variable exponent Lebesgue spaces.

Definition 1 Let (X, d) be a metric space and $\varepsilon > 0$. A subset K in X is called a ε -net (or ε -cover) for X if for every $x \in X$ there is a $x_\varepsilon \in K$ such that $d(x, x_\varepsilon) < \varepsilon$. Moreover, a metric space is called totally bounded if it admits a finite ε -net.

It is known that a subset of a complete metric space relatively compact (i.e. its closure is compact) if and only if it is totally bounded, see [38].

Theorem 1 Assume that X is a metric space and $K \subset X$. Then the following conditions are equivalent.

- (i) K is totally bounded (or, precompact) and complete
- (ii) K is compact.

Definition 2 Let X and Y be metric spaces. A family $\mathcal{F}$ of functions from X to Y is said to be equicontinuous if given $\varepsilon > 0$ there exists a number $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$ and all $x, y \in X$ satisfying $d_X(x, y) < \delta$.

The following theorem is quite useful for several compactness results.

Theorem 2 [18] *Let X be a metric space. Assume that, for every $\varepsilon > 0$, there exists some $\delta > 0$, a metric space W , and a mapping $\Phi : X \rightarrow W$ so that $\Phi[X]$ is totally bounded, and whenever $x, y \in X$ are such that $d(\Phi(x), \Phi(y)) < \delta$, then $d(x, y) < \varepsilon$. Then X is totally bounded.*

Definition 3 [25] For a measurable function $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ (called the variable exponent on $\mathbb{R}^n$ by the symbol $\mathcal{P}(\mathbb{R}^n)$), we put

$$p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\},$$

where $\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$.

If $p^+ < \infty$, then $f \in L^{p(\cdot)}(\mathbb{R}^n)$ iff $\varrho_{p(\cdot)}(f) < \infty$. The set $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{p(\cdot)}$. Moreover, the norm $\|\cdot\|_{p(\cdot)}$ coincides with the usual Lebesgue norm $\|\cdot\|_p$ whenever $p(\cdot) = p$ is a constant function.

Definition 4 A measurable and locally integrable function $w : \mathbb{R}^n \rightarrow (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\varrho_{p(\cdot), w}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) dx.$$

The weighted variable exponent Lebesgue space $L_w^{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ for which $\|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} = \left\| f w^{\frac{1}{p(\cdot)}} \right\|_{p(\cdot)} < \infty$. Also, $L_w^{p(\cdot)}(\mathbb{R}^n)$ is a uniformly convex Banach space, thus reflexive.

The relations between the modular $\varrho_{p(\cdot), w}(\cdot)$ and $\|\cdot\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}$ as follows

$$\begin{aligned} \min \left\{ \varrho_{p(\cdot), w}(f)^{\frac{1}{p^-}}, \varrho_{p(\cdot), w}(f)^{\frac{1}{p^+}} \right\} &\leq \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \leq \max \left\{ \varrho_{p(\cdot), w}(f)^{\frac{1}{p^-}}, \varrho_{p(\cdot), w}(f)^{\frac{1}{p^+}} \right\} \\ \min \left\{ \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{p^+}, \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{p^-} \right\} &\leq \varrho_{p(\cdot), w}(f) \leq \max \left\{ \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{p^+}, \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{p^-} \right\}. \end{aligned}$$

Moreover, if $0 < C \leq w$, then we have $L_w^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$, since one easily sees that

$$C \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \leq \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) dx$$

and $C \|f\|_{p(\cdot)} \leq \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}$, see [2].

Theorem 3 Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. Then for $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L_{w^*}^{q(\cdot)}(\mathbb{R}^n)$, we have $fg \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq C \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L_{w^*}^{q(\cdot)}(\mathbb{R}^n)},$$

where $w^* = w^{1-q(\cdot)}$.

Proof By the Hölder inequality for variable exponent Lebesgue spaces, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)g(x)| dx &= \int_{\mathbb{R}^n} |f(x)g(x)| w(x)^{\frac{1}{p(\cdot)} - \frac{1}{p(\cdot)}} dx \\ &\leq C \left\| f w^{\frac{1}{p(\cdot)}} \right\|_{p(\cdot)} \left\| g w^{-\frac{1}{p(\cdot)}} \right\|_{q(\cdot)} \end{aligned}$$

for some $C > 0$. That is the desired result. $\square$

The space $L_{loc}^1(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. It is a topological vector space with the family of seminorms $f \mapsto \|f\chi_K\|_{L^1}$. A Banach function space (shortly, BF-space) on $\mathbb{R}^n$ is a Banach space $(B, \|\cdot\|_B)$ of measurable functions which is continuously embedded into $L_{loc}^1(\mathbb{R}^n)$, briefly $B \hookrightarrow L_{loc}^1(\mathbb{R}^n)$, i.e. for any compact subset $K \subset \mathbb{R}^n$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_{L^1} \leq C_K \|f\|_B$ for all $f \in B$.

The dual space of $L_w^{p(\cdot)}(\mathbb{R}^n)$ is $L_{w^*}^{q(\cdot)}(\mathbb{R}^n)$, where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$ and $w^* = w^{1-q(\cdot)}$, see [27]. Also, it is known that if X is a BF-space, then the dual space X^* consisting of g such that

$$\|g\|_{X^*} = \sup_{\substack{f \in X \\ \|f\|_X \leq 1}} \int_{\mathbb{R}^n} |g(x)f(x)| dx$$

is also a BF-space. If using Hölder inequality for variable Lebesgue spaces, then we have

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq C \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L_{w^*}^{q(\cdot)}(\mathbb{R}^n)}$$

and

$$\|g\|_{(L_w^{p(\cdot)}(\mathbb{R}^n))^*} \leq \|g\|_{L_{w^*}^{q(\cdot)}(\mathbb{R}^n)}$$

for some $C > 0$. Therefore, the norm $\|\cdot\|_{(L_w^{p(\cdot)}(\mathbb{R}^n))^*}$ is well defined. Moreover, $\|\cdot\|_{(L_w^{p(\cdot)}(\mathbb{R}^n))^*}$ and $\|\cdot\|_{L_{w^*}^{q(\cdot)}(\mathbb{R}^n)}$ are equivalent by the similar methods for dual spaces of $L^{p(\cdot)}$, see [25]. Therefore, there is a isometric isomorphism between $(L_w^{p(\cdot)}(\mathbb{R}^n))^*$ and $L_{w^*}^{q(\cdot)}(\mathbb{R}^n)$. This yields that $(L_w^{p(\cdot)}(\mathbb{R}^n))^* = L_{w^*}^{q(\cdot)}(\mathbb{R}^n)$.

Remark 1 Let $K \subset \mathbb{R}^n$ with $|K| < \infty$. Then we have $\|\chi_K\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} < \infty$.

Proof Fix $\lambda \geq 1$. Since weight function w is locally integrable function $\mathbb{R}^n$, then

$$\begin{aligned} \varrho_{p(\cdot), w} \left(\frac{\chi_K}{\lambda} \right) &= \int_{\mathbb{R}^n} \frac{|\chi_K(x)|^{p(x)} w(x)}{\lambda^{p(x)}} dx = \int_K \lambda^{-p(x)} w(x) dx \\ &\leq \lambda^{-p^-} \int_K w(x) dx \leq \lambda^{-1} C_K, \end{aligned}$$

where $C_K = \int_K w(x) dx < \infty$. If we take $\lambda = C_K + 1 > 0$, then we have $\|\chi_K\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \leq C_K + 1$. $\square$

Definition 5 For $x \in \mathbb{R}^n$ and $r > 0$, we denote an open ball with center x and radius r by $B(x, r)$. For $f \in L_{loc}^1(\mathbb{R}^n)$, we denote the (centered) Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy$$

where the supremum is taken over all balls $B(x, r)$.

Hästö and Diening defined the class $A_{p(\cdot)}$ to consist of those weights w such that

$$\|w\|_{A_{p(\cdot)}} = \sup_{B \in \mathcal{B}} |B|^{-p_B} \|w\|_{L^1(B)} \left\| \frac{1}{w} \right\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(B)} < \infty,$$

where $\mathcal{B}$ denotes the set of all balls in $\mathbb{R}^n$, $p_B = \left(\frac{1}{|B|} \int_B \frac{1}{p(x)} dx \right)^{-1}$ and $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Note that this class is ordinary Muckenhoupt class $A_{p(\cdot)}$ if $p(\cdot)$ is a constant function, see [7].

Definition 6 We say that $p(\cdot)$ satisfies the local log-Hölder continuity condition if

$$|p(x) - p(y)| \leq \frac{C}{\log \left(e + \frac{1}{|x-y|} \right)}$$

for all $x, y \in \mathbb{R}^n$. If the inequality

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}$$

holds for some $p_\infty > 1$, $C > 0$ and all $x \in \mathbb{R}^n$, then we say that $p(\cdot)$ satisfies the log-Hölder decay condition. We denote by the symbol $P^{\log}(\mathbb{R}^n)$ the class of variable exponents which are log-Hölder continuous, i.e. which satisfy the local log-Hölder continuity condition and the log-Hölder decay condition.

Let $p(\cdot), q(\cdot) \in P^{\log}(\mathbb{R}^n)$, $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$. If $q(\cdot) \leq p(\cdot)$, then there exists a constant $C > 0$ depending on the characteristics of $p(\cdot)$ and $q(\cdot)$ such that $\|w\|_{A_{p(\cdot)}} \leq C \|w\|_{A_{q(\cdot)}}$. This yields that

$$A_1 \subset A_{p^-} \subset A_{p(\cdot)} \subset A_{p^+} \subset A_\infty$$

for $p(\cdot) \in P^{\log}(\mathbb{R}^n)$ and $1 < p^- \leq p(\cdot) \leq p^+ < \infty$.

Let $p(\cdot) \in P^{\log}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ < \infty$. Then $M : L_w^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_w^{p(\cdot)}(\mathbb{R}^n)$ if and only if $w \in A_{p(\cdot)}$, see [7].

We use the notation

$$\Phi(\mathbb{R}^n) = \left\{ p(\cdot) : 1 < p^- \leq p(\cdot) \leq p^+ < \infty, \|Mf\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \right\},$$

i.e. the maximal operator M is bounded on $L_w^{p(\cdot)}(\mathbb{R}^n)$. Hence we can find a sufficient condition for $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

Proposition 1 [2] *Let w be a weight function and $1 < p^- \leq p(\cdot) \leq p^+ < \infty$. If $w^{-\frac{1}{p(\cdot)-1}} \in L_{loc}^1(\mathbb{R}^n)$, then $L_w^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{loc}^1(\mathbb{R}^n)$.*

Definition 7 Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a nonnegative, radial, decreasing function belonging to $C_0^\infty(\mathbb{R}^n)$ and having the properties

- (i) $\varphi(x) = 0$ if $|x| \geq 1$,
- (ii) $\int_{\mathbb{R}^n} \varphi(x) dx = 1$.

Let $\varepsilon > 0$. If the function $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\frac{x}{\varepsilon})$ is nonnegative, belongs to $C_0^\infty(\mathbb{R}^n)$, and satisfies

- (i) $\varphi_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$ and
- (ii) $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = 1$,

then φ_ε is called a mollifier and we define the convolution by

$$\varphi_\varepsilon * f(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) f(y) dy.$$

The following proposition was proved in [8, Proposition 2.7].

Proposition 2 *Let φ_ε be a mollifier and $f \in L_{loc}^1(\mathbb{R}^n)$. Then*

$$\sup_{\varepsilon > 0} |\varphi_\varepsilon * f(x)| \leq Mf(x).$$

Proposition 3 [2] *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$ and $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$. Then $\varphi_\varepsilon * f \rightarrow f$ in $L_w^{p(\cdot)}(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0^+$.*

As a direct consequence of Proposition 3 there follows.

Corollary 1 *The class $C_0^\infty(\mathbb{R}^n)$ denotes continuous functions having continuous derivatives of all orders with compact support on $\mathbb{R}^n$. Now, let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $L_w^{p(\cdot)}(\mathbb{R}^n)$.*

Definition 8 Let $w = \{w_k\}$ be a sequence of positive numbers. The weighted variable sequence Lebesgue spaces $l_{p_n}(w)$ is defined by

$$l_{p_n}(w) = \left\{ x = \{x_k\} : \exists \lambda > 0, \sum_{k=1}^{\infty} (\lambda |x_k|)^{p_k} w_k < \infty \right\}$$

equipped with the norm

$$\|x\|_{l_{p_n}(w)} := \left\| x w^{\frac{1}{p_n}} \right\|_{l_{p_n}(w)} = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \left(\frac{|x_k|}{\lambda} \right)^{p_k} w_k \leq 1 \right\}.$$

The following theorem was proved for weighted variable exponent sequence spaces by [5] and [13]. Also, this theorem for the case of a constant exponent was obtained by [18].

Theorem 4 Let $\mathcal{F} \subset l_{p_n}(w)$, $p^+ < \infty$. Then the subset $\mathcal{F}$ is precompact in $l_{p_n}(w)$ if and only if

- (i) $\mathcal{F}$ is bounded, i.e. $\forall x = \{x_k\} \in \mathcal{F}$, $\exists C > 0$, $\sum_{k=1}^{\infty} |x_k|^{p_k} w_k \leq C$
 (ii) For every $\varepsilon > 0$ there is a $K = K(\varepsilon) > 0$ such that for every $x = \{x_k\}$ in $\mathcal{F}$

$$\left\| x w^{\frac{1}{p_k}} \right\|_{l_{p_k}(k > K)} < \varepsilon$$

or equivalently

$$\sum_{k=K+1}^{\infty} |x_k|^{p_k} w_k < \varepsilon.$$

The following theorem is an extension to the weighted variable exponent Lebesgue spaces of the classical Riesz–Kolmogorov Theorem, see [5].

Theorem 5 Let $p(\cdot) \in P^{\log}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ < \infty$. Assume that w is a weight function and $w \in A_{p(\cdot)}$. Then $\mathcal{F} \subset L_w^{p(\cdot)}(\mathbb{R}^n)$ is relatively compact if and only if

- (i) $\mathcal{F}$ is bounded in $L_w^{p(\cdot)}(\mathbb{R}^n)$, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} < \infty$
 (ii) For every $\varepsilon > 0$ there is a $\gamma > 0$ such that for all $f \in \mathcal{F}$

$$\|f\|_{L_w^{p(\cdot)}(|x| > \gamma)} < \varepsilon$$

or equivalently

$$\int_{|x| > \gamma} |f(x)|^{p(x)} w(x) dx < \varepsilon.$$

- (iii) $\lim_{\varepsilon \rightarrow 0^+} \|f * \varphi_\varepsilon - f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} = 0$ uniformly for $f \in \mathcal{F}$, where φ_ε is a mollifier function.

The following theorem can be proved for the spaces $L_w^{p(\cdot)}(\mathbb{R}^n)$ as Theorem 3 and Theorem 4 in [13].

Theorem 6 Let $\mathcal{F} \subset L_w^{p(\cdot)}(\mathbb{R}^n)$, $p(\cdot) \in P^{\log}(\mathbb{R}^n)$, $1 < p^- \leq p^+ < \infty$ and $w \in A_{p(\cdot)}$. Then the family $\mathcal{F} \subset L_w^{p(\cdot)}(\mathbb{R}^n)$ is precompact in $L_w^{p(\cdot)}(\mathbb{R}^n)$ if and only if

- (i) $\mathcal{F}$ is bounded in $L^{p(\cdot)}(\mathbb{R}^n, w)$, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} < \infty$
 (ii) For every $\varepsilon > 0$ there is a $R > 0$ such that for all $f \in \mathcal{F}$

$$\int_{|x| > R} |f(x)|^{p(x)} w(x) dx < \varepsilon$$

- (iii) For every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $f \in \mathcal{F}$ and $\forall |h| < \delta$

$$\|f_h - f\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} < \varepsilon$$

or equivalently

$$\int_{\mathbb{R}^n} |f_h(x) - f(x)|^{p(x)} w(x) dx < \varepsilon$$

$$\text{where } f_h(x) = (f)_{B(x,h)} = \frac{1}{|B(x,h)|} \int_{B(x,h)} f(t) dt.$$

3 Weighted variable exponent amalgam spaces

Definition 9 The space $L_{loc,w}^{p(\cdot)}(\mathbb{R}^n)$ is to be space of functions on $\mathbb{R}^n$ such that f restricted to any compact subset K of $\mathbb{R}^n$ belongs to $L_w^{p(\cdot)}(\mathbb{R}^n)$. Note that the embeddings $L_w^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{loc,w}^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{loc}^1(\mathbb{R}^n)$ hold.

Let $1 \leq p(\cdot), q < \infty$ and $J_k = [k, k+1)$, $k \in \mathbb{Z}$. The weighted variable exponent amalgam spaces $(L_w^{p(\cdot)}, \ell^q)$ are defined by

$$(L_w^{p(\cdot)}, \ell^q) = \left\{ f \in L_{loc,w}^{p(\cdot)}(\mathbb{R}) : \|f\|_{(L_w^{p(\cdot)}, \ell^q)} < \infty \right\},$$

where

$$\|f\|_{(L_w^{p(\cdot)}, \ell^q)} = \left(\sum_{k \in \mathbb{Z}} \|f \chi_{J_k}\|_{L_w^{p(\cdot)}(\mathbb{R})}^q \right)^{\frac{1}{q}}.$$

It is well known that $(L_w^{p(\cdot)}, \ell^q)$ is a Banach space and does not depend on the particular choice of J_k , that is, J_k can be equal to $[k, k+1)$, $[k, k+1]$ or $(k, k+1)$. If the weight w is a constant function, then the space $(L_w^{p(\cdot)}, \ell^q)$ coincides with $(L^{p(\cdot)}, \ell^q)$. Moreover, if the exponent $p(\cdot)$ and the weight w are constant functions, then we have the usual amalgam space (L^p, ℓ^q) , see [3,22,37]. The dual space of $(L_w^{p(\cdot)}, \ell^q)$ is isometrically isomorphic to $(L_{w^*}^{r(\cdot)}, \ell^t)$ where $\frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} = 1$, $\frac{1}{q} + \frac{1}{t} = 1$ and $w^* = w^{1-r(\cdot)}$. Also, the space $(L_w^{p(\cdot)}, \ell^q)$ is reflexive. Moreover, it is known that $(L_w^{p(\cdot)}, \ell^q)$ is a solid Banach function space, see [4].

In 2014, Meskhi and Zaighum [29] proved the boundedness of maximal operator for weighted variable exponent amalgam spaces under some conditions, see [29, Theorem 3.3], [29, Theorem 3.4]. Throughout this paper, we assume that $p(\cdot) \in P^{\log}(\mathbb{R}^n)$, $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, $w \in A_{p(\cdot)}$ and the maximal operator is bounded in weighted variable exponent amalgam spaces.

Remark 2 Let $\frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} = 1$ and $\frac{1}{q} + \frac{1}{s} = 1$. Then there exists a constant $C > 0$ such that

$$\|fg\|_{(L^1, \ell^1)} \leq C \|f\|_{(L_w^{p(\cdot)}, \ell^q)} \|g\|_{(L_{w^*}^{r(\cdot)}, \ell^s)}$$

for $f \in (L_w^{p(\cdot)}, \ell^q)$ and $g \in (L_{w^*}^{r(\cdot)}, \ell^s)$. Moreover, the expression

$$(L_w^{p(\cdot)}, \ell^q) (L_{w^*}^{r(\cdot)}, \ell^s) \subset (L^1, \ell^1) = L^1$$

is satisfied.

Proof Let $f \in (L_w^{p(\cdot)}, \ell^q)$ and $g \in (L_{w^*}^{r(\cdot)}, \ell^s)$. Using Hölder inequality for variable exponent Lebesgue and classical sequences spaces, we have

$$\begin{aligned} \|fg\|_{(L^1, \ell^1)} &= \sum_{k \in \mathbb{Z}} \|fg\chi_{J_k}\|_{L^1(\mathbb{R})} \\ &\leq C \sum_{k \in \mathbb{Z}} \left(\|f\chi_{J_k}\|_{L_w^{p(\cdot)}(\mathbb{R})} \|g\chi_{J_k}\|_{L_{w^*}^{r(\cdot)}(\mathbb{R})} \right) \\ &\leq C \left(\sum_{k \in \mathbb{Z}} \|f\chi_{J_k}\|_{L_w^{p(\cdot)}(\mathbb{R})}^q \right)^{\frac{1}{q}} \left(\sum_{k \in \mathbb{Z}} \|g\chi_{J_k}\|_{L_{w^*}^{r(\cdot)}(\mathbb{R})}^s \right)^{\frac{1}{s}} \\ &\leq C \|f\|_{(L_w^{p(\cdot)}, \ell^q)} \|g\|_{(L_{w^*}^{r(\cdot)}, \ell^s)}. \end{aligned}$$

This completes the proof. $\square$

Definition 10 [3,33] $L_{c,w}^{p(\cdot)}(\mathbb{R})$ denotes the functions f in $L_w^{p(\cdot)}(\mathbb{R})$ such that $\text{supp } f \subset \mathbb{R}$ is compact, that is,

$$L_{c,w}^{p(\cdot)}(\mathbb{R}) = \left\{ f \in L_w^{p(\cdot)}(\mathbb{R}) : \text{supp } f \text{ compact} \right\}.$$

Let $K \subset \mathbb{R}$ be given. The cardinality of the set

$$S(K) = \{J_k : J_k \cap K \neq \emptyset\}$$

is denoted by $|S(K)|$, where $\{J_k\}_{k \in \mathbb{Z}}$ is a collection of intervals.

Proposition 4 [3] If g belongs to $L_{c,w}^{p(\cdot)}(\mathbb{R})$, then

- (i) $\|g\|_{(L_w^{p(\cdot)}, \ell^q)} \leq |S(K)|^{\frac{1}{q}} \|g\|_{L_w^{p(\cdot)}(\mathbb{R})}$ for $1 \leq q < \infty$,
- (ii) $\|g\|_{(L_w^{p(\cdot)}, \ell^\infty)} \leq |S(K)| \|g\|_{L_w^{p(\cdot)}(\mathbb{R})}$ for $q = \infty$,
- (iii) $L_{c,w}^{p(\cdot)}(\mathbb{R}) \subset (L_w^{p(\cdot)}, \ell^q)$ for $1 \leq q \leq \infty$,

where K is the compact support of g .

Theorem 7 $L_{c,w}^{p(\cdot)}(\mathbb{R})$ is dense subspace of $(L_w^{p(\cdot)}, \ell^q)$ for $1 \leq p(\cdot), q < \infty$.

Proof If we use the similar techniques of Theorem 7 in [22] or Theorem 3.6 in [33], then we can prove the theorem similarly. $\square$

Proposition 5 $C_c(\mathbb{R})$, which consists of continuous functions on $\mathbb{R}$ whose support is compact, is dense in $(L_w^{p(\cdot)}, \ell^q)$ for $1 \leq p(\cdot), q < \infty$.

Proof It is clear that $C_c(\mathbb{R})$ is included in $(L_w^{p(\cdot)}, \ell^q)$. Let $f \in (L_w^{p(\cdot)}, \ell^q)$. By Theorem 7, given $\varepsilon > 0$ there exists $g \in L_{c,w}^{p(\cdot)}(\mathbb{R})$ such that

$$\|f - g\|_{(L_w^{p(\cdot)}, \ell^q)} < \frac{\varepsilon}{2}. \quad (1)$$

If E is the compact support of g , then there exists h in $C_c(E)$ such that

$$\|g - h\|_{L_w^{p(\cdot)}(E)} < \frac{\varepsilon}{2 |S(E)|^{\frac{1}{q}}}$$

since $C_c(E)$ is dense in $L_w^{p(\cdot)}(E)$, see [1]. Hence by Proposition 4, we have

$$\|g - h\|_{(L_w^{p(\cdot)}, \ell^q)} \leq |S(E)|^{\frac{1}{q}} \|g - h\|_{L_w^{p(\cdot)}(E)} < \frac{\varepsilon}{2}. \quad (2)$$

If we consider the (1) and (2), then

$$\begin{aligned} \|f - h\|_{(L_w^{p(\cdot)}, \ell^q)} &\leq \|f - g\|_{(L_w^{p(\cdot)}, \ell^q)} + \|g - h\|_{(L_w^{p(\cdot)}, \ell^q)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof. $\square$

By the Corollary 1, the proof of the following result can be obtained similar to Proposition 5.

Corollary 2 *The class $C_0^\infty(\mathbb{R})$ is dense in $(L_w^{p(\cdot)}, \ell^q)$ for $1 \leq p(\cdot), q < \infty$.*

Now we give the proof of the Kolmogorov–Riesz Theorem for $\mathcal{F} \subset (L_w^{p(\cdot)}, \ell^q)$.

Theorem 8 *A subset $\mathcal{F} \subset (L_w^{p(\cdot)}, \ell^q)$ is totally bounded if and only if*

- (i) $\mathcal{F}$ is bounded in $(L_w^{p(\cdot)}, \ell^q)$, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{(L_w^{p(\cdot)}, \ell^q)} < \infty$
- (ii) For every $\varepsilon > 0$ there is some $\gamma > 0$ such that for all $f \in \mathcal{F}$

$$\|f\|_{(L_w^{p(\cdot)}, \ell^q)(|x| > \gamma)} < \varepsilon$$

- (iii) $\lim_{\varepsilon \rightarrow 0^+} \|f * \varphi_\varepsilon - f\|_{(L_w^{p(\cdot)}, \ell^q)} = 0$ uniformly for $f \in \mathcal{F}$, where φ_ε is a mollifier function.

Proof Assume that $\mathcal{F}$ is totally bounded in $(L_w^{p(\cdot)}, \ell^q)$. Then, for every $\varepsilon > 0$ there exists a finite ε -cover for the set $\mathcal{F}$. This implies plainly the boundedness of $\mathcal{F}$, and then we get (i). To prove condition (ii), let $\varepsilon > 0$ be given. If we take the set $\{V_1, V_2, \dots, V_m\}$ as an ε -cover of $\mathcal{F}$, and $h_j \in V_j$ for $j = 1, \dots, m$, then for a $\gamma > 0$ we have

$$\|h_j\|_{(L_w^{p(\cdot)}, \ell^q)(|x| > \gamma)} < \varepsilon.$$

If $f \in V_j$, then we have $\|f - h_j\|_{(L_w^{p(\cdot)}, \ell^q)} \leq \varepsilon$. This follows that

$$\begin{aligned} \|f\|_{(L_w^{p(\cdot)}, \ell^q)(|x| > \gamma)} &\leq \|f - h_j\|_{(L_w^{p(\cdot)}, \ell^q)(|x| > \gamma)} + \|h_j\|_{(L_w^{p(\cdot)}, \ell^q)(|x| > \gamma)} \\ &\leq \|f - h_j\|_{(L_w^{p(\cdot)}, \ell^q)} + \|h_j\|_{(L_w^{p(\cdot)}, \ell^q)(|x| > \gamma)} < 2\varepsilon. \end{aligned}$$

This implies the condition (ii). Finally, we show the condition (iii). Let $f \in \mathcal{F}$ be given. Then by Theorem 7, given $\varepsilon > 0$ there exists $g \in L_{c,w}^{p(\cdot)}(\mathbb{R})$ such that

$$\|f - g\|_{(L_w^{p(\cdot)}, \ell^q)} < \frac{\varepsilon}{2 \max\{1, c\}}. \quad (3)$$

If E is the compact support of g , then we have

$$\|g - g * \varphi_\varepsilon\|_{L_w^{p(\cdot)}(E)} < \frac{\varepsilon}{2 |S(E)|^{\frac{1}{q}}} \quad (4)$$

by Proposition 3. Also by Proposition 4, we get

$$\|g - g * \varphi_\varepsilon\|_{(L_w^{p(\cdot)}, \ell^q)} \leq |S(E)|^{\frac{1}{q}} \|g - g * \varphi_\varepsilon\|_{L_w^{p(\cdot)}(E)} < \frac{\varepsilon}{2}. \quad (5)$$

If we consider the Proposition 2, the boundedness of maximal operator, (3) and (5), then we have

$$\begin{aligned} & \|f - f * \varphi_\varepsilon\|_{(L_w^{p(\cdot)}, \ell^q)} \\ & \leq \|f - g\|_{(L_w^{p(\cdot)}, \ell^q)} + \|g - g * \varphi_\varepsilon\|_{(L_w^{p(\cdot)}, \ell^q)} + \|g * \varphi_\varepsilon - f * \varphi_\varepsilon\|_{(L_w^{p(\cdot)}, \ell^q)} \\ & \leq \max\{1, c\} \|f - g\|_{(L_w^{p(\cdot)}, \ell^q)} + \|g - g * \varphi_\varepsilon\|_{(L_w^{p(\cdot)}, \ell^q)} \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This finishes the proof of necessity. Now, we assume that $\mathcal{F} \subset (L_w^{p(\cdot)}, \ell^q)$ satisfies all three conditions. Let $\varepsilon > 0$ be given. Denote

$$\mathcal{F}_\varepsilon = \{f_\varepsilon : f \in \mathcal{F}, f_\varepsilon = f * \varphi_\varepsilon\}.$$

By Proposition 2, we have

$$|f_\varepsilon(x)| \leq Mf(x),$$

where Mf is the maximal function. Since the condition (i) hold, we have uniformly boundedness of all functions in $\mathcal{F}_\varepsilon$. Now, we denote

$$\mathcal{F}_{\varepsilon\varepsilon} = \{f_{\varepsilon\varepsilon} : f_\varepsilon \in \mathcal{F}_\varepsilon, f_{\varepsilon\varepsilon} = f_\varepsilon * \varphi_\varepsilon\}.$$

If we consider the Proposition 2 and the monotonicity of the maximal operator, then we get

$$\begin{aligned} |f_{\varepsilon\varepsilon}(x)| & \leq |(f_\varepsilon * \varphi_\varepsilon)(x)| \leq M(f_\varepsilon)(x) \\ & \leq M(Mf)(x). \end{aligned}$$

This yields

$$\sup_{f_\varepsilon \in \mathcal{F}_\varepsilon} \|f_{\varepsilon\varepsilon}\|_{(L_w^{p(\cdot)}, \ell^q)} < \infty.$$

Therefore, we get that all functions in $\mathcal{F}_{\varepsilon\varepsilon}$ are uniformly bounded. If we use the techniques of Theorem 11 in [32] for the rest of proof, then we can prove the theorem similarly. $\square$

The following theorem has been given us a different characterization of precompactness in $(L_w^{p(\cdot)}, \ell^q)$ similar to Theorem 3 and Theorem 4 in [13].

Theorem 9 *The family $\mathcal{F} \subset (L_w^{p(\cdot)}, \ell^q)$ is totally bounded in $(L_w^{p(\cdot)}, \ell^q)$ if and only if*

- (i) $\mathcal{F}$ is bounded in $(L_w^{p(\cdot)}, \ell^q)$, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{(L_w^{p(\cdot)}, \ell^q)} < \infty$
- (ii) For every $\varepsilon > 0$, $r \rightarrow 0^+$ and for all $f \in \mathcal{F}$ we have

$$\|f - (f)_{B(\cdot, r)}\|_{(L_w^{p(\cdot)}, \ell^q)} < \varepsilon$$

or equivalently

$$\lim_{r \rightarrow 0^+} \|f - (f)_{B(\cdot, r)}\|_{(L_w^{p(\cdot)}, \ell^q)} = 0$$

where $(f)_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(t) dt$.

(iii) For every $\varepsilon > 0$ there is a $\gamma > 0$ such that for all $f \in \mathcal{F}$

$$\|f\|_{(L_w^{p(\cdot)}, \ell^q)(|x| > \gamma)} < \varepsilon.$$

Proof Assume that $\mathcal{F} \subset (L_w^{p(\cdot)}, \ell^q)$ is totally bounded. Then, for every $\varepsilon > 0$ there exists a finite ε -cover for the set $\mathcal{F}$. Thus, we take $\{f_l\}_{l=1, \dots, m}$ ε -cover in $\mathcal{F}$ such that

$$\mathcal{F} \subset \bigcup_{l=1}^m B(f_l, \varepsilon).$$

The totally boundedness of $\mathcal{F}$ implies plainly the boundedness of $\mathcal{F}$. Hence we get (i). By Theorem 7, given $\varepsilon > 0$ there exists $g \in L_{c, w}^{p(\cdot)}(\mathbb{R})$ such that

$$\|f - g\|_{(L_w^{p(\cdot)}, \ell^q)} < \frac{\varepsilon}{2 \max\{1, c\}} \quad (6)$$

where $c > 0$. Let E be the compact support of g . Now, we will show that

$$\|g - (g)_{B(\cdot, r)}\|_{L_w^{p(\cdot)}(E)} \rightarrow 0$$

or equivalently

$$\int_E |g(x) - (g)_{B(x, r)}|^{p(x)} w(x) dx \rightarrow 0$$

as $r \rightarrow 0^+$. By the Proposition 1, we have $L_w^{p(\cdot)} \hookrightarrow L_{loc}^1$. Therefore, we have $(g)_{B(x, r)} \rightarrow g(x)$ for $x \in E$ as $r \rightarrow 0^+$ by the Lebesgue differentiation theorem, see [21]. If we use the boundedness of the Hardy-Littlewood maximal operator Mg for $g \in L_w^{p(\cdot)}(E)$, then we get

$$\begin{aligned} |g(x) - (g)_{B(x, r)}|^{p(x)} w(x) &\leq 2^{p^+-1} \left(|g(x)|^{p(x)} + |(g)_{B(x, r)}|^{p(x)} \right) w(x) \\ &\leq 2^{p^+-1} \left(|g(x)|^{p(x)} + |M(g)(x)|^{p(x)} \right) w(x) \in L^1(E). \end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\int_E |g(x) - (g)_{B(x, r)}|^{p(x)} w(x) dx < \frac{\varepsilon}{2 |S(E)|^{\frac{1}{q}}}.$$

for sufficiently small $r > 0$. Also by Proposition 4, we get

$$\|g - (g)_{B(\cdot, r)}\|_{(L_w^{p(\cdot)}, \ell^q)} \leq |S(E)|^{\frac{1}{q}} \|g - (g)_{B(\cdot, r)}\|_{L_w^{p(\cdot)}(E)} < \frac{\varepsilon}{2}. \quad (7)$$

By Proposition 2, (6) and (7), we have

$$\begin{aligned} & \|f - (f)_{B(.,r)}\|_{(L_w^{p(.,\ell q)})} \\ & \leq \|f - g\|_{(L_w^{p(.,\ell q)})} + \|g - (g)_{B(.,r)}\|_{(L_w^{p(.,\ell q)})} + \|(g)_{B(.,r)} - (f)_{B(.,r)}\|_{(L_w^{p(.,\ell q)})} \\ & \leq \|f - g\|_{(L_w^{p(.,\ell q)})} + \|M(g - f)\|_{(L_w^{p(.,\ell q)})} + \|g - (g)_{B(.,r)}\|_{(L_w^{p(.,\ell q)})} \\ & \leq \max\{1, c\} \|f - g\|_{(L_w^{p(.,\ell q)})} + \|g - (g)_{B(.,r)}\|_{(L_w^{p(.,\ell q)})} < \varepsilon. \end{aligned}$$

This completes the proof of (ii). If we use similar method in Theorem 8, then we get (iii).

Now, we assume that the conditions (i), (ii) and (iii) are satisfied. Since $(L_w^{p(.,\ell q)})$ is a solid Banach function space, the proof is completed by [14, Theorem 3.1]. $\square$

Remark 3 Let $\Omega \subset \mathbb{R}^n$ be an open set. The set $L_{loc,w}^{p(.,\ell q)}(\Omega)$ is defined by

$$L_{loc,w}^{p(.,\ell q)}(\Omega) = \left\{ f : f\chi_K \in L_w^{p(.,\ell q)}(\Omega) \text{ for any compact subset } K \subset \Omega \right\}$$

with the usual identification of functions that are equal almost everywhere. Moreover, by [15, Lemma 2.2] it is well known that there exists a sequence of compact subsets $\{K_j\}_{j \in \mathbb{N}}$ such that

$$K_1 \subset K_2 \subset \cdots \subset K_j \subset \cdots \text{ and } \Omega = \bigcup_{j \in \mathbb{N}} K_j$$

where $K_j = \left\{ x \in \Omega : |x| \leq j \text{ and } \text{dist}(x, \mathbb{C}\Omega) \geq \frac{1}{j} \right\}$. Here, the complement of Ω is denoted by $\mathbb{C}\Omega$.

$L_{loc,w}^{p(.,\ell q)}(\Omega)$ is equipped with topology of $L_w^{p(.,\ell q)}(\Omega)$ convergence on compact subsets of Ω . In addition, any compact subset of Ω is contained in some K_j , and then the space $L_{loc,w}^{p(.,\ell q)}(\Omega)$ is a topological vector space with the countable family of seminorms,

$$p_j(f) = \|f\chi_{K_j}\|_{L_w^{p(.,\ell q)}(\Omega)}, \quad j = 1, 2, \dots$$

Moreover, the space $L_{loc,w}^{p(.,\ell q)}(\Omega)$ is a complete with respect to the metric $(f, g) \rightarrow \sum_{j=1}^{\infty} \min(2^{-j}, p_j(f - g))$. Hence it is obtained that $L_{loc,w}^{p(.,\ell q)}(\Omega)$ is a Fréchet space.

The following theorem is proved by [18] for constant exponent.

Theorem 10 A subset $\mathcal{F} \subset L_{loc,w}^{p(.,\ell q)}(\Omega)$ is totally bounded if and only if

(i) For every compact $K \subset \Omega$ there is some $C > 0$ such that

$$\int_{\Omega} |f_K(x)|^{p(x)} w(x) dx < C, \quad f \in \mathcal{F}$$

$$\text{where } f_K(x) = \begin{cases} f(x), & x \in K \\ 0, & \text{otherwise} \end{cases}.$$

(ii) For every $\varepsilon > 0$ and every compact $K \subset \Omega$ there is some $r > 0$ such that

$$\|f_K * \varphi_\varepsilon - f_K\|_{L_w^{p(.,\ell q)}(\Omega)} < \varepsilon, \quad f \in \mathcal{F}$$

$$\text{where } f_K(x) = \begin{cases} f(x), & x \in K \\ 0, & \text{otherwise} \end{cases}.$$

Proof The subset $\mathcal{F} \subset L_{loc,w}^{p(\cdot)}(\Omega)$ is totally bounded in $L_{loc,w}^{p(\cdot)}(\Omega)$ if and only if $\mathcal{F}_j = \{f_{K_j} : f \in \mathcal{F}\}$ is totally bounded for every j , with K_j as defined above. $\square$

Remark 4 Let $\{A_k\}_{k \in \mathbb{Z}}$ be a family of Banach spaces. We define the space $\ell^q(A_k)$ given by

$$\ell^q(A_k) = \{x = (x_k) : x_k \in A_k, \|x\| < \infty\},$$

where $\|x\| = \left(\sum_{k \in \mathbb{Z}} \|x_k\|_{A_k}^q \right)^{\frac{1}{q}}$. It can be seen that $\ell^q(A_k)$ is a Banach space with respect to the norm $\|\cdot\|$. Moreover, $(L_w^{p(\cdot)}, \ell^q)$ is a particular case of $\ell^q(A_k)$. Indeed, if we define the amalgam space as

$$(L_w^{p(\cdot)}, \ell^q) = \left\{ f \in L_{loc,w}^{p(\cdot)}(\mathbb{R}) : \left\{ \|f \chi_{J_k}\|_{L_w^{p(\cdot)}(\mathbb{R})} \right\}_{k \in \mathbb{Z}} \in \ell^q \right\}$$

take $A_k = L_w^{p(\cdot)}(J_k)$, $J_k = [k, k+1)$, then the map $f \rightarrow (f_k)$, $f_k = f \chi_{J_k}$ is an isometric isomorphism from $(L_w^{p(\cdot)}, \ell^q)$ to $\ell^q(L_w^{p(\cdot)}(J_k))$, see [1, 10]. Hence, for the total boundedness of $\mathcal{F} \subset (L_w^{p(\cdot)}, \ell^q)$ we can use the Theorem 4. Note that $\mathcal{F}$ is totally bounded in $(L_w^{p(\cdot)}, \ell^q)$ if and only if the set $\left\{ \left\{ \|f \chi_{J_k}\|_{L_w^{p(\cdot)}(\mathbb{R})} \right\}_{k \in \mathbb{Z}} : f \in \mathcal{F} \right\}$ is totally bounded in $\ell^q(L_w^{p(\cdot)}(J_k))$ for $k \in \mathbb{Z}$ with $1 \leq q < \infty$.

4 Weighted variable exponent Sobolev spaces

Let $\vartheta^{-\frac{1}{p(\cdot)-1}} \in L_{loc}^1(\mathbb{R}^n)$. Since every function in $L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)$ has distributional derivatives by Proposition 1, we get that the weighted variable exponent Sobolev spaces $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ are well defined.

Definition 11 Let $1 < p^- \leq p(x) \leq p^+ < \infty$, $\vartheta^{-\frac{1}{p(\cdot)-1}} \in L_{loc}^1(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We define the weighted variable Sobolev spaces $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ by

$$W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n) : D^{\alpha} f \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n), 0 \leq |\alpha| \leq k \right\}$$

equipped with the norm

$$\|f\|_{W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)} = \sum_{0 \leq |\alpha| \leq k} \|D^{\alpha} f\|_{L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)},$$

where $\alpha \in \mathbb{N}_0^n$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ and $D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. Moreover, the space $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ is a reflexive Banach space.

The space $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$ is defined by

$$W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n) : |\nabla f| \in L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n) \right\}.$$

The dual space of $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$ is denoted by $W_{\vartheta^*}^{-1,q(\cdot)}(\mathbb{R}^n)$ where $\vartheta^* = \vartheta^{1-q(\cdot)}$. The function $\varrho_{1,p(\cdot),\vartheta} : W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n) \rightarrow [0, \infty)$ is defined as $\varrho_{1,p(\cdot),\vartheta}(f) = \varrho_{p(\cdot),\vartheta}(f) + \varrho_{p(\cdot),\vartheta}(\nabla f)$

for every $f \in W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$. Also, the norm $\|f\|_{W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)} = \|f\|_{L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)} + \|\nabla f\|_{L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)}$ makes the space $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$ a Banach space, see [25].

Theorem 11 *A subset $\mathcal{F} \subset W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ is totally bounded if and only if*

- (i) *$\mathcal{F}$ is bounded in $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$, i.e. there is a $C > 0$ such that for $f \in \mathcal{F}$ and $0 \leq |\alpha| \leq k$*

$$\int_{\Omega} |D^{\alpha} f|^{p(x)} \vartheta(x) dx < C.$$

- (ii) *For every $\varepsilon > 0$ there is a $\gamma > 0$ such that for all $f \in \mathcal{F}$ and $0 \leq |\alpha| \leq k$*

$$\|D^{\alpha} f\|_{p(\cdot), \vartheta(|x|>\gamma)} < \varepsilon$$

or equivalently

$$\int_{|x|>\gamma} |D^{\alpha} f(x)|^{p(x)} \vartheta(x) dx < \varepsilon.$$

- (iii) $\lim_{\varepsilon \rightarrow 0^+} \|D^{\alpha}(f * \varphi_{\varepsilon}) - D^{\alpha} f\|_{L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)} = 0$ *uniformly for $f \in \mathcal{F}$ and $0 \leq |\alpha| \leq k$ where φ_{ε} is a mollifier function.*

Proof Note that $\mathcal{F}$ is totally bounded in $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ if and only if $D^{\alpha}[\mathcal{F}] = \{D^{\alpha} f : f \in \mathcal{F}\}$ is totally bounded in $L_{\vartheta}^{p(\cdot)}(\mathbb{R}^n)$ for every multi-index α with $0 \leq |\alpha| \leq k$ by Theorem 5. $\square$

Using [13, Theorem 5] we have an extension of [18, Corollary 9] to weighted variable exponent Sobolev spaces $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$.

Theorem 12 *Let $\mathcal{F} \subset W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$ be given. If the following conditions are satisfied*

- (i) *$\mathcal{F}$ is bounded in $W_{\vartheta}^{k,p(\cdot)}(\mathbb{R}^n)$, i.e. there is $C > 0$ such that for $f \in \mathcal{F}$ and $0 \leq |\alpha| \leq k$*

$$\int_{\Omega} |D^{\alpha} f|^{p(x)} \vartheta(x) dx < C.$$

- (ii) *for every $\varepsilon > 0$ there is a $\gamma > 0$ such that for all $f \in \mathcal{F}$ and $0 \leq |\alpha| \leq k$*

$$\|D^{\alpha} f\|_{L_{\vartheta}^{p(\cdot)}(|x|>\gamma)} < \varepsilon$$

or equivalently

$$\int_{|x|>\gamma} |D^{\alpha} f(x)|^{p(x)} \vartheta(x) dx < \varepsilon.$$

- (iii) *for every $\varepsilon > 0$ there is a $\rho > 0$ such that*

$$\int_{\mathbb{R}^n} |D^{\alpha} f(x+y) - D^{\alpha} f(x)|^{p(x)} \vartheta(x) dx < \varepsilon,$$

for $f \in \mathcal{F}$, $0 \leq |\alpha| \leq k$ and $|y| < \rho$,

then, $\mathcal{F}$ is totally bounded.

5 Applications

In this section, using a continuous embedding theorem for weighted variable exponent Sobolev spaces $W_{\vartheta}^{1,p(\cdot)}(\mathbb{R}^n)$, we discuss totally bounded subsets of $L_{\vartheta}^{q(\cdot)}(\mathbb{R}^n)$.

Theorem 13 [28] *Let $p(\cdot)$, $q(\cdot)$, ϑ_1 and ϑ_2 satisfy hypotheses in [28, Corollary 3.1]. Then the continuous embedding $W_{\vartheta_1}^{1,p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)$ is satisfied.*

Theorem 14 *Assume that hypotheses of Theorem 13 hold. Also, let $\vartheta_2 \in A_{q(\cdot)}$ and $\mathcal{F}$ be a bounded subset of $W_{\vartheta_1}^{1,p(\cdot)}(\mathbb{R}^n)$. If, for every $\varepsilon > 0$, there is a $\gamma > 0$ such that for all $f \in \mathcal{F}$*

$$\varrho_{W_{\vartheta_1}^{1,p(\cdot)}(|x|>\gamma)}(f) = \int_{|x|>\gamma} \left(|f(x)|^{p(x)} + |\nabla f(x)|^{p(x)} \right) \vartheta_1(x) dx < \varepsilon, \quad (8)$$

then $\mathcal{F}$ is a totally bounded subset of $L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)$.

Proof For the proof, we will show that $\mathcal{F}$ satisfies the hypotheses of Theorem 6 with $p(\cdot)$ replaced by $q(\cdot)$. By Theorem 13, there exists a $C > 0$ such that

$$\|f\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{W_{\vartheta_1}^{1,p(\cdot)}(\mathbb{R}^n)} \quad (9)$$

for all $f \in \mathcal{F} \subset W_{\vartheta_1}^{1,p(\cdot)}(\mathbb{R}^n)$. This yields the condition (i) of Theorem 6. Now, we set a function $u(\cdot) = f(\cdot) \chi(|\cdot| - \gamma)$ where $\chi(|\cdot| - \gamma) = \begin{cases} 0, & |\cdot| \leq \gamma \\ 1, & |\cdot| > \gamma \end{cases}$. If we consider the (9) and [28, Proposition 2.3], then we have

$$\|u\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} \leq C \max \left\{ \left(\varrho_{W_{\vartheta_1}^{1,p(\cdot)}(\mathbb{R}^n)}(u) \right)^{\frac{1}{p^-}}, \left(\varrho_{W_{\vartheta_1}^{1,p(\cdot)}(\mathbb{R}^n)}(u) \right)^{\frac{1}{p^+}} \right\}. \quad (10)$$

By the expression (8), we get

$$\begin{aligned} \varrho_{W_{\vartheta_1}^{1,p(\cdot)}(\mathbb{R}^n)}(u) &= \int_{|x|>\gamma} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) \vartheta_1(x) dx \\ &\quad + \int_{|x|\leq\gamma} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) \vartheta_1(x) dx \\ &= \int_{|x|>\gamma} \left(|f(x)|^{p(x)} + |\nabla f(x)|^{p(x)} \right) \vartheta_1(x) dx < \varepsilon. \end{aligned}$$

Since ε is sufficiently small, we obtain

$$\varrho_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)}(u) \leq \|u\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} < \varepsilon^*.$$

Therefore, we have

$$\begin{aligned} \varrho_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)}(u) &= \int_{|x|>\gamma} |u(x)|^{q(x)} \vartheta_2(x) dx + \int_{|x|\leq\gamma} |u(x)|^{q(x)} \vartheta_2(x) dx \\ &= \int_{|x|>\gamma} |f(x)|^{q(x)} \vartheta_2(x) dx < \varepsilon^* \end{aligned}$$

for all $f \in \mathcal{F}$. This completes the condition of (ii) of Theorem 6. Now, we will consider the rest of proof. Let $f \in \mathcal{F}$. Since the space $C_c(\mathbb{R}^n)$ is dense in $L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)$, given $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R}^n)$ such that

$$\|f - g\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} < \frac{\varepsilon}{2}. \quad (11)$$

Now, we will show that

$$\|g - (g)_{B(\cdot, h)}\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad (12)$$

or equivalently

$$\int_{\mathbb{R}^n} |g(x) - (g)_{B(x, h)}|^{q(x)} \vartheta_2(x) dx \longrightarrow 0$$

as $h \longrightarrow 0^+$. By the Proposition 1, we have $L_{\vartheta_2}^{q(\cdot)} \hookrightarrow L_{loc}^1$. Therefore, we have $(g)_{B(x, h)} \longrightarrow g(x)$ for $x \in \mathbb{R}^n$ as $h \longrightarrow 0^+$ by the Lebesgue differentiation theorem, see [21]. If we use the boundedness of the Hardy-Littlewood maximal operator Mg for $g \in L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)$, then we get

$$\begin{aligned} |g(x) - (g)_{B(x, h)}|^{q(x)} \vartheta_2(x) &\leq 2^{q^+-1} \left(|g(x)|^{q(x)} + |(g)_{B(x, h)}|^{q(x)} \right) \vartheta_2(x) \\ &\leq 2^{q^+-1} \left(|g(x)|^{q(x)} + |M(g)(x)|^{q(x)} \right) \vartheta_2(x) \in L^1(\mathbb{R}^n). \end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^n} |g(x) - (g)_{B(x, h)}|^{q(x)} \vartheta_2(x) dx < \frac{\varepsilon}{2}.$$

for sufficiently small $h > 0$. Therefore, if we consider (11) and (12), then we obtain

$$\begin{aligned} \|f - (f)_{B(\cdot, h)}\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} &\leq \|f - g\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} + \|g - (g)_{B(\cdot, h)}\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} + \|(g)_{B(\cdot, h)} - (f)_{B(\cdot, h)}\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq \|f - g\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} + \|M(g - f)\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} + \|g - (g)_{B(\cdot, h)}\|_{L_{\vartheta_2}^{q(\cdot)}(\mathbb{R}^n)} < \varepsilon \end{aligned}$$

This completes the proof. $\square$

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