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ORIGINAL RESEARCH PAPER

Almost all weak solutions of the weighted *p*(.)-biharmonic problem

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Abstract

We give a new compact embedding theorem, and use an equivalent norm to obtain some different solutions of the weighted $p(.)$-biharmonic eigenvalue problem

$$\begin{cases} \Delta\Big(v(x)|\Delta u|^{p(x)-2}\Delta u\Big) = \lambda\omega(x)|u|^{q(x)-2}u, & \text{in } \Omega \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

Using Mountain Pass Theorem we show that the problem has a nontrivial weak solution. Moreover, we obtain infinite many pairs of solutions of the problem due to the Fountain Theorem.

Keywords Weighted $p(.)$-biharmonic operator · Poincare inequality · Variational methods

Mathematics Subject Classification 35J35 · 46E35 · 35J60 · 35J70

1 Introduction

Let v and ω be two different weight functions. In this paper, we are interested in the following nonhomogeneous problem

$$\begin{cases} \Delta^2_{p(.),v} u = \lambda\omega(x)|u|^{q(x)-2}u, & \text{in } \Omega \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, p, q are

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continuous functions on $\overline{\Omega}$, i.e. $p, q \in C(\overline{\Omega})$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, and $\Delta^2_{p(.),v} u = \Delta\left(v(x)|\Delta u|^{p(x)-2}\Delta u\right)$ is the weighted $p(.)$-biharmonic operator of fourth order.

In recent years, variational mathematical problems with $p(x)$-growth have been studied in several topics [1–5]. Variable exponent Sobolev spaces (weighted or unweighted) are very useful to investigate weak solutions for elliptic and parabolic problems involving $p(x)$-Laplacian operator. Especially, when we need to find solutions to nonlinear and degenerate problems, using compact embedding theorems and equivalent norms in weighted Sobolev spaces gives very good results. For more detailed information, see [6–18].

Note that there is a crucial difference between p-biharmonic and $p(x)$-biharmonic problems. Since the problem involving $p(x)$-biharmonic operator, which is not homogeneous, possesses more complicated nonlinearities than the constant case, several variational methods, such as the Lagrange Multiplier Theorem, may be not practible [9, 19–26].

In 2001, the problem (1) was studied by Drábek and Ôtani [27] for $p(x) = q(x) = p =$ constant and $v(x) = \omega(x) = 1$. They proved that the problem has a principal positive eigenvalue which is simple and isolated. Later, a non decreasing sequence of positive eigenvalues of the weighted p-biharmonic operator was obtained, and the simplicity and the isolation of the first positive eigenvalue was proved by Talbi and Tsouli [28] for $p(x) = q(x) = p =$ constant and two weight functions.

In 2009, Ayoujil and El Amrouss [19] obtained the eigenvalues of the following fourth order elliptic equation, which is the particular case of (1),

$$\begin{cases} \Delta^2_{p(.)} u = \lambda |u|^{p(x)-2} u, & \text{in } \Omega \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

in the space $W^{2,p(.)}(\Omega) \cap W_0^{1,p(.)}(\Omega)$ for $p(x) = q(x)$ and $v(x) = \omega(x) = 1$. The problem (1) is also examined for $v(x) = 1$ by Ge, Zhou and Wu [20]. They demonstrate the existence of a continuous family of eigenvalues of the problem.

Inspired by the articles mentioned above, we prove a new compact embedding theorem, and introduce a more general norm in $W^{2,p(.)}(\Omega; v)$ compared to the norm given by Zang and Fu (Theorem 4.4 in [29]). Moreover, we display the presence of different weak solutions of the problem (1). So we improve the results presented in [10, 11, 19, 20, 27, 28] under more general conditions.

2 Notation and preliminaries

In this section, we recall some required information about variable exponent Lebesgue and Sobolev spaces (weighted or unweighted).

Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \inf_{x \in \overline{\Omega}} p(x) > 1 \right\},$$

Let $p \in C_+(\overline{\Omega})$, we indicate

$$p^- = ess \inf_{x \in \Omega} p(x) \text{ and } p^+ = ess \sup_{x \in \Omega} p(x).$$

Define the modular function by

$$\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Then the space $L^{p(.)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u(x)|^{p(x)} dx < \infty$. Moreover, the modular space $\left(L^{p(.)}(\Omega), \|.\|_{p(.)}\right)$ is a Banach space, where

$$\|u\|_{p(.)} = \inf\left\{ \lambda > 0 : \varrho_{p(.)}\left(\frac{u}{\lambda}\right) \le 1 \right\},$$

for $p \in C_+(\overline{\Omega})$ and $1 < p^- \le p^+ < \infty$. It is known that u belongs to the weighted space $L^{p(.)}(\Omega; v)$ if and only if $uv^{\frac{1}{p(.)}} \in L^{p(.)}(\Omega)$ and $\|u\|_{p(.),v} = \left\| uv^{\frac{1}{p(.)}} \right\|_{p(.)} < \infty$. If $v \in L^\infty(\Omega)$, then $L^{p(.)}(\Omega; v) = L^{p(.)}(\Omega)$.

For $k \in \mathbb{Z}^+$, let

$$W^{k,p(.)}(\Omega; v) = \left\{ u \in L^{p(.)}(\Omega; v) : D^\alpha u \in L^{p(.)}(\Omega; v), 0 \le |\alpha| \le k \right\},$$

where $\alpha \in \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}}$ for $v^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\Omega)$. Hence $W^{k,p(.)}(\Omega; v)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{W^{k,p(.)}(\Omega; v)} = \|u\|_v^{k,p(.)} = \sum_{0 \le |\alpha| \le k} \|D^\alpha u\|_{p(.),v}.$$

Define the space

$$W^{1,p(.)}(\Omega; v) = \left\{ u \in L^{p(.)}(\Omega; v) : |\nabla u| \in L^{p(.)}(\Omega; v) \right\}$$

with the norm

$$\|u\|_v^{1,p(.)} = \|u\|_{p(.),v} + \|\nabla u\|_{p(.),v}.$$

The space $W^{-1,p'(.)}(\Omega; v')$ is dual space of $W^{1,p(.)}(\Omega; v)$, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$, $v' = v^{1-p'(.)} = v^{-\frac{1}{p(.)-1}}$.

Moreover, the space $C_0^\infty(\Omega)$ is a subspace of $W^{1,p(.)}(\Omega;v)$. So we can obtain that the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega;v)$, which is denoted by $W_0^{1,p(.)}(\Omega;v)$. The space $W_0^{1,p(.)}(\Omega;v)$ can also be defined as

$$W_0^{1,p(.)}(\Omega;v) = \left\{u \in W^{1,p(.)}(\Omega;v) : u|_{\partial\Omega} = 0\right\}.$$

Finally, the space $W_0^{1,p(.)}(\Omega;v)$ is equipped with the norm

$$\|u\|_{W_0^{1,p(.)}} = \|\nabla u\|_{p(.),v}$$

by the Poincaré inequality (see Proposition 3.4 in [17]). So the norms $\|.\|_v^{1,p(.)}$ and $\|.\|_{W_0^{1,p(.)}}$ are equivalent on $W_0^{1,p(.)}(\Omega;v)$. The space $\left(W_0^{1,p(.)}(\Omega;v), \|.\|_{W_0^{1,p(.)}}\right)$ is also a separable and reflexive Banach space [1, 17, 30–35].

Let $f \in L^1_{loc}(\Omega)$. Then, the maximal function of f is defined by

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|} \int\limits_{B\cap\Omega} |f(y)|dy,$$

where the supremum is taken over all balls centered at x. It is proved that the operator Mf is bounded on $L^{p(.)}(\Omega;v)$. In addition, $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in $W^{1,p(.)}(\mathbb{R}^N;v)$ and $L^{p(.)}(\mathbb{R}^N;v)$ [30, 36].

Proposition 1 *If* $1 < q(x) \le p(x) < \infty$, $0 < \omega(x) \le v(x)$ *a.e.* $x \in \Omega$, *and* $|\Omega| < \infty$, *then the inequality*

$$\|u\|_{q(.),\omega} \le C\|u\|_{p(.),v}$$

holds for $u \in L^{p(.)}(\Omega;v)$, i.e. the embeddings $L^{p(.)}(\Omega;v) \hookrightarrow L^{q(.)}(\Omega;\omega)$ is continuous, where C is independent of u [37].

Theorem 1 *Let* $q(x) < p(x)$. *Then, the space* $L^{p(.)}(\Omega;v)$ *is embedded continuously in* $L^{q(.)}(\Omega;\omega)$ *if and only if*

$$\left\|\frac{\omega}{v}\right\|_{\frac{p(.)}{p(.)-q(.)},v} < \infty$$

(the measure of Ω may be not finite) [38].

Let E and F be Banach spaces. Define the sum norm on the space $Y = E \cap F$ as $\|u\|_Y = \|u\|_E + \|u\|_F$ for $u \in Y$. To solve the problem (1.1) we call for an equivalent norm, and a new compact embedding theorem for $X = W^{2,p(.)}(\Omega;v) \cap W_0^{1,p(.)}(\Omega;v)$. For $u \in X$ we have

$$\|u\|_X = \|\nabla u\|_{p(.),v} + \|u\|_v^{2,p(.)} = \|u\|_{p(.),v} + \|\nabla u\|_{p(.),v} + \sum_{|\alpha|=2} \|D^\alpha u\|_{p(.),v}.$$

In 2008, Zang and Fu [29] showed that the norms $\|u\|_X$ and $\|\Delta u\|_{p(.)}$ are equivalent on X for $v(x) = 1$. To improve and prove the weighted version of this theorem we have required conditions for X. So if we use the techniques and methods in Theorem 4.4 in [29], then we get the following more general norm for X.

Theorem 2 *Let Ω be a bounded domain with Lipschitz boundary. If the operator Mf is bounded on $L^{p(.)}(\Omega; v)$, then the norm $\|\Delta u\|_{p(.),v}$ is equivalent to the norm $\|u\|_X$ in $W^{2,p(.)}(\Omega; v) \cap W_0^{1,p(.)}(\Omega; v)$.*

After now, we take the norm for the space X as $\|u\|_X = \|\Delta u\|_{p(.),v}$, where

$$\|u\|_X = \inf\left\{ \gamma > 0 : \int_\Omega v(x) \left| \frac{\Delta u}{\gamma} \right|^{p(x)} dx \le 1 \right\}$$

for $u \in X$. Note that the space $\left(X, \|.\|_X\right)$ is a separable and reflexive Banach space. Throughout the paper we will take the norm $\|.\|_X$ and the modular $\varrho^v_{p(.)}$, which is defined as $\varrho^v_{p(.)} : X \to \mathbb{R}$ by

$$\varrho^v_{p(.)}(u) = \int_\Omega v(x) |\Delta u(x)|^{p(x)} dx.$$

Proposition 2 (see [20, 23, 39]) *For any $u, u_k \in X$ $(k = 1, 2, \ldots)$, we have*

(i) $\|u\|_X < 1 (= 1, > 1) \Leftrightarrow \varrho^v_{p(.)}(u) < 1 (= 1, > 1)$

(ii) $\|u\|_X^{p^-} \le \varrho^v_{p(.)}(u) \le \|u\|_X^{p^+}$ with $\|u\|_X \ge 1$

(iii) $\|u\|_X^{p^+} \le \varrho^v_{p(.)}(u) \le \|u\|_X^{p^-}$ with $\|u\|_X \le 1$,

(iv) $\min\left\{ \|u\|_X^{p^-}, \|u\|_X^{p^+} \right\} \le \varrho^v_{p(.)}(u) \le \max\left\{ \|u\|_X^{p^-}, \|u\|_X^{p^+} \right\}$,

(v) $\|u - u_k\|_X \to 0 \Leftrightarrow \varrho^v_{p(.)}(u - u_k) \to 0$ as $k \to \infty$,

(vi) $\|u_k\|_X \to \infty$ if and only if $\varrho^v_{p(.)}(u_k) \to \infty$ as $k \to \infty$.

Let

$$p_2^*(x) = \begin{cases} \dfrac{Np(x)}{N - 2p(x)}, & \text{if } p(x) < \dfrac{N}{2}, \\ +\infty, & \text{if } p(x) \ge \dfrac{N}{2}, \end{cases}$$

for every $x \in \overline{\Omega}$. It is obvious that $p(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$. In 2009, Ayoujil and

Amrouss [19] proved that the space $W^{2,p(.)}(\Omega) \cap W_0^{1,p(.)}(\Omega)$ (non-weighted) is continuously and compactly embedded in $L^{q(.)}(\Omega)$ for $p, q \in C_+(\overline{\Omega})$ such that $q(x) < p_2^*(x)$ for any $x \in \overline{\Omega}$ (Theorem 3.2, [19]). Now, we will show that the embedding from X into $L^{q(.)}(\Omega; v)$ is compact under some suitable conditions. Assume that the conditions in Lemma 3.1 and Lemma 3.2 in [40], Corollary 3.1 in [37], Corollary 2.4 and Theorem 2.5 in [41], Lemma 4.3 in [17]. Hence, we have the following compact embedding with respect to $p(.), q(.), \omega$ and v.

Theorem 3 *Let $1 \le t \le k$ and $k, t \in \mathbb{Z}^+$. Then the compact embedding*

$$W^{k,p(.)}(\Omega; v) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega; \omega)$$

holds.

Proof It is known that for $k \ge t \ge 1$ the following inequality

$$\sum_{0 \le |\alpha| \le t} \|D^\alpha u\|_{p(.),v} \le \sum_{0 \le |\alpha| \le k} \|D^\alpha u\|_{p(.),v}$$

is valid for $u \in W^{k,p(.)}(\Omega; v)$ [34]. Hence, we have the continuous embedding $W^{k,p(.)}(\Omega; v) \hookrightarrow W^{1,p(.)}(\Omega; v)$. So we can write

$$W^{1,p(.)}(\Omega; v) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega; \omega)$$

and

$$W^{k,p(.)}(\Omega; v) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega; \omega).$$

$\square$

Theorem 4 *Let $1 < q(x) \le p(x) < \infty$, $0 < \omega(x) \le v(x)$ a.e. $x \in \Omega$, or let $\left\|\frac{\omega}{v}\right\|_{\frac{p(.)}{p(.)-q(.)}, v} < \infty$ for $q(x) < p(x)$. Then the compact embedding*

$$W^{k,p(.)}(\Omega; v) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega; \omega)$$

is satisfied.

Proof By Proposition 1 or Theorem 1 it is easy to see that the embedding $W^{k,p(.)}(\Omega; v) \hookrightarrow W^{k,q(.)}(\Omega; \omega)$ is continuous. Moreover, by Theorem 3 we can get the embeddings

$$W^{k,q(.)}(\Omega; \omega) \hookrightarrow W^{1,q(.)}(\Omega; \omega)$$

and

$$W^{1,q(.)}(\Omega; \omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega; \omega).$$

Finally, we obtain that the embedding

$$W^{k,p(.)}(\Omega;v)\hookrightarrow\hookrightarrow L^{q(.)}(\Omega;\omega)$$

is compact for some conditions with respect to $p(.), q(.), \omega$ and v by [37] and [41]. □

Corollary 1 *Suppose that all conditions in Theorem 3 or Theorem 4 are satisfied. Then, the embedding* $X\hookrightarrow L^{q(.)}(\Omega;\omega)$ *is compact.*

Now, we prove Rellich-Kondrachov compact embedding theorem in $W^{k,p}(\Omega;v)$ for the constant and weighted case. This theorem is well known for $k=1$ (see [42]), but not for $k>1$, as far as we know, there is no directly relevant paper about this compact embedding $W^{k,p}(\Omega;v)\hookrightarrow\hookrightarrow L^{q}(\Omega;\omega)$. So using this result we prove more general compact embedding result than Theorem 3.2 in [19].

Theorem 5 (*Rellich-Kondrachov Compact Embedding Theorem*) *Let* $k\geq 1$ *be an integer, and let* $1\leq p<\infty$. *Then the following compact embedding holds;*

$$W^{k,p}(\Omega;v)\hookrightarrow\hookrightarrow L^{q}(\Omega;\omega)$$

for all q with $1\leq p\leq q<p^*$ if $p<\frac{N}{k}$, where

$$p^*=\begin{cases}\dfrac{Np}{N-kp}, & \text{if } p<\dfrac{N}{k},\\[2ex] +\infty, & \text{if } p\geq\dfrac{N}{k},\end{cases}.$$

Proof Let $p^*=Np\Big/N-kp$. Firstly, we show by induction on k that $W^{k,p}(\Omega;v)\hookrightarrow L^{p^*}(\Omega;\omega)$. For $k=1$ the embedding $W^{1,p}(\Omega;v)\hookrightarrow L^{q}(\Omega;\omega)$ is known by [42]. Suppose that the embedding $W^{k-1,p}(\Omega;v)\hookrightarrow L^{r}(\Omega;\omega)$ is satisfied for $r=Np\Big/(N-kp+p)$ when $p<\frac{N}{k-1}$. Since $u\in W^{k,p}(\Omega;v)$, then u, $D_iu=\frac{\partial u}{\partial_{x_i}}\in W^{k-1,p}(\Omega;v)$ for all $i=1,2,..,N$, where $p<\frac{N}{k}$. So it is easy to see that $u\in W^{1,r}(\Omega;v)$ and

$$\|u\|_v^{1,r}\leq C_1\|u\|_{W^{k,p}(\Omega;v)}.$$

Due to $kp<N$, we get $r<N$ and $W^{1,r}(\Omega;v)\hookrightarrow L^{p^*}(\Omega;\omega)$, where $p^*=Nr\Big/N-r=Np\Big/N-kp$ and

$$\|u\|_{p^*,\omega}\leq C_2\|u\|_v^{1,r}\leq C_3\|u\|_{W^{k,p}(\Omega;v)}, \tag{2}$$

i.e. the embedding $W^{k,p}(\Omega;v)\hookrightarrow L^{p^*}(\Omega;\omega)$ is continuous. Take $s=(p^*-q)p\Big/(p^*-p)$, $t=\frac{p}{s}>1$ and $\frac{1}{t}+\frac{1}{t'}=1$ for $p\leq q<p^*$. Then we have $st=p$ and $(q-s)t'=p^*$. So, we get

$$\begin{aligned}\|u\|_{q,\omega}^{q}&=\int_{\Omega}\omega(x)|u(x)|^{q}dx=\int_{\Omega}\omega(x)^{\frac{1}{t}}|u(x)|^{s}\omega(x)^{\frac{1}{t'}}|u(x)|^{q-s}dx\\&\leq\left[\int_{\Omega}\omega(x)|u(x)|^{st}dx\right]^{\frac{1}{t}}\left[\int_{\Omega}\omega(x)|u(x)|^{(q-s)t'}dx\right]^{\frac{1}{t'}}\\&=\|u\|_{p,\omega}^{p/t}\|u\|_{p^{*},\omega}^{p^{*}/t}\\&\leq C_{3}^{p^{*}/t}\|u\|_{W^{k,p}(\Omega;v)}^{q}\end{aligned}$$

by (2) and Hölder inequality. Hence, the continuous embedding $W^{k,p}(\Omega;v)\hookrightarrow L^{q}(\Omega;\omega)$ is obtained. Now let $\{u_n\}\subset W^{k,p}(\Omega;v)$ and $u_n\rightharpoonup 0$ (weakly) in $W^{k,p}(\Omega;v)$. Since the embedding $W^{k,p}(\Omega;v)\hookrightarrow W^{1,p}(\Omega;v)$ is continuous and the embedding $W^{1,p}(\Omega;v)\hookrightarrow\hookrightarrow L^{q}(\Omega;\omega)$ is compact by [42], then $u_n\to 0$ (strongly) in $L^{q}(\Omega;\omega)$. □

Theorem 6 *Assume that* $p,q\in C_{+}\left(\overline{\Omega}\right)$ *such that* $p(x)\leq q(x)<p_{2}^{*}(x)$ *for* $x\in\overline{\Omega}$ *Then, there is a continuous and compact embedding* $X\hookrightarrow L^{q(.)}(\Omega;\omega)$, *where*

$$p_{2}^{*}(x)=\begin{cases}\dfrac{Np}{N-2p}, & \text{if } p<\dfrac{N}{2},\\ +\infty, & \text{if } p\geq\dfrac{N}{2},\end{cases}.$$

Proof Since $q(x)<p_{2}^{*}(x)$ for $x\in\overline{\Omega}$, then there exists a neighborhood K_x in $\overline{\Omega}$ such that

$$q^{+}(K_x)<\left(p_{2}^{-}(K_x)\right)^{*}=\frac{Np^{-}(K_x)}{N-2p^{-}(K_x)},$$

where $p^{-}(K_x)=\inf\{p(y):y\in K_x\}$ and $q^{+}(K_x)=\sup\{q(y):y\in K_x\}$. If we take a finite sub-covering $\{K_j:j=1,2,3,...,n\}$ of the open covering $\{K_x\}_{x\in\overline{\Omega}}$ of compact set $\overline{\Omega}$ and displaying $p_j^{-}=p^{-}(K_j)$, $q_j^{+}=q^{+}(K_j)$. It is obvious that the embedding $W^{2,p(.)}(K_j;v)\hookrightarrow W^{2,p^{-}}(K_j;v)$ is valid for $p^{-}\leq p(x)$ by Theorem 2.8 in [33] (or Proposition 1). Moreover, we obtain continuous and compact embedding $W^{2,p^{-}}(K_j;v)\hookrightarrow L^{q^{+}}(K_j;\omega)$ for $q^{+}(K_j)<\left(p_{2}^{-}(K_j)\right)^{*}$ and $j=1,2,3,...,n$ by Theorem 5. Again by Theorem 2.8 in [33] there is a continuous embedding $L^{q^{+}}(K_j;\omega)\hookrightarrow L^{q(.)}(K_j;\omega)$ so for every K_j, $j=1,2,3,...,n$. So we show that the embeddings

$$W^{2,p(.)}(\Omega;v)\hookrightarrow\hookrightarrow L^{q(.)}(\Omega;\omega)$$

and

$$W^{2,p(.)}(\Omega; v) \cap W_0^{1,p(.)}(\Omega; v) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega; \omega)$$

are continuous and compact. This completes the proof. $\square$

We need the following Proposition.

Proposition 3 *(see [19–21, 39, 43]) Let us define the functional $L_{v(x)} : X \to \mathbb{R}$ by*

$$L_{v(x)}(u) = \int_{\Omega} \frac{v(x)}{p(x)} |\Delta u|^{p(x)} dx$$

for all $u \in X$. Then $L_{v(x)} \in C^1(X, \mathbb{R})$ and its derivative (in the weak sense) is denoted by

$$L'_{v(x)}(u)(y) = <L'_{v(x)}(u), y> = \int_{\Omega} v(x)|\Delta u|^{p(x)-2} \Delta u \Delta y dx$$

for any $u, y \in X$. In addition, we have the following properties:

(i) $L'_{v(x)} : X \longrightarrow X^*$ is continuous, bounded and strictly monotone operator,

(ii) $L'_{v(x)} : X \longrightarrow X^*$ is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in X and $\limsup_{n \longrightarrow \infty} L'_{v(x)}(u_n)(u_n - u) \leq 0$, then $u_n \longrightarrow u$ in X,

(iii) $L'_{v(x)} : X \longrightarrow X^*$ is a homeomorphism.

Definition 1 We say that the function $u \in X$ is a weak solution of the problem (1) if it satisfies the following condition

$$\int_{\Omega} v(x)|\Delta u|^{p(x)-2} \Delta u \Delta y dx - \lambda \int_{\Omega} \omega(x)|u|^{q(x)-2} u y dx = 0$$

for all $y \in X$. We recall that if $\lambda \in \mathbb{R}$ is an eigenvalue of the problem (1), then the corresponding $u \in X - \{0\}$ is a weak solution of (1).

Note that Zhikov and Surnachev [44] prove a sufficient condition for the density of smooth functions in the weighted Sobolev space with variable exponent is obtained. Therefore, the weak solution of the problem (1) is well defined.

To obtain a weak solution to (1), define the energy functional $\Psi_\lambda : X \to \mathbb{R}$ denoted by

$$\Psi_\lambda(u) = \int_{\Omega} \frac{v(x)}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_{\Omega} \frac{\omega(x)}{q(x)} |u|^{q(x)} dx$$

for $\lambda > 0$. It is known that Ψ_λ is well defined from X to $\mathbb{R}$ and $\Psi_\lambda \in C^1(X, \mathbb{R})$, and its derivative is given by

$$\Psi'_{\lambda}(u)(y) = <\Psi'_{\lambda}(u), y> = \int_{\Omega} v(x)|\Delta u|^{p(x)-2}\Delta u \Delta y dx - \lambda \int_{\Omega} \omega(x)|u|^{q(x)-2} u y dx$$

for all $u, y \in X$. Moreover, the function $u \in X$ is a weak solution of the problem (1) if and only if u is a critical point of Ψ_{λ}.

3 Main results

Lemma 1 *The functional Ψ_{λ} is coercive on X.*

Proof Let $u \in X$ and $\|u\|_X > 1$. Using Proposition 2, Proposition 2.2 in [37] and Theorem 6, we have

$$\begin{aligned}
\Psi_{\lambda}(u) &= \int_{\Omega} \frac{v(x)}{p(x)}|\Delta u|^{p(x)} dx - \lambda \int_{\Omega} \frac{\omega(x)}{q(x)}|u|^{q(x)} dx \\
&\geq \frac{1}{p^+}\int_{\Omega} v(x)|\Delta u|^{p(x)} dx - \frac{\lambda}{q^-}\int_{\Omega} \omega(x)|u|^{q(x)} dx \\
&\geq \frac{1}{p^+}\varrho^{v}_{p(.)}(u) - \frac{\lambda}{q^-}\max\left\{\|u\|^{q^-}_{q(.),\omega}, \|u\|^{q^+}_{q(.),\omega}\right\} \\
&\geq \frac{1}{p^+}\|u\|^{p^-}_X - \frac{\lambda}{q^-}\max\left\{\|u\|^{q^-}_X, \|u\|^{q^+}_X\right\} \\
&= \frac{1}{p^+}\|u\|^{p^-}_X - \frac{\lambda}{q^-}\|u\|^{q^+}_X.
\end{aligned}$$

Since $q^+ < p^-$, we deduce that $\Psi_{\lambda}(u) \longrightarrow \infty$ as $\|u\|_X \longrightarrow \infty$, i.e. Ψ_{λ} is coercive on X. □

Lemma 2 (*see* [21, 39]) *The energy functional Ψ_{λ} is weakly lower semicontinuous.*

Lemma 3 *There exists $\phi \in X$ such that $\phi \geq 0$, $\phi \neq 0$ and $\Psi_{\lambda}(t\phi) < 0$, for $t > 0$ small enough and $1 < q^- < p^- < q^+ < p^+$.*

Proof By $q^- < p^-$ we write $q^- + \varepsilon_0 < p^-$ for a $\varepsilon_0 > 0$. Since $q \in C(\overline{\Omega})$, then there exists an open subset $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \varepsilon_0$ for any $x \in \Omega_0$. So, we deduce that

$$q(x) \leq q^- + \varepsilon_0 < p^- \tag{3}$$

for any $x \in \Omega_0$. Take $\phi \in C_0^{\infty}(\Omega)$ such that $\overline{\Omega_0} \subset \mathrm{supp}\phi$, $\phi(x) = 1$ for all $x \in \overline{\Omega_0}$ and $0 \leq \phi \leq 1$ in Ω. In addition, we take $\|\phi\|_X = \|\Delta\phi\|_{p(.),v} = 1$, that is

$$\int_{\Omega} v(x)|\Delta\phi(x)|^{p(x)} dx = 1. \tag{4}$$

Using (3) and (4) we obtain

$$
\begin{aligned}
\Psi_\lambda(t\phi) &= \int\limits_\Omega \frac{v(x)t^{p(x)}}{p(x)}|\Delta\phi|^{p(x)}dx - \lambda\int\limits_\Omega \frac{\omega(x)t^{q(x)}}{q(x)}|\phi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-}\varrho^v_{p(.)}(\phi) - \frac{\lambda}{q^+}\int\limits_\Omega \omega(x)t^{q(x)}|\phi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+}\int\limits_{\Omega_0} \omega(x)t^{q(x)}|\phi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^-+\varepsilon_0}}{q^+}\int\limits_{\Omega_0} \omega(x)|\phi|^{q(x)}dx
\end{aligned}
$$

for any $t \in (0,1)$. Then, for any $t < \delta^{\frac{1}{p^- - q^- - \varepsilon_0}}$ with $0 < \delta < \min\left\{1, \frac{\lambda p^-}{q^+}\int\limits_{\Omega_0}\omega(x)|\phi|^{q(x)}dx\right\}$, we conclude that $\Psi_\lambda(t\phi) < 0$. □

Theorem 7 *The problem* (1) *has at least one nontrivial weak solution.*

Proof Using Lemma 1 and Lemma 2 the functional Ψ_λ is coercive and weakly lower semicontinuous. So Ψ_λ has a global minimizer u, which is a critical point of Ψ_λ [21]. Due to Lemma 3, it is obtained that $u \neq 0$. If we indicate $\Delta u = 0$ on $\partial\Omega$, then u is a solution of problem (1). As $u \in X - \{0\}$ is a critical point of Ψ_λ, we can write that

$$
\int\limits_\Omega v(x)|\Delta u|^{p(x)-2}\Delta u\Delta y dx = \lambda\int\limits_\Omega \omega(x)|u|^{q(x)-2}uy dx \tag{5}
$$

for all $y \in X$. It is easy to see that the relation (5) is valid for any $y \in C_0^\infty(\Omega)$. Assume that ξ is the unique solution of the problem

$$
\begin{cases} \Delta\xi = \lambda\omega(x)|u|^{q(x)-2}u, & \text{in } \Omega \\ \xi = 0, & \text{on } \partial\Omega. \end{cases}
$$

Due to (5) we get

$$
\int\limits_\Omega v(x)|\Delta u|^{p(x)-2}\Delta u\Delta y dx = \int\limits_\Omega \Delta\xi y dx
$$

for all $y \in C_0^\infty(\Omega)$. If we use the Green formula, then we obtain

$$
\int\limits_\Omega (\Delta\xi)y dx = \int\limits_\Omega \xi\Delta y dx.
$$

Hence

$$\int_{\Omega} v(x)|\Delta u|^{p(x)-2}\Delta u \Delta y dx = \int_{\Omega} \xi \Delta y dx \tag{6}$$

for any $y \in C_0^{\infty}(\Omega)$. It is well known that there is a unique $y \in C_0^{\infty}(\Omega)$ such that $\Delta y = w$ in Ω for all $w \in C_0^{\infty}(\Omega)$. Thus, the relation (6) take the form

$$\int_{\Omega} \left(v(x)|\Delta u|^{p(x)-2}\Delta u - \xi \right) w dx = 0$$

for all $w \in C_0^{\infty}(\Omega)$. If we implement the fundamental of the calculus of variations, then we achieve

$$v(x)|\Delta u|^{p(x)-2}\Delta u - \xi = 0 \quad \text{in } \Omega.$$

Since $\xi = 0$ on $\partial\Omega$, we find that $\Delta u = 0$ on $\partial\Omega$. □

Lemma 4 *Assume that there is a constant $\eta > 0$ such that $p^+ < \eta < q^-$. Then the operator Ψ_λ satisfies the (PS) condition.*

Proof Let $\{u_n\} \subset X$ be such that

$$\sup|\Psi_\lambda(u_n)| \le M, \quad \Psi'_\lambda(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty, \tag{7}$$

i.e. $\{u_n\}$ is a (PS) sequence. We will first show that the sequence $\{u_n\}$ is bounded in X. Assume that $\{u_n\}$ is not bounded in X. Therefore, we may take $\|u_n\|_X > 1$ for any $n \in \mathbb{N}$. Hence

$$\begin{aligned}
M + \|u_n\|_X &\ge \Psi_\lambda(u_n) - \frac{1}{\eta}\left\langle \Psi'_\lambda(u_n), u_n \right\rangle \\
&= \int_{\Omega} \frac{v(x)}{p(x)} |\Delta u_n|^{p(x)} dx - \lambda \int_{\Omega} \frac{\omega(x)}{q(x)} |u_n|^{q(x)} dx \\
&\quad - \frac{1}{\eta} \int_{\Omega} v(x)|\Delta u_n|^{p(x)} dx + \frac{\lambda}{\eta} \int_{\Omega} \omega(x)|u_n|^{q(x)} dx \\
&\ge \left(\frac{1}{p^+} - \frac{1}{\eta} \right) \varrho^{v}_{p(.)}(u_n) + \lambda \left(\frac{1}{\eta} - \frac{1}{q^-} \right) \int_{\Omega} \omega(x)|u_n|^{q(x)} dx \\
&\ge \left(\frac{1}{p^+} - \frac{1}{\eta} \right) \|u\|_X^{p^-}.
\end{aligned}$$

Dividing the above inequality by $\|u_n\|_X^{p^-}$ and taking to the limit as $n \to \infty$ we obtain a contradiction because of $\eta \le p^+$. So $\{u_n\}$ is bounded in X. Due the fact that X is a reflexive Banach space, there exists a function $u \in X$ such that $u_n \rightharpoonup u$ (weakly) in X. Due to Theorem 6, there is a compact embedding $X \hookrightarrow L^{q(.)}(\Omega;\omega)$ such that $u_n \to u$ (strongly) in $L^{q(.)}(\Omega;\omega)$ and $\sup_n \|u_n\|_X < \infty$. By the H ölder inequality for the exponential case, we obtain

$$\left|\int_{\Omega} \omega(x)|u|^{q(x)-2}u_n(u_n-u)dx\right| \leq \int_{\Omega} \omega(x)^{\frac{1}{q(x)}+\frac{1}{r(x)}}|u_n|^{q(x)-1}|u_n-u|dx$$
$$\leq C\left\||u_n|^{q(x)-1}\omega(x)^{\frac{1}{r(x)}}\right\|_{r(.)}\left\||u_n-u|\omega(x)^{\frac{1}{q(x)}}\right\|_{q(.)}$$
$$=C\|u_n\|_{q(.),\omega}\|u_n-u\|_{q(.),\omega}$$
$$\leq C^*\|u_n\|_X\|u_n-u\|_{q(.),\omega}\longrightarrow 0$$

as $n\to\infty$, where $\frac{1}{q(.)}+\frac{1}{r(.)}=1$ and $C_1\geq 1$. So

$$\lim_{n\longrightarrow\infty}\int_{\Omega} \omega(x)|u|^{q(x)-2}u_n(u_n-u)dx=0. \tag{8}$$

Therefore, we have

$$\Psi'_{\lambda}(u_n)(u_n-u)\to 0 \tag{9}$$

by (7) and boundedness of $\{u_n-u\}$ in X. If we collect (8) and (9), we achieve the following result

$$\lim_{n\longrightarrow\infty}\int_{\Omega} v(x)|\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)dx=0$$

or

$$\lim_{n\longrightarrow\infty} L'_{v(x)}(u_n)(u_n-u)=0. \tag{10}$$

So, we deduce that the sequence $u_n\to u$ (strongly) in X by (10) and Proposition 3. The proof is completed. □

Theorem 8 *The problem* (1) *has a nontrivial weak solution for* $p^+<\eta<q^-$.

Proof Due to Lemma 4, (PS) condition on X is satisfied for the functional Ψ_λ. Since $p^+<\eta<q^-$ and the compact embedding $X\hookrightarrow L^{q(.)}(\Omega;\omega)$, then we get

$$\Psi_\lambda(u)=\int_{\Omega}\frac{v(x)}{p(x)}|\Delta u|^{p(x)}dx-\lambda\int_{\Omega}\frac{\omega(x)}{q(x)}|u|^{q(x)}dx$$
$$\geq\frac{1}{p^+}\int_{\Omega} v(x)|\Delta u|^{p(x)}dx-\frac{\lambda}{q^-}\int_{\Omega}\omega(x)|u|^{q(x)}dx$$
$$\geq\frac{1}{p^+}\varrho^{v}_{p(.)}(u)-\frac{\lambda}{q^-}\max\left\{\|u\|^{q^-}_{q(.),\omega},\|u\|^{q^+}_{q(.),\omega}\right\}$$
$$\geq\frac{1}{p^+}\|u\|^{p^+}_X-\frac{\lambda}{q^-}\max\left\{\|u\|^{q^-}_X,\|u\|^{q^+}_X\right\}$$
$$\geq\frac{1}{p^+}\|u\|^{p^+}_X-\frac{\lambda}{q^-}\|u\|^{q^-}_X$$
$$\geq\left(\frac{1}{p^+}-\frac{\lambda}{q^-}\right)\|u\|^{p^+}_X$$

for $\|u\|_X\leq 1$ and $p^+\lambda<q^-$. Thus, we obtain $\Psi_\lambda(u)>0$ for $\|u\|_X=\rho$ sufficiently small. Moreover, we see that

$$\Psi_\lambda(t\phi) = \int\limits_\Omega \frac{v(x)t^{p(x)}}{p(x)}|\Delta\phi|^{p(x)}dx - \lambda \int\limits_\Omega \frac{\omega(x)t^{q(x)}}{q(x)}|\phi|^{q(x)}dx$$
$$\leq t^{p^+}\varrho^v_{p(.)}(\phi) - \lambda t^{q^-}\int\limits_\Omega \omega(x)|\phi|^{q(x)}dx$$
$$\leq t^{p^+}\varrho^v_{p(.)}(\phi) - \lambda t^{\eta^-}\int\limits_\Omega \omega(x)|\phi|^{q(x)}dx \longrightarrow -\infty$$

as $t \longrightarrow \infty$ for $\phi \in X - \{0\}$.Using $\Psi_\lambda(0) = 0$ and Mountain Pass Theorem (see [45]), we finish the proof of the theorem. □

Lemma 5 *Suppose that* $1 < q^- < p^- < q^+ < p^+$. *Then, there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$ *there exist* $a, b > 0$ *such that* $\Psi_\lambda(u) \geq b > 0$ *for any* $u \in X$ *with* $\|u\|_X = a$.

Proof Because of the embedding $X \hookrightarrow L^{q(.)}(\Omega;\omega)$, there is a positive constant $C_2 > 0$ such that

$$\|u\|_{q(.),\omega} \leq C_2\|u\|_X \tag{11}$$

for any $u \in X$. Let $a \in (0, 1)$ such that $a < \frac{1}{C_2}$. From (11) implies $\|u\|_{q(.),\omega} < 1$ and

$$\int\limits_\Omega \omega(x)|u|^{q(x)}dx \leq \|u\|^{q^-}_{q(.),\omega} \tag{12}$$

for all $u \in X$ with $\|u\|_X = a$. Combining (11) and (12), we get

$$\int\limits_\Omega \omega(x)|u|^{q(x)}dx \leq C_2^{q^-}\|u\|_X^{q^-} \tag{13}$$

for all $u \in X$ with $\|u\|_X = a$. Using the inequality (13), we conclude that

$$\Psi_\lambda(u) \geq \frac{1}{p^+}\varrho^v_{p(.)}(u) - \frac{\lambda}{q^-}\int\limits_\Omega \omega(x)|u|^{q(x)}dx$$
$$\geq \frac{1}{p^+}\|u\|_X^{p^+} - \frac{\lambda}{q^-}C_2^{q^-}\|u\|_X^{q^-}$$
$$= \frac{1}{p^+}a^{p^+} - \frac{\lambda}{q^-}C_2^{q^-}a^{q^-}$$
$$= a^{q^-}\left(\frac{1}{p^+}a^{p^+-q^-} - \frac{\lambda}{q^-}C_2^{q^-}\right).$$

Taking

$$\lambda^* = \frac{a^{p^+-q^-}}{2p^+} \cdot \frac{q^-}{C_2^{q^-}}, \tag{14}$$

then there exists $b = \frac{a^{p^+}}{2p^+} > 0$ such that $\Psi_\lambda(u) \geq b > 0$ for all $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\|_X = a$. □

Theorem 9 *Let* $1 < q^- < p^- < q^+ < p^+$. *Then, there exists* $\lambda^* > 0$ *such that any* $\lambda \in (0, \lambda^*)$ *is an eigenvalue for the problem* (1).

Proof Take $\lambda \in (0, \lambda^*)$, where λ^* is given by (14). By Lemma 5 it follows that on the boundary of the ball centered at the origin and of radius a in X, denoted by $B_a(0)$, we have

$$\inf_{\partial B_a(0)} \Psi_\lambda > 0.$$

From Lemma 3, there exists $\phi \in X$ such that $\Psi_\lambda(t\phi) < 0$ for $t > 0$ small enough. In addition, we get

$$\Psi_\lambda(u) \geq \frac{1}{p^+} \|u\|_X^{p^+} - \frac{\lambda}{q^-} C_2^{q^-} \|u\|_X^{q^-}$$

for any $u \in X$. This follows that

$$-\infty < \underline{c} = \inf_{\overline{B_a(0)}} \Psi_\lambda < 0.$$

Let choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \inf_{\partial B_a(0)} \Psi_\lambda - \inf_{B_a(0)} \Psi_\lambda.$$

If we implement the Ekeland's variational principle to the functional $\Psi_\lambda : \overline{B_a(0)} \longrightarrow \mathbb{R}$, then we discover $u_\varepsilon \in \overline{B_a(0)}$ such that

$$\Psi_\lambda(u_\varepsilon) < \inf_{\overline{B_a(0)}} \Psi_\lambda + \varepsilon,$$
$$\Psi_\lambda(u_\varepsilon) < \Psi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_X, \; u \neq u_\varepsilon.$$

Since

$$\Psi_\lambda(u_\varepsilon) \leq \inf_{\overline{B_a(0)}} \Psi_\lambda + \varepsilon \leq \inf_{B_a(0)} \Psi_\lambda + \varepsilon < \inf_{\partial B_a(0)} \Psi_\lambda,$$

then we can deduce that $u_\varepsilon \in B_a(0)$. Let take a function $\psi : \overline{B_a(0)} \longrightarrow \mathbb{R}$ denoted by $\psi(u) = \Psi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_X$. Hence, we obtain that u_ε is a minimum point of ψ. Thus we see that

$$\frac{\psi(u_\varepsilon + ty) - \psi(u_\varepsilon)}{t} \geq 0$$

and

$$\frac{\Psi_\lambda(u_\varepsilon + ty) - \Psi_\lambda(u_\varepsilon)}{t} + \varepsilon\|y\|_X \geq 0.$$

for small $t > 0$, $y \in B_a(0)$. Taking $t \longrightarrow 0$, we have

$$\langle \Psi'_\lambda(u_\varepsilon), y \rangle + \varepsilon\|y\|_X \geq 0. \tag{15}$$

If we substitute $-y$ in (15) for y, we have $|\langle \Psi'_\lambda(u_\varepsilon), y \rangle| \leq \varepsilon\|y\|_X$. It is known that $\Psi'_\lambda(u_\varepsilon) \in X^*$. If we use the definition on operator norm for X^*, we have $\|\Psi'_\lambda(u_\varepsilon)\|_{X^*} \leq \varepsilon$. Thus, we achieve that there is a sequence $\{z_n\} \subset B_a(0)$ such that

$$\Psi_\lambda(z_n) \longrightarrow \underline{c} \text{ and } \Psi'_\lambda(z_n) \longrightarrow 0. \tag{16}$$

Since $\{z_n\} \subset B_a(0)$ and $\|z_n\|_X < a$, then $\{z_n\}$ is bounded in X. This follows that $\{z_n\}$ converges strongly to z in X by Lemma 4. So, by (16),

$$\Psi_\lambda(z) = \underline{c} < 0 \text{ and } \Psi'_\lambda(z) = 0.$$

Thus any $\lambda \in (0, \lambda^*)$ is an eigenvalue of the problem (1). □

Lemma 6 (*see* [43]) *There exist* $\{\phi_k\} \subset X$ *and* $\{\phi_k^*\} \subset X^*$ *such that*

$$X = \overline{span\{\phi_k | k = 1, 2, \ldots\}}, \quad X^* = \overline{span\{\phi_k^* | k = 1, 2, \ldots\}}$$

and

$$\langle \phi_k^*, \phi_l \rangle = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

for the space, which is a reflexive and separable Banach space, where $\langle \cdot, \cdot \rangle$ denotes the duality product between X and X^*.

For convenience, we write $X_l = span\{\phi_l\}$, $Y_i = \bigoplus_{l=1}^{i} X_l$, $Z_i = \bigoplus_{l=i}^{\infty} X_l$.

Lemma 7 *If we define the set*

$$\alpha_k = \sup\left\{ \|u\|_{q(.),\omega} : \|u\|_X = 1,\ u \in Z_k \right\},$$

then $\lim_{k \longrightarrow \infty} \alpha_k = 0$.

Proof Using the continuous embedding $X \hookrightarrow L^{q(\cdot)}(\Omega; \omega)$ and Lemma 4.9 in [43], then $\lim_{k \longrightarrow \infty} \alpha_k = 0$. □

Using Lemma 7 we obtain the following theorem.

Theorem 10 *Let* $p^+ < q^-$. *There are infinite many pairs of solutions of the problem* (1), *i.e., the functional* Ψ_λ *has a sequence of critical point* $\{u_n\}$ *such that* $\Psi_\lambda(u_n) \longrightarrow +\infty$.

Proof Firstly, we want to show that the conditions (A_1) and (A_2) are satisfied, where

$$(A_1)\quad b_k := \inf\left\{\Psi_\lambda(u) \middle| u \in Z_k,\ \|u\|_X = \gamma_k\right\} \longrightarrow \infty \text{ as } k \longrightarrow \infty$$
$$(A_2)\quad a_k := \max\left\{\Psi_\lambda(u) \middle| u \in Y_k,\ \|u\|_X = \eta_k\right\} \le 0$$

for the reel numbers η_k and γ_k such that $\eta_k > \gamma_k > 0$ when k is large enough. Moreover, it is noted that

(A_1) Let $u \in Z_k$ such that $\|u\|_X = \gamma_k > 1$. Then, we obtain

$$\begin{aligned}\Psi_\lambda(u) &\ge \frac{1}{p^+}\varrho^v_{p(.)}(u) - \frac{\lambda}{q^-}\int_\Omega \omega(x)|u|^{q(x)}dx\\ &\ge \frac{1}{p^+}\varrho^v_{p(.)}(u) - \frac{\lambda}{q^-}\max\left\{\|u\|^{q^-}_{q(.),\omega}, \|u\|^{q^+}_{q(.),\omega}\right\}\\ &\ge \begin{cases} \dfrac{1}{p^+}\|u\|^{p^-}_X - \lambda, & \|u\|_{q(.),\omega} \le 1\\ \dfrac{1}{p^+}\|u\|^{p^-}_X - \lambda\alpha_k^{q^+}\|u\|^{q^+}_X, & \|u\|_{q(.),\omega} > 1\end{cases}\\ &\ge \frac{1}{p^+}\|u\|^{p^-}_X - \lambda\alpha_k^{q^+}\|u\|^{q^+}_X\\ &= \frac{1}{p^+}\gamma_k^{p^-} - \lambda\alpha_k^{q^+}\gamma_k^{q^+}\end{aligned}$$

If we choose $\gamma_k = \left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{1}{p^- - q^+}}$, then we get

$$\begin{aligned}\Psi_\lambda(u) &\ge \frac{1}{p^+}\left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} - \lambda\alpha_k^{q^+}\left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{q^+}{p^- - q^+}}\\ &= \left(\frac{1}{p^+} - \frac{1}{q^+}\right)\left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} \longrightarrow \infty\end{aligned}$$

as $k \longrightarrow \infty$ because $p^+ < q^+$ and $\alpha_k \longrightarrow 0$. So (A_1) holds.

(A_2) Let $u \in Y_k$ be such that $\|u\|_X = \eta_k > \gamma_k > 1$. Then, we obtain

$$\begin{aligned}\Psi_\lambda(u) &\le \frac{1}{p^-}\|u\|^{p^+}_X - \frac{\lambda}{q^+}\int_\Omega \omega(x)|u|^{q(x)}dx\\ &\le \frac{1}{p^-}\|u\|^{p^+}_X - \frac{\lambda}{q^+}\min\left\{\|u\|^{q^-}_{q(.),\omega}, \|u\|^{q^+}_{q(.),\omega}\right\}.\end{aligned}$$

Furthermore the norms $\|.\|_X$ and $\|.\|_{q(.),\omega}$ are equivalent due to the fact that Y_k has finite dimension. Finally,

$$\Psi_\lambda(u) \to -\infty \text{ as } \|u\|_X \to +\infty,\ u \in Y_k$$

by $p^+ < q^-$ and the Fountain Theorem (see [45, Theorem 3.6]). □

Data availability We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

Declarations

Conflict of interest Author has not any conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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