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ORIGINAL RESEARCH

Existence of Weak Solutions for Weighted Robin Problem Involving $p(.)$-biharmonic operator

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Abstract

Using Mountain Pass Theorem, we consider the existence of weak solutions of weighted Robin problem involving $p(.)$-biharmonic operator

$$\begin{cases} a(x)\Delta^2_{p(x)}u = \lambda b(x)|u|^{q(x)-2}u, & in\ \Omega \\ a(x)|\Delta u|^{p(x)-2}\frac{\partial u}{\partial \upsilon} + \beta(x)|u|^{p(x)-2}u = 0, & on\ \partial\Omega \end{cases}$$

under some conditions in the space $W^{2,p(.)}_{a,b}(\Omega)$.

Keywords Weak solution · $p(.)$-Biharmonic equation · Mountain pass theorem

Mathematics Subject Classification 35J35 · 46E35 · 35J60 · 35J70

Introduction

Let a and b be two different weight functions. Here, we are interested in the following fourth-order quasilinear weighted elliptic equation with Robin boundary conditions

$$\begin{cases} a(x)\Delta^2_{p(x)}u = \lambda b(x)|u|^{q(x)-2}u, & in\ \Omega \\ a(x)|\Delta u|^{p(x)-2}\frac{\partial u}{\partial \upsilon} + \beta(x)|u|^{p(x)-2}u = 0, & on\ \partial\Omega \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \upsilon}$ is the outer unit normal derivative of u with respect to $\partial\Omega$, p, q are continuous functions on $\overline{\Omega}$, i.e. $p, q \in C\left(\overline{\Omega}\right)$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, $\beta \in L^{\infty}(\partial\Omega)$ such that

$$\beta^{-} = \inf_{x\in\partial\Omega} \beta(x) > 0$$

where $\Delta^2_{p(x)} u = \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right)$ is the $p(x)$ -biharmonic operator of fourth order.

Before giving our main results, we recall literature involving with $p(x)$-biharmonic operator. Recently, variable exponent function spaces, especially Sobolev spaces, have been very useful for exploring the existence and multiplicity of solutions of partial differential equations involving $p(x)$-biharmonic operator. Besides this, electrorheological fluids, image processing, elastic mechanics and fluid dynamics are some of the applications in the area. In this context, we refer [1–7] and the references therein among others.

This paper is organized as follows. In "Notation and preliminaries", we investigate some preliminaries and basic informations about the weighted variable exponent Lebesgue spaces $L^{p(.)}_a$ and double weighted variable exponent Sobolev space $W^{2,p(.)}_{a,b}(\Omega)$ that will be used in "Main results". In the last section, we discuss the major conclusions of our main results.

Notation and Preliminaries

The set is given by

$$C_{+}\left(\overline{\Omega}\right) = \left\{ p \in C\left(\overline{\Omega}\right) : \inf_{x\in\overline{\Omega}} p(x) > 1 \right\},$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let $p \in C_{+}\left(\overline{\Omega}\right)$, we denote

$$p^{-} = \inf_{x\in\Omega} p(x) \text{ and } p^{+} = \sup_{x\in\Omega} p(x).$$

For $p \in C_{+}\left(\overline{\Omega}\right)$ and $1 < p^{-} \le p^{+} < \infty$. The space $L^{p(.)}(\Omega)$ is defined by

$$L^{p(.)}(\Omega) = \left\{ u \,\middle|\, u : \Omega \to \mathbb{R} \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}$$

with the (Luxemburg) norm

$$\|u\|_{p(.)} = \inf\left\{ \lambda > 0 : \varrho_{p(.)}\left(\frac{u}{\lambda}\right) \le 1 \right\},$$

where

$$\varrho_{p(.)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx.$$

Moreover, it is known that the space $\left(L^{p(.)}(\Omega), \|.\|_{p(.)}\right)$ is a reflexive Banach space, see [8].

The weighted Lebesgue space $L_a^{p(.)}(\Omega)$ is defined by

$$L_a^{p(.)}(\Omega) = \left\{ u \middle| u : \Omega \longrightarrow \mathbb{R} \text{ measurable and } \int\limits_{\Omega} |u(x)|^{p(x)} a(x) dx < \infty \right\}$$

such that $\|u\|_{p(.),a} = \left\| u a^{\frac{1}{p(.)}} \right\|_{p(.)} < \infty$ for $u \in L_a^{p(.)}(\Omega)$, where a is a weight function from Ω to $(0, \infty)$. The space $\left(L_a^{p(.)}(\Omega), \|.\|_{p(.),a}\right)$ becomes a Banach space. Also the space $L_a^{p(.)}(\partial\Omega)$ is given by

$$L_a^{p(.)}(\partial\Omega) = \left\{ u \middle| u : \partial\Omega \longrightarrow \mathbb{R} \text{ measurable and } \int\limits_{\partial\Omega} |u(x)|^{p(x)} a(x) d\sigma < +\infty \right\}$$

with the norm

$$\|u\|_{p(.),a,\partial\Omega} = \inf\left\{ \tau > 0 : \int\limits_{\partial\Omega} \left| \frac{u(x)}{\tau} \right|^{p(x)} a(x) d\sigma \leq 1 \right\}$$

for $u \in L_a^{p(.)}(\partial\Omega)$, where $d\sigma$ is the measure on the boundary of Ω. The space $\left(L_a^{p(.)}(\partial\Omega), \|.\|_{p(.),a,\partial\Omega}\right)$ is a Banach space. If $a \in L^{\infty}(\Omega)$, then $L_a^{p(.)} = L^{p(.)}$, see [9–12].

Proposition 1 (see [13–17]) *For all* $u, v \in L_a^{p(.)}(\Omega)$, *we have*

(i) $\|u\|_{p(.),a} < 1$ (resp.$= 1, > 1$) if and only if $\varrho_{p(.),a}(u) < 1$ (resp.$= 1, > 1$),
(ii) $\|u\|_{p(.),a}^{p^-} \leq \varrho_{p(.),a}(u) \leq \|u\|_{p(.),a}^{p^+}$ with $\|u\|_{p(.),a} > 1$,
(iii) $\|u\|_{p(.),a}^{p^+} \leq \varrho_{p(.),a}(u) \leq \|u\|_{p(.),a}^{p^-}$ with $\|u\|_{p(.),a} < 1$
(iv) $\min\left\{ \|u\|_{p(.),a}^{p^-}, \|u\|_{p(.),a}^{p^+} \right\} \leq \varrho_{p(.),a}(u) \leq \max\left\{ \|u\|_{p(.),a}^{p^-}, \|u\|_{p(.),a}^{p^+} \right\}$,
(v) $\min\left\{ \varrho_{p(.),a}(u)^{\frac{1}{p^-}}, \varrho_{p(.),a}(u)^{\frac{1}{p^+}} \right\} \leq \|u\|_{p(.),a} \leq \max\left\{ \varrho_{p(.),a}(u)^{\frac{1}{p^-}}, \varrho_{p(.),a}(u)^{\frac{1}{p^+}} \right\}$,
(vi) $\varrho_{p(.),a}(u - v) \to 0$ if and only if $\|u - v\|_{p(.),a} \to 0$.

For $k \in \mathbb{Z}^+$, let

$$W_{\nu}^{k,p(.)}(\Omega) = \left\{ u \in L_{\nu}^{p(.)}(\Omega) : D^{\alpha} u \in L_{\nu}^{p(.)}(\Omega), 0 \leq |\alpha| \leq k \right\},$$

where $\alpha \in \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}}$ for $\nu^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\Omega)$. Hence the space $W_\nu^{k,p(.)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{W_\nu^{k,p(.)}(\Omega)} = \|u\|_\nu^{k,p(.)} = \sum_{0 \le |\alpha| \le k} \|D^\alpha u\|_{p(.),\nu}.$$

Define the space

$$W_\nu^{1,p(.)}(\Omega) = \left\{u \in L_\nu^{p(.)}(\Omega) : |\nabla u| \in L_\nu^{p(.)}(\Omega)\right\}$$

with the norm

$$\|u\|_\nu^{1,p(.)} = \|u\|_{p(.),\nu} + \|\nabla u\|_{p(.),\nu}.$$

The space $W_{\nu^*}^{-1,p'(.)}(\Omega)$ is dual space of $W_\nu^{1,p(.)}(\Omega)$ where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1, \nu^* = \nu^{1-p'(.)} = \nu^{-\frac{1}{p(.)-1}}$.

Moreover, the space $C_0^\infty(\Omega)$ is a subspace of $W_\nu^{1,p(.)}(\Omega)$. Thus we can conclude that the closure of $C_0^\infty(\Omega)$ in $W_\nu^{1,p(.)}(\Omega)$ which is denoted by $W_{0,\nu}^{1,p(.)}(\Omega)$. The space $W_{0,\nu}^{1,p(.)}(\Omega)$ can also be defined as

$$W_{0,\nu}^{1,p(.)}(\Omega) = \left\{u \in W_\nu^{1,p(.)}(\Omega) : u\big|_{\partial\Omega} = 0\right\}$$

equipped with the norm

$$\|u\|_{W_{0,\nu}^{1,p(.)}} = \|\nabla u\|_{p(.),\nu}$$

by the Poincaré inequality, see [6, Proposition 3.4]. This follows that the norms $\|.\|_\nu^{1,p(.)}$ and $\|.\|_{W_{0,\nu}^{1,p(.)}}$ are equivalent on $W_{0,\nu}^{1,p(.)}(\Omega)$. The space $\left(W_{0,\nu}^{1,p(.)}(\Omega), \|.\|_{W_{0,\nu}^{1,p(.)}}\right)$ is also a separable and reflexive Banach space, see [6, 8, 13, 18–21].

Let $a^{-\frac{1}{p(.)-1}}, b^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\Omega)$. The double weighted variable exponent Sobolev space $W_{a,b}^{2,p(.)}(\Omega)$ is defined as

$$W_{a,b}^{2,p(.)}(\Omega) = \left\{u \in L_b^{p(.)}(\Omega) : D^\alpha u \in L_a^{p(.)}(\Omega),\ 1 \le |\alpha| \le 2\right\}$$

where $\alpha \in N_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}}$. Endowed with the norm

$$\|u\|_{a,b}^{2,p(.)} = \|u\|_{p(.),b} + \sum_{0 \le |\alpha| \le 2} \|D^\alpha u\|_{p(.),a},$$

$W_{a,b}^{2,p(.)}(\Omega)$ is also separable and reflexive Banach space. For $a = b = 1$, we get $W_{a,b}^{2,p(.)}(\Omega) = W^{2,p(.)}(\Omega)$. The dual space of $W_{a,b}^{2,p(.)}(\Omega)$ is $W_{a^*,b^*}^{-2,p'(.)}(\Omega)$ where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$, $a^* = a^{1-p'(.)} = a^{-\frac{1}{p(.)-1}}$ and $b^* = b^{1-p'(.)} = b^{-\frac{1}{p(.)-1}}$. If $a \in L^\infty(\Omega)$, then $L_a^{p(.)} = L^{p(.)}$, see [17, 18, 22–24].

Let $f \in L^1_{loc}(\Omega)$. Then, the maximal function of f is defined by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy,$$

where the supremum is taken over all ball centers at x. It is proved that the operator Mf is bounded on $L_v^{p(.)}(\Omega)$. In addition, $C_0^{\infty}\left(\mathbb{R}^N\right)$ is dense in $W_v^{1,p(.)}(\mathbb{R}^N)$ and $L_v^{p(.)}(\mathbb{R}^N)$, see [13, 25].

Denote

$$p^{\partial}(x) = (p(x))^{\partial} = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N \\ \infty, & \text{if } p(x) \geq N \end{cases}$$

and

$$p^{\partial}_{r(x)}(x) = \frac{r(x) - 1}{r(x)} p^{\partial}(x)$$

for any $x \in \partial\Omega$ and $r \in C(\partial\Omega, \mathbb{R})$ with $r^- = \inf_{x \in \partial\Omega} r(x) > 1$.

In this paper, we assume that $p(x) < p^{\partial}_{*,r(x)}(x) < p^{\partial}_{*}(x)$ for all $x \in \partial\Omega$ and $1 \leq q(x) < \frac{r(x)-1}{r(x)} \left(p_*\right)^{\theta}(x)$ for all $x \in \overline{\Omega}$ where $p_*(x) = \frac{\alpha(x)p(x)}{\alpha(x)+1}$ and

$$p^{\theta}(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases}.$$

Theorem 1 *Let $a^{-\alpha(.)} \in L^1(\Omega)$ with $\alpha(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$. Then there exists a compact embedding between $W_{a,b}^{2,p(.)}(\Omega)$ and $L_a^{p(.)}(\partial\Omega)$.*

Proof Take any $u \in W_{a,b}^{2,p(.)}(\Omega)$. Then we have

$$\begin{aligned} \|u\|_{a,b}^{1,p(.)} \leq & \|u\|_{p(.),b} + \|\nabla u\|_{p(.),a} + \sum_{|\alpha|=2} \|D^{\alpha} u\|_{p(.),a} \\ = & \|u\|_{a,b}^{2,p(.)}. \end{aligned}$$

Thus, we get that the continuous embedding between $W_{a,b}^{2,p(.)}(\Omega)$and $W_{a,b}^{1,p(.)}(\Omega)$ hold. Also by [26, Corollary 2], there exists the compact embedding between $W_{a,b}^{1,p(.)}(\Omega)$ and $L_a^{p(.)}(\partial\Omega)$. Therefore, we obtain that the space $W_{a,b}^{2,p(.)}(\Omega)$ is compactly embedded into space $L_a^{p(.)}(\partial\Omega)$.

Using the continuous embedding between $W_{a,b}^{2,p(.)}(\Omega)$and $W_{a,b}^{1,p(.)}(\Omega)$ and by [26, Corollary 3], the proof of the following theorem is easy to see.

Theorem 2 *The space $W_{a,b}^{2,p(.)}(\Omega)$ is compactly embedded in the space $L_b^{q(.)}(\Omega)$.*

Theorem 3 *Let Ω be a bounded domain with Lipschitz boundary. If the maximal operator is bounded on $L_a^{p(.)}(\Omega)$. Then the norms $\|u\|_{\partial}$ and $\|u\|_{a,b}^{2,p(.)}$ are equivalent on $W_{a,b}^{2,p(.)}(\Omega)$, where*

$$\|u\|_{\partial} = \|\Delta u\|_{p(.),a} + \|u\|_{p(.),a,\partial\Omega}.$$

Proof If we use the embedding $W^{2,p(.)}_{a,b}(\Omega) \hookrightarrow L^{p(.)}_{a}(\partial\Omega)$ in Theorem 1, then there exists a $C_1 > 0$ such that

$$\|u\|_{p(.),a,\partial\Omega} \leq \left(C_1 - 1\right)\|u\|^{2,p(.)}_{a,b}$$

for all $u \in W^{2,p(.)}_{a,b}(\Omega)$. Thus, we have

$$\|u\|_{\partial} \leq C_1\|u\|^{2,p(.)}_{a,b} \tag{2}$$

for all $u \in W^{2,p(.)}_{a,b}(\Omega)$. Then we have $u \in W^{1,p(.)}_{a,b}(\Omega)$. By [26, Theorem 4], there exists $C_2 > 0$ such that

$$\|u\|_{p(.),b} \leq C_2\left\{\|\nabla u\|_{p(.),a} + \|u\|_{p(.),a,\partial\Omega}\right\}. \tag{3}$$

By [27], it is known that $\frac{\partial^2 u}{\partial x_i \partial x_j} = -R_iR_j(\Delta u)$ where $R_i u$ is called the Riesz transform of u. Since the maximal operator is bounded on $L^{p(.)}_{a}(\Omega)$, the Riesz transform is bounded on $L^{p(.)}_{a}(\Omega)$, [28]. This yields that there is a $C_3 > 0$ such that

$$\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{p(.),a} = \left\|R_iR_j(\Delta u)\right\|_{p(.),a} \leq C_3\|\Delta u\|_{p(.),a}$$

for $i, j = 1, 2, ..., N$. In addition, we can get

$$\sum_{|\alpha|=2}\|D^{\alpha}u\|_{p(.),a} \leq C_4\|\Delta u\|_{p(.),a} \tag{4}$$

for $C_4 > 0$. Also by [29, Theorem 4.4], we get

$$|\nabla u| \leq C_5|M(\Delta u)|,$$

where C_5 is only depend on Ω and N. Since the maximal operator is bounded on $L^{p(.)}_{a}(\Omega)$, there is a $C_6 > 0$ such that

$$\|\nabla u\|_{p(.),a} \leq C_6\|\Delta u\|_{p(.),a}. \tag{5}$$

Combining the inequalities (3), (4) and (5), there exists a $C_7 > 0$ such that

$$\begin{aligned}\|u\|^{2,p(.)}_{a,b} &= \|u\|_{p(.),b} + \|\nabla u\|_{p(.),a} + \sum_{|\alpha|=2}\|D^{\alpha}u\|_{p(.),a} \\ &\leq C_7\left\{\|\Delta u\|_{p(.),a} + \|u\|_{p(.),a,\partial\Omega}\right\} = C_7\|u\|_{\partial}.\end{aligned} \tag{6}$$

Hence by (2) and (6), we get the desired result.

Let $\beta \in L^{\infty}(\partial\Omega)$ such that

$$\beta^{-} = \inf_{x\in\partial\Omega}\beta(x) > 0.$$

Define

$$\|u\|_{\beta(x)} = \left\{ t > 0 \,:\, \psi_{\beta(x)}\left(\frac{u}{t}\right) \leq 1 \right\}$$

where

$$\psi_{\beta(x)}(u) = \int_{\Omega} a(x)|\Delta u(x)|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u(x)|^{p(x)}d\sigma$$

for any $u \in W^{2,p(.)}_{a,b}(\Omega)$ where $d\sigma$ is measure on boundary $\partial\Omega$. Also, if we consider the Theorem 3, the norms $\|.\|_{\beta(x)}$ and $\|.\|^{2,p(.)}_{a,b}$ are equivalent on $W^{2,p(.)}_{a,b}(\Omega)$.

The following Proposition can be easily seen by the references [16, 17, 30, 31].

Proposition 2 *The following statements are true.*

(i) $\|u\|^{p^-}_{\beta(x)} \leq \psi_{\beta(x)}(u) \leq \|u\|^{p^+}_{\beta(x)}$ with $\|u\|_{\beta(x)} \geq 1$,
(ii) $\|u\|^{p^+}_{\beta(x)} \leq \psi_{\beta(x)}(u) \leq \|u\|^{p^-}_{\beta(x)}$ with $\|u\|_{\beta(x)} \leq 1$,
(iii) $\min\left\{ \|u\|^{p^-}_{\beta(x)}, \|u\|^{p^+}_{\beta(x)} \right\} \leq \psi_{\beta(x)}(u) \leq \max\left\{ \|u\|^{p^-}_{\beta(x)}, \|u\|^{p^+}_{\beta(x)} \right\}$,
(iv) $\left\|u - u_k\right\|_{\beta(x)} \to 0$ if and only if $\psi_{\beta(x)}(u - u_k) \to 0$ as $k \to \infty$,
(v) $\left\|u_k\right\|_{\beta(x)} \to \infty$ if and only if $\psi_{\beta(x)}(u_k) \to \infty$ as $k \to \infty$.

The following Proposition can be proved by [30, Proposition 3.2].

Proposition 3 *Let us define the functional* $L_{\beta(x)} \,:\, W^{2,p(.)}_{a,b}(\Omega) \to \mathbb{R}$ *by*

$$L_{\beta(x)}(u) = \int_{\Omega} \frac{a(x)}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u(x)|^{p(x)}d\sigma$$

for all $u \in W^{2,p(.)}_{a,b}(\Omega)$. *Then we have* $L_{\beta(x)} \in C^1\left(W^{2,p(.)}_{a,b}(\Omega), \mathbb{R}\right)$ *and*

$$L'_{\beta(x)}(u)(v) =< L'_{\beta(x)}(u), v >= \int_{\Omega} a(x)|\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_{\partial\Omega} \beta(x)|u(x)|^{p(x)-2}uv d\sigma$$

for any $u, v \in W^{2,p(.)}_{a,b}(\Omega)$. *Moreover, we have*

(i) $L'_{\beta(x)} \,:\, W^{2,p(.)}_{a,b}(\Omega) \longrightarrow W^{-2,p'(.)}_{a^*,b^*}(\Omega)$ is continuous, bounded and strictly monotone operator,
(ii) $L'_{\beta(x)} \,:\, W^{2,p(.)}_{a,b}(\Omega) \longrightarrow W^{-2,p'(.)}_{a^*,b^*}(\Omega)$ is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in $W^{2,p(.)}_{a,b}(\Omega)$ and $\limsup_{n\longrightarrow\infty} L'_{\beta(x)}(u_n)(u_n - u) \leq 0$, then $u_n \longrightarrow u$ in $W^{2,p(.)}_{a,b}(\Omega)$,
(iii) $L'_{\beta(x)} \,:\, W^{2,p(.)}_{a,b}(\Omega) \longrightarrow W^{-2,p'(.)}_{a^*,b^*}(\Omega)$ is a homeomorphism.

Definition 1 If the function $u \in W^{2,p(.)}_{a,b}(\Omega)$ has the following condition

$$\int_{\Omega} a(x)|\nabla u|^{p(x)-2}\Delta u\Delta v dx + \int_{\partial\Omega} \beta(x)|u(x)|^{p(x)-2}uv d\sigma - \lambda \int_{\Omega} b(x)|u|^{q(x)-2}uv dx = 0$$

for all $v \in W^{2,p(.)}_{a,b}(\Omega)$, then u is called a weak solution of the problem (1).

Zhikov and Surnachev [32] prove that a sufficient condition for the density of smooth functions in the weighted Sobolev space with variable exponent is obtained. Moreover, it is known that the space $C^{\infty}_0(\Omega)$ is a subspace of $W^{1,p(.)}_{a,b}(\Omega)$, and $W^{1,p(.)}_{a,b}(\Omega) = W^{1,p(.)}(\Omega)$ for $0 < a_1 \le a(x) \le a_2, 0 < b_1 \le b(x) \le b_2$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Thus we obtain that the spaces $C^{\infty}_0(\Omega)$ and $C^{\infty}(\Omega) \cap W^{1,p(.)}_{a,b}(\Omega)$ is dense in $W^{1,p(.)}_{a,b}(\Omega)$ since log-Hölder continuity of the exponent is sufficient for the density of smooth functions, see [28, Theorem 9.1.8], [33, 34].

The class $A_{p(.)}$ is set of weights w such that

$$\|w\|_{A_{p(\cdot)}} = \sup_{B\in\beta} |B|^{-p_B}\|w\|_{L^1(B)}\left\|\frac{1}{w}\right\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}(B)} < \infty$$

where ß denotes the set of all balls in Ω, $p_B = \left(\frac{1}{|B|}\int_B \frac{1}{p(x)}dx\right)^{-1}$ and $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$. If $a \in A_{p(.)}$, $0 < b_1 \le b(x) \le b_2$ and $p(.) \in P^{\log}(\mathbb{R}^n)$, i.e. $p(.)$ satisfy log-Hölder continuity condition, then $C^{\infty}_0(\Omega)$ is dense in $W^{2,p(.)}_{a,b}(\Omega)$ [13, 35]. So, the weak solution of the problem (1) is well defined. In this work, we assume that $a \in A_{p(.)}$, $0 < b_1 \le b(x) \le b_2$ and $p(.) \in P^{\log}(\Omega)$.

For a weak solution to (1), we will introduce the functional $\Phi_\lambda : W^{1,p(.)}_{a,b}(\Omega) \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \int_{\Omega} \frac{a(x)}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u(x)|^{p(x)}d\sigma - \lambda \int_{\Omega} \frac{b(x)}{q(x)}|u|^{q(x)}dx$$

for any $\lambda > 0$. The function Φ_λ is well defined from $W^{2,p(.)}_{a,b}(\Omega)$ to $\mathbb{R}$ and $\Phi_\lambda \in C^1\left(W^{2,p(.)}_{a,b}(\Omega), \mathbb{R}\right)$ with the derivative given by

$$\Phi'_\lambda(u)(v) = \int_{\Omega} a(x)|\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_{\partial\Omega} \beta(x)|u(x)|^{p(x)-2}uv d\sigma - \lambda \int_{\Omega} b(x)|u|^{q(x)-2}uv dx$$

for all $u, v \in W^{2,p(.)}_{a,b}(\Omega)$. The operator $\Phi'_\lambda(u)$ is in the dual space $W^{-2,p'(.)}_{a^*,b^*}(\Omega)$. It is also note that $u \in W^{1,p(.)}_{a,b}(\Omega)$ is a weak solution of the problem (1) if and only if u is a critical point of Φ_λ.

Main Results

Lemma 1 *If there is a constant $\eta > 0$ such that $p^+ < \eta < q^-$, then the operator Φ_λ satisfies the (PS) condition.*

Proof Let $\left(u_n\right)_{n\in\mathbb{N}} \subset W^{2,p(.)}_{a,b}(\Omega)$ be such that

$$\sup\left|\Phi_\lambda\left(u_n\right)\right| \le M,\ \Phi'_\lambda\left(u_n\right) \to 0 \text{ in } W^{-1,p'(.)}_{a^*,b^*}(\Omega) \text{ as } n \to \infty, \tag{7}$$

i.e. $\left(u_n\right)_{n\in\mathbb{N}}$ is a (PS) sequence. Now, we will show that the sequence $\left(u_n\right)_{n\in\mathbb{N}}$ is bounded in $W^{2,p(.)}_{a,b}(\Omega)$. Assume that the sequence $\left(u_n\right)_{n\in\mathbb{N}}$ is not bounded. Thus we can say $\|u_n\|_{\beta(x)} > 1$ for all $n \in \mathbb{N}$. Then we conclude

$$\begin{aligned}
M + \|u_n\|_{\beta(x)} \ge& \Phi_\lambda\left(u_n\right) - \frac{1}{\eta}\left\langle \Phi'_\lambda\left(u_n\right), u_n\right\rangle \\
=& \int_\Omega \frac{a(x)}{p(x)}\left|\Delta u_n\right|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}\left|u_n(x)\right|^{p(x)}d\sigma - \lambda\int_\Omega \frac{b(x)}{q(x)}\left|u_n\right|^{q(x)}dx \\
&- \frac{1}{\eta}\int_\Omega a(x)\left|\Delta u_n\right|^{p(x)}dx - \frac{1}{\eta}\int_{\partial\Omega} \beta(x)\left|u_n(x)\right|^{p(x)}d\sigma + \frac{\lambda}{\eta}\int_\Omega b(x)\left|u_n\right|^{q(x)}dx \\
\ge& \left(\frac{1}{p^+} - \frac{1}{\eta}\right)\Psi_{\beta(x)}(u_n) + \lambda\left(\frac{1}{\eta} - \frac{1}{q^-}\right)\int_\Omega b(x)\left|u_n\right|^{q(x)}dx \\
\ge& \left(\frac{1}{p^+} - \frac{1}{\eta}\right)\|u_n\|^{p^-}_{\beta(x)}.
\end{aligned}$$

If the we divide the last inequality by $\|u_n\|^{p^-}_{\beta(x)}$ and passe to the limit as $n \to \infty$, we find $\eta \le p^+$. That is a contradiction. Hence, we achieve that $\left(u_n\right)$ is bounded in $W^{2,p(.)}_{a,b}(\Omega)$. On the other hand, since $W^{2,p(.)}_{a,b}(\Omega)$ is a reflexive Banach space, there exists $u \in W^{2,p(.)}_{a,b}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{2,p(.)}_{a,b}(\Omega)$. Use the fact that the embedding $W^{2,p(.)}_{a,b}(\Omega) \hookrightarrow L^{q(.)}_b(\Omega)$ is compact. So, we get $u_n \to u$ in $L^{q(.)}_b(\Omega)$ and $\sup_n \|u_n\|_{\beta(x)} < \infty$. Also, by the Hölder inequality, there exist $C_8, C_9 > 0$ such that

$$\begin{aligned}
\left|\int_\Omega b(x)\left|u_n\right|^{q(x)-2}u_n\left(u_n - u\right)dx\right| &\le \int_\Omega b(x)^{\frac{1}{q(x)}+\frac{1}{r(x)}}\left|u_n\right|^{q(x)-1}\left|u_n - u\right|dx \\
&\le C_8\left\|\left|u_n\right|^{q(x)-1}b(x)^{\frac{1}{r(x)}}\right\|_{r(.)}.\left\|\left|u_n - u\right|b(x)^{\frac{1}{q(x)}}\right\|_{q(.)} \\
&= C_8\|u_n\|_{q(.),b}.\|u_n - u\|_{q(.),b} \\
&\le C_9\|u_n\|_{\beta(x)}.\|u_n - u\|_{q(.),b} \longrightarrow 0
\end{aligned}$$

where $\frac{1}{q(.)} + \frac{1}{r(.)} = 1$. Thus

$$\lim_{n\longrightarrow\infty}\int_\Omega b(x)\left|u_n\right|^{q(x)-2}u_n\left(u_n - u\right)dx = 0. \tag{8}$$

Using (7) and the boundedness of $\left\{u_n - u\right\}$ in $W^{2,p(.)}_{a,b}(\Omega)$, then we find

$$\Phi'_\lambda\left(u_n\right)\left(u_n - u\right) \to 0. \tag{9}$$

Combining (8) and (9), we obtain the

$$\lim_{n\longrightarrow\infty}\left(\int_{\Omega} a(x)\left|\Delta u_n\right|^{p(x)-2}\Delta u_n\left(\Delta u_n-\Delta u\right)dx+\int_{\partial\Omega}\beta(x)\left|u_n(x)\right|^{p(x)-2}u_n\left(u_n-u\right)d\sigma\right)=0.$$

That means

$$\lim_{n\longrightarrow\infty} L'_{\beta(x)}\left(u_n\right)(u_n-u)=0. \tag{10}$$

Using (10) and Proposition 3, we get that the sequence $\left\{u_n\right\}$ converges strongly to u in $W^{2,p(.)}_{a,b}(\Omega)$. This completes the proof.

Theorem 4 *Let $p^+<\eta<q^-$ for some $\eta>0$. Then, there is a nontrivial weak solution of the problem (1) for $p^+<\eta<q^-$.*

Proof From Lemma 1, the functional Φ_λ satisfies (PS) condition on $W^{2,p(.)}_{a,b}(\Omega)$. By the assumption $p^+<\eta<q^-$ and the embedding $W^{2,p(.)}_{a,b}(\Omega)\hookrightarrow L^{q(.)}_b(\Omega)$, we get

$$\begin{aligned}
\Phi_\lambda(u)&=\int_{\Omega}\frac{a(x)}{p(x)}|\Delta u|^{p(x)}dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u(x)|^{p(x)}d\sigma-\lambda\int_{\Omega}\frac{b(x)}{q(x)}|u|^{q(x)}dx\\
&\geq\frac{1}{p^+}\left(\int_{\Omega}a(x)|\Delta u|^{p(x)}dx+\int_{\partial\Omega}\beta(x)|u(x)|^{p(x)}d\sigma\right)-\frac{\lambda}{q^-}\int_{\Omega}b(x)|u|^{q(x)}dx\\
&\geq\frac{1}{p^+}\psi_{\beta(x)}(u)-\frac{\lambda}{q^-}\max\left\{\|u\|^{q^-}_{q(.),b},\|u\|^{q^+}_{q(.),b}\right\}\\
&\geq\frac{1}{p^+}\psi_{\beta(x)}(u)-\frac{\lambda}{q^-}\max\left\{\|u\|^{q^-}_{\beta(x)},\|u\|^{q^+}_{\beta(x)}\right\}\\
&\geq\frac{1}{p^+}\|u\|^{p^+}_{\beta(x)}-\frac{\lambda}{q^-}\|u\|^{q^-}_{\beta(x)}\\
&\geq\left(\frac{1}{p^+}-\frac{\lambda}{q^-}\right)\|u\|^{p^+}_{\beta(x)}
\end{aligned}$$

for $\|u\|_{\beta(x)}\leq 1$ and $p^+\lambda<q^-$. So, we obtain $\Phi_\lambda(u)>0$ for $\|u\|_{\beta(x)}=\rho$ sufficiently small. Moreover, we get

$$\begin{aligned}
\Phi_\lambda(tv)&=\int_{\Omega}\frac{a(x)t^{p(x)}}{p(x)}|\Delta v|^{p(x)}dx+\int_{\partial\Omega}\frac{\beta(x)t^{p(x)}}{p(x)}|v|^{q(x)}d\sigma-\lambda\int_{\Omega}\frac{b(x)t^{q(x)}}{q(x)}|v|^{q(x)}dx\\
&\leq t^{p^+}\psi_{\beta(x)}(v)-\lambda t^{q^-}\int_{\Omega}b(x)|v|^{q(x)}dx\\
&\leq t^{p^+}\psi_{\beta(x)}(v)-\lambda t^{\eta^-}\int_{\Omega}b(x)|v|^{q(x)}dx\longrightarrow-\infty
\end{aligned}$$

as $t\longrightarrow\infty$ for $v\in W^{2,p(.)}_{a,b}(\Omega)-\{0\}$. Also, we have $\Phi_\lambda(0)=0$. If we consider the Mountain Pass Theorem (see [36]), then there is a nontrivial weak solution of the problem (1).

To consider Ekeland's variational principle to the problem (1), we need the following two lemmas.

Lemma 2 *Assume that* $1 < q^- < p^- < q^+ < p^+$ *is satisfied. Then, there is a* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$ *there exist* $\mu_1, \mu_2 > 0$ *such that* $\Phi_\lambda(u) \geq \mu_2 > 0$ *for any* $u \in W^{1,p(.)}_{a,b}(\Omega)$ *with* $\|u\|_{\beta(x)} = \mu_1$.

Proof Since $W^{1,p(.)}_{a,b}(\Omega) \hookrightarrow L^{q(.)}_b(\Omega)$ holds, there exists a $C_{10} > 0$ such that

$$\|u\|_{q(.),b} \leq C_{10}\|u\|_{\beta(x)} \tag{11}$$

for any $u \in W^{1,p(.)}_{a,b}(\Omega)$. Now, we choose $\mu_1 \in (0, 1)$ such that $\mu_1 < \frac{1}{C_{10}}$. By (11), we have $\|u\|_{q(.),b} < 1$ and

$$\int_\Omega |u|^{q(x)} b(x)dx \leq \|u\|^{q^-}_{q(.),b} \tag{12}$$

for all $u \in W^{1,p(.)}_{a,b}(\Omega)$ with $\|u\|_{\beta(x)} = \mu_1$. If we consider the relations (11) and (12), then we have

$$\int_\Omega |u|^{q(x)} b(x)dx \leq C^{q^-}_{10}\|u\|^{q^-}_{\beta(x)}$$

for all $u \in W^{1,p(.)}_{a,b}(\Omega)$ with $\|u\|_{\beta(x)} = \mu_1$. This yields that

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+}\Psi_{\beta(x)}(u) - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)} b(x)dx \\
&\geq \frac{1}{p^+}\|u\|^{p^+}_{\beta(x)} - \frac{\lambda}{q^-}C^{q^-}_{10}\|u\|^{q^-}_{\beta(x)} \\
&\geq \frac{1}{p^+}\|u\|^{p^+}_{\beta(x)} - \frac{\lambda}{q^-}C^{q^-}_{2}\|u\|^{q^-}_{\beta(x)} \\
&= \frac{1}{p^+}\mu_1^{p^+} - \frac{\lambda}{q^-}C_2^{q^- \mu_1 q^-} \\
&= \mu_1^{q^-}\left(\frac{1}{p^+}\mu_1^{p^+-q^-} - \frac{\lambda}{q^-}C^{q^-}_{10}\right).
\end{aligned}$$

Taking

$$\lambda^* = \frac{\rho^{p^+-q^-}}{2p^+}.\frac{q^-}{C_2^{q^-}}, \tag{13}$$

then there exists $\mu_2 = \frac{\rho^{p^+}}{2p^+} > 0$ such that $\Phi_\lambda(u) \geq \mu_2 > 0$ for all $\lambda \in (0, \lambda^*)$ and $u \in W^{1,p(.)}_{a,b}(\Omega)$ with $\|u\|_{\beta(x)} = \mu_1$.

Lemma 3 *Let* $1 < q^- < p^- < q^+ < p^+$. *There is a function u in* $W^{1,p(.)}_{a,b}(\Omega)$ *satisfying* $u \geq 0$, $u \neq 0$ *and* $\Phi_\lambda(tu) < 0$ *when t is small enough.*

Proof Since $q^- < p^-$, we get $q^- + \varepsilon_0 < p^-$ for a $\varepsilon_0 > 0$. From $q(.) \in C\left(\overline{\Omega}\right)$, there is an open subset $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \varepsilon_0$ for any $x \in \Omega_0$. This yields that

$$q(x) \leq q^- + \varepsilon_0 < p^- \tag{14}$$

for any $x \in \Omega_0$. Let us consider $u \in C_0^\infty(\Omega)$ satisfying $\overline{\Omega_0} \subset \operatorname{supp} u$, $u(x) = 1$ for all $x \in \overline{\Omega_0}$ and $0 \leq u \leq 1$ in Ω. Now, we take $\|u\|_{\beta(x)} = \|\Delta u\|_{p(.),a} = 1$, or equivalently,

$$\int_\Omega |\Delta u|^{p(x)} a(x)dx = 1. \tag{15}$$

By (14) and (15), we have

$$\begin{aligned}
\Phi_\lambda(tu) &= \int_\Omega \frac{a(x)t^{p(x)}}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)t^{p(x)}}{p(x)}|u|^{p(x)}d\sigma - \lambda \int_\Omega \frac{b(x)t^{q(x)}}{q(x)}|u|^{q(x)}dx \\
&\leq \frac{t^{p^-}}{p^-}\psi_{\beta(x)}(u) - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|u|^{q(x)}b(x)dx \\
&\leq \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+}\int_{\Omega_0} t^{q(x)}|u|^{q(x)}b(x)dx \\
&\leq \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^-+\varepsilon_0}}{q^+}\int_{\Omega_0} |u|^{q(x)}b(x)dx
\end{aligned}$$

for all $t \in (0,1)$ and $\|u\|_{\beta(x)} = \psi_{\beta(x)}(u) = 1$. Moreover, we obtain that $\Phi_\lambda(tu) < 0$ for all $t < \delta^{\frac{1}{p^- - q^- - \varepsilon_0}}$ with $0 < \delta < \min\left\{1, \frac{\lambda p^-}{q^+}\int_{\Omega_0} |u|^{q(x)}b(x)dx\right\}$.

Theorem 5 *Suppose that* $1 < q^- < p^- < q^+ < p^+$ *is satisfied. Then, there is a* $\lambda^* > 0$ *satisfying any* $\lambda \in (0, \lambda^*)$ *is an eigenvalue for the problem* (1).

Proof Let $\lambda \in (0, \lambda^*)$ where λ^* is given by (13). By Lemma 2 it follows that on the boundary of the ball centered at the origin and of radius μ in $W_{a,b}^{2,p(.)}(\Omega)$ denoted by $B_\mu(0)$, we have

$$\inf_{\partial B_\mu(0)} \Phi_\lambda > 0.$$

If we consider the Lemma 3, there is a $u \in W_{a,b}^{2,p(.)}(\Omega)$ such that $\Phi_\lambda(tu) < 0$ for $t > 0$ small enough. Moreover, we get

$$\Phi_\lambda(u) \geq \frac{1}{p^+}\|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^-}C_{10}^{q^-}\|u\|_{\beta(x)}^{q^-}$$

for any $u \in W_{a,b}^{2,p(.)}(\Omega)$. This yields that

$$-\infty < \underline{\gamma} = \inf_{B_\mu(0)} \Phi_\lambda < 0.$$

Now, let us take $\varepsilon > 0$. This yields that

$$0 < \varepsilon < \inf_{\partial B_\mu(0)} \Phi_\lambda - \inf_{B_\mu(0)} \Phi_\lambda.$$

Using the Ekeland's variational principle to the functional $\Phi_\lambda : \overline{B_\mu(0)} \longrightarrow \mathbb{R}$, then we find $u_\varepsilon \in \overline{B_\mu(0)}$ such that

$$\Phi_\lambda\left(u_\varepsilon\right) < \inf_{\overline{B_\mu(0)}} \Phi_\lambda + \varepsilon,$$
$$\Phi_\lambda\left(u_\varepsilon\right) < \Phi_\lambda(u) + \varepsilon\left\|u - u_\varepsilon\right\|_{\beta(x)},\ u \neq u_\varepsilon.$$

Since the relations

$$\Phi_\lambda\left(u_\varepsilon\right) \le \inf_{\overline{B_\mu(0)}} \Phi_\lambda + \varepsilon \le \inf_{B_\mu(0)} \Phi_\lambda + \varepsilon < \inf_{\partial B_\mu(0)} \Phi_\lambda$$

hold, we have $u_\varepsilon \in B_\mu(0)$. Define a function $\tau : \overline{B_\mu(0)} \longrightarrow \mathbb{R}$ as $\tau(u) = \Phi_\lambda(u) + \varepsilon\left\|u - u_\varepsilon\right\|_{\beta(x)}$. It is obvious that u_ε is a minimum point of τ. This follows that

$$\frac{\tau\left(u_\varepsilon + t\vartheta\right) - \tau\left(u_\varepsilon\right)}{t} \ge 0$$

and

$$\frac{\Phi_\lambda\left(u_\varepsilon + t\vartheta\right) - \Phi_\lambda\left(u_\varepsilon\right)}{t} + \varepsilon\|\vartheta\|_{\beta(x)} \ge 0$$

for small $t > 0$ and all $\vartheta \in B_\mu(0)$. As $t \longrightarrow 0$, we have

$$\left\langle \Phi'_\lambda\left(u_\varepsilon\right), \vartheta \right\rangle + \varepsilon\|\vartheta\|_{\beta(x)} \ge 0. \tag{16}$$

If we replace ϑ in (16) with $-\vartheta$, then we get $\left|\left\langle \Phi'_\lambda\left(u_\varepsilon\right), \vartheta \right\rangle\right| \le \varepsilon\|\vartheta\|_{\beta(x)}$. It is known that $\Phi'_\lambda\left(u_\varepsilon\right) \in W^{-2,p'(.)}_{a^*,b^*}(\Omega)$. If we consider the definition of operator norm for $W^{-2,p'(.)}_{a^*,b^*}(\Omega)$, then we obtain $\left\|\Phi'_\lambda\left(u_\varepsilon\right)\right\|_{W^{-2,p'(.)}_{a^*,b^*}(\Omega)} \le \varepsilon$. Therefore, we get that there exists a sequence $\left\{z_n\right\} \subset B_\mu(0)$ such that

$$\Phi_\lambda\left(z_n\right) \longrightarrow \underline{\gamma} \text{ and } \Phi'_\lambda\left(z_n\right) \longrightarrow 0. \tag{17}$$

Since $\left\{z_n\right\} \subset B_\mu(0)$ and $\left\|z_n\right\|_{\beta(x)} < \mu$, the sequence $\left\{z_n\right\}$ is bounded in $W^{2,p(.)}_{a,b}(\Omega)$. By Lemma 1, we get that $\left\{z_n\right\}$ converges strongly to z in $W^{2,p(.)}_{a,b}(\Omega)$. This follows by (17) that

$$\Phi_\lambda(z) = \underline{\gamma} < 0 \text{ and } \Phi'_\lambda(z) = 0.$$

Finally, we obtain that z is a non-trivial weak solution for the problem (1) and thus any $\lambda \in (0, \lambda^*)$ is an eigenvalue of the problem (1).

Now, we give some well-known assertions for the multiplicity solutions of the problem (1).

Lemma 4 (see [2]) *Assume that X is a reflexive and separable Banach space. Then there are $\left\{x_k\right\} \subset X$ and $\left\{y_k\right\} \subset X^*$ such that*

$$X = \overline{span\left\{x_k | k = 1, 2, \dots\right\}},\ X^* = \overline{span\left\{y_k | j = 1, 2, \dots\right\}}$$

and

$$\langle y_k, x_l \rangle = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

where $\langle \cdot, \cdot \rangle$ *define as the duality product between* X *and* X^*.

Briefly, we denote $X_k = span\{x_k\}, Y_l = \bigoplus_{k=1}^{l} X_k, Z_l = \bigoplus_{k=l}^{\infty} X_k$.

Lemma 5 *Let us denote*

$$c_l = \sup\left\{ \|u\|_{q(.),b} : \|u\|_{\beta(x)} = 1,\ u \in Z_l \right\}.$$

Then, we have $\lim_{l \longrightarrow \infty} c_l = 0$.

Proof If we use the continuous embedding $W^{2,p(.)}_{a,b}(\Omega) \hookrightarrow L^{q(.)}_b(\Omega)$ and the approach in [2, Lemma 4.9], then we get the desired result.

By Lemma 5, we conclude the following result.

Theorem 6 *Let* $p^+ < q^-$. *Then there exist infinite many pairs of solutions of the problem* (1), *i.e., the functional* Φ_λ *has a sequence of critical points* $\{u_n\}$ *such that* $\Phi_\lambda(u_n) \longrightarrow +\infty$ *as* $n \longrightarrow +\infty$.

Proof It is known that Φ_λ is an even functional and satisfies the (PS) condition. Now, we will prove that if l is large enough, then there are $\eta_l > \gamma_l > 0$ such that

$$\begin{aligned} &(A_1)\ d_l := \inf\left\{ \Phi_\lambda(u) \middle| u \in Z_l,\ \|u\|_{\beta(x)} = \gamma_l \right\} \longrightarrow \infty \text{ as } k \longrightarrow \infty \\ &(A_2)\ e_l := \max\left\{ \Phi_\lambda(u) \middle| u \in Y_l,\ \|u\|_{\beta(x)} = \eta_l \right\} \leq 0. \end{aligned}$$

(A_1) Let $u \in Z_l$ such that $\|u\|_{\beta(x)} = \gamma_l > 1$. Then we conclude that

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{p^+} \Psi_{\beta(x)}(u) - \frac{\lambda}{q^-} \int_\Omega |u|^{q(x)} b(x) dx \\ &\geq \frac{1}{p^+} \Psi_{\beta(x)}(u) - \frac{\lambda}{q^-} \max\left\{ \|u\|^{q^-}_{q(.),b}, \|u\|^{q^+}_{q(.),b} \right\} \\ &\geq \frac{1}{p^+} \|u\|^{p^-}_{\beta(x)} - \frac{\lambda}{q^-} \max\left\{ \|u\|^{q^-}_{q(.),b}, \|u\|^{q^+}_{q(.),b} \right\} \\ &\geq \begin{cases} \frac{1}{p^+} \|u\|^{p^-}_{\beta(x)} - \lambda, & \|u\|_{q(.),b} \leq 1 \\ \frac{1}{p^+} \|u\|^{p^-}_{\beta(x)} - \lambda \alpha_l^{q^+} \|u\|^{q^+}_{\beta(x)}, & \|u\|_{q(.),b} > 1 \end{cases} \\ &\geq \frac{1}{p^+} \|u\|^{p^-}_{\beta(x)} - \lambda \alpha_l^{q^+} \|u\|^{q^+}_{\beta(x)} \\ &= \frac{1}{p^+} \gamma_l^{p^-} - \lambda \alpha_l^{q^+} \gamma_l^{q^+} \end{aligned}$$

If we define $\gamma_l = \left(\lambda q^+ \alpha_l^{q^+} \right)^{\frac{1}{p^- - q^+}}$, then we have

$$\Phi_\lambda(u) \geq \frac{1}{p^+}\left(\lambda q^+ \alpha_l^{q^+}\right)^{\frac{p^-}{p^- - q^+}} - \lambda \alpha_l^{q^+}\left(\lambda q^+ \alpha_l^{q^+}\right)^{\frac{q^+}{p^- - q^+}}$$
$$= \left(\frac{1}{p^+} - \frac{1}{q^+}\right)\left(\lambda q^+ \alpha_l^{q^+}\right)^{\frac{p^-}{p^- - q^+}} \longrightarrow \infty$$

as $l \longrightarrow \infty$ due to $p^+ < q^+$ and $c_l \longrightarrow 0$. So $\left(A_1\right)$ holds.

$\left(A_2\right)$ Suppose that $u \in Y_l$ is a function satisfying $\|u\|_{\beta(x)} = \eta_l > \gamma_l > 1$. This follows that

$$\Phi_\lambda(u) \leq \frac{1}{p^-}\|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^+}\int_\Omega |u|^{q(x)} b(x)dx$$
$$\leq \frac{1}{p^-}\|u\|_{\beta(x)}^{p^+} - \frac{\lambda}{q^+}\min\left\{\|u\|_{q(.),b}^{q^-}, \|u\|_{q(.),b}^{q^+}\right\}.$$

If we consider the fact that the space Y_l has finite dimension, the norms $\|.\|_{\beta(x)}$ and $\|.\|_{q(.),b}$ are equivalent. By the Fountain Theorem (see [36, Theorem 3.6]), we have

$$\Phi_\lambda(u) \to -\infty \text{ as } \|u\|_{\beta(x)} \to +\infty,\ u \in Y_l$$

due to $p^+ < q^-$.

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