ANNALI DELL'UNIVERSITA' DI FERRARA (2025) 71:37
https://doi.org/10.1007/s11565-025-00593-1

On a nonlinear partial differential equation with a $p(.)$-triharmonic operator

Ismail Aydın[1] · **Khaled Kefi**[2]

Received: 5 November 2024 / Accepted: 31 March 2025 / Published online: 30 April 2025

Abstract
We study the following $p(.)$-triharmonic problem

$$\begin{cases} \Delta^3_{p(.)}u + a(x)\,|u|^{p(x)-2}\,u = \lambda(V_1(x)\,|u|^{q(x)-2}\,u - V_2(x)\,|u|^{\alpha(x)-2}\,u), & \text{in } \Omega \\ |\nabla\Delta u|^{p(x)-2}\,\frac{\partial u}{\partial \upsilon} + \beta(x)\,|u|^{p(x)-2}\,u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, and $\lambda > 0$ is a parameter. Using some variational methods and compact embedding results for variable exponent third-order Sobolev space, we obtain the existence of weak solutions for the problem.

Keywords $p(.)$-triharmonic operator · Variational methods · Variable exponent Sobolev spaces

Mathematics Subject Classification 35J35 · 46E35 · 35J60 · 35J70

1 Introduction

In this paper, we are concerned with the existence of weak solutions for the $p(.)$-triharmonic problem

$$\begin{cases} \Delta^3_{p(.)}u + a(x)\,|u|^{p(x)-2}\,u = \lambda(V_1(x)\,|u|^{q(x)-2}\,u - V_2(x)\,|u|^{\alpha(x)-2}\,u), & \text{in } \Omega \\ |\nabla\Delta u|^{p(x)-2}\,\frac{\partial u}{\partial \upsilon} + \beta(x)\,|u|^{p(x)-2}\,u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

✉ Ismail Aydın
iaydin@sinop.edu.tr

Khaled Kefi
khaled_kefi@yahoo.fr

1 Department of Mathematics, Sinop University, Korucuk Street, 57100 Sinop, Turkey

2 Center for Scientific Research and Entrepreneurship, Northern Border University, Arar 73213, Saudi Arabia

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with smooth boundary, $a, V_1, V_2, q, \alpha \in L^\infty(\Omega)$ are nonnegative functions, $\beta \in L^\infty(\partial\Omega)$ such that $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$, $p \in C\left(\overline{\Omega}\right)$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, $\frac{\partial u}{\partial \upsilon}$ is the outer unit normal derivative of u on $\partial\Omega$, $\Delta^3_{p(.)}u = \operatorname{div}(\Delta(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u)$ is the $p(.)$-triharmonic operator of sixth order, $\lambda, \mu > 0$ are real numbers.

Partial differential equations involving $p(.)$-Laplacian operators have been received considerable attention in recent years. Especially, in application there are several interesting and difficult problems, such as electrorlogical fluids, image processing, flow in porous media, elasticity and calculus of variation (see [2, 3, 8, 9, 11, 15, 23, 24, 26, 29, 32]).

In 2014 Kong [20] found the existence of one nontrivial weak solution of the following fourth order elliptic equation with a $p(.)$ -biharmonic operator and Navier boundary conditions

$$\begin{cases} \Delta^2_{p(.)}u + a(x)\,|u|^{p(x)-2}\,u = \lambda\left(b(x)\,|u|^{\gamma(x)-2}\,u - c(x)\,|u|^{\beta(x)-2}\,u\right), & \text{in } \Omega \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary, $p, a, b, c, \beta, \gamma \in C\left(\overline{\Omega}\right)$ with $p(x) > 1$ on $\overline{\Omega}$, $\lambda > 0$ is a parameter, $\Delta^2_{p(.)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(.)$-biharmonic operator. In addition, Kefi and Rădulescu [18] investigate weak solutions of the problem (2) under more general conditions by Ekeland's variational principle.

In 2019, using Mountain Pass Theorem, Rahal [25] studied the existence of weak solutions to a class of $p(.)$-triharmonic operator with Navier boundary condition

$$\begin{cases} -M(\int\limits_\Omega \frac{1}{p(x)}\,|\nabla\Delta u|^{p(x)}\,dx)\Delta^3_{p(.)}u = \lambda\zeta(x)\,|u|^{\alpha(x)-2}\,u - \lambda\xi(x)\,|u|^{\beta(x)-2}\,u, & \text{in } \Omega \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with smooth boundary, $p, \zeta, \xi, \alpha, \beta \in C\left(\overline{\Omega}\right)$ are nonnegative functions such that $1 < p(x) < \frac{N}{3}$ for any $x \in \overline{\Omega}$. Moreover, Shokooh [27] showed the existence of infinitely many weak solutions under suitable conditions on his parameters. Precisely the author studied the following problem $p(.)$-triharmonic operator problem

$$\begin{cases} -\Delta^3_{p(.)}u = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases}$$

where μ is a nonnegative parameter, $\lambda > 0$, $f, g \in C^0\left(\overline{\Omega}\times\mathbb{R}\right)$ and $p \in C^0\left(\overline{\Omega}\right)$ with $\max\left\{3, \frac{N}{3}\right\} < p^- = \inf\limits_{x\in\overline{\Omega}} p(x) \leq p^+ = \sup\limits_{x\in\overline{\Omega}} p(x) < \infty$. For $p(.)$-triharmonic problems, see ([5, 6, 16, 27, 31]).

Inspired by the articles mentioned above, our aim is to show that problem (1) has nontrivial weak solutions under some suitable conditions. Moreover the multiplicity is also discussed.

2 Notation and preliminaries

Let $C_+\left(\overline{\Omega}\right) = \left\{p \in C\left(\overline{\Omega}\right) : \inf_{x\in\overline{\Omega}} p(x) > 1\right\}$. For any $p \in C_+\left(\overline{\Omega}\right)$, let

$$1 < p^- = \inf_{x\in\Omega} p(x) \leq p^+ = \sup_{x\in\Omega} p(x) < \infty$$

and

$$L^{p(.)}(\Omega) = \left\{ u \left| u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right. \right\}$$

with the (Luxemburg) norm

$$\|u\|_{p(.)} = \inf\left\{\xi > 0 : \varrho_{p(.)}\left(\frac{u}{\xi}\right) \leq 1\right\},$$

where the space $\left(L^{p(.)}(\Omega), \|.\|_{p(.)}\right)$ is a separable and reflexive Banach space.

The variable exponent Sobolev space $W^{k,p(.)}(\Omega)$ is defined by

$$W^{k,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) : D^\alpha u \in L^{p(.)}(\Omega), 0 \leq |\alpha| \leq k\right\},$$

where $\alpha \in \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}...\partial_{x_N}^{\alpha_N}}$. Then, $W^{k,p(.)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(.)} = \sum_{0\leq|\alpha|\leq k} \left\|D^\alpha u\right\|_{p(.)}$$

(see [12, 19]).

Theorem 1 (see [Theorem 2.3 in [12]]) *Let $p, q \in C_+\left(\overline{\Omega}\right)$ such that $q(x) < p_k^*(x)$ for any $x \in \overline{\Omega}$. Then there is a continuous and compact embedding $W^{k,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$, where*

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)}, & \text{if } p(x) < \frac{N}{k}, \\ +\infty, & \text{if } p(x) \geq \frac{N}{k}. \end{cases}$$

Theorem 2 (see [Theorem 2.4 in [13]]) *Assume that the set Ω is an open bounded domain with Lipschitz boundary, $k \in \mathbb{Z}^+$, $p \in C^0\left(\overline{\Omega}\right)$ with $p^- > 1$ and $kp^+ < N$. If $q \in S(\partial\Omega)$, where $S(\partial\Omega)$ is the set of all measurable real functions defined on Ω, and there exists a positive constant ε such that*

$$1 \leq q(x) < q(x) + \varepsilon \leq \frac{(N-1)\,p(x)}{N - kp(x)} \quad \text{for } x \in \partial\Omega,$$

then the boundary trace embedding $W^{k,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$ is compact.

Corollary 1 *Due to $p(x) > \frac{1}{3}$ for $x \in \Omega$, then the continuous embedding $W^{3,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$ is compact.*

Throughout this paper, we let the space $X = W^{3,p(x)}(\Omega)$. Define a norm $\|.\|$ of X by

$$\|u\| = \|u\|_{1,p(.)} + \|u\|_{2,p(.)} + \|u\|_{3,p(.)} .$$

To obtain the existence of a weak solution for problem (1), we introduce a norm for the space X, which will be used later. For $u \in X$, define

$$\|u\|_X = \inf\left\{\tau > 0 : \int_\Omega (\left|\frac{\nabla\Delta u}{\tau}\right|^{p(x)} + a(x)\left|\frac{u}{\tau}\right|^{p(x)})dx + \int_{\partial\Omega} \beta(x)\left|\frac{u}{\tau}\right|^{p(x)} d\sigma \leq 1\right\}.$$

Then, by Theorem 2.1 in [7], $\|u\|_X$ is also a norm on X which is equivalent to $\|u\|$ due to $\beta \in L^\infty(\partial\Omega)$ and $\beta^- = \inf_{x\in\partial\Omega}\beta(x) > 0$ (see [2, 7, 14, 17, 20, 21, 25, 27]).

Proposition 1 (see [7, 14, 21, 25]) *Let $\psi_{\beta(x)} = \int_\Omega (|\nabla\Delta u|^{p(x)} + a(x)|u|^{p(x)})dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} d\sigma$ with $\beta^- > 0$, where $d\sigma$ is the measure on the boundary of Ω. Hence, we have*

(i) $\|u\|_X \leq 1 \Rightarrow \|u\|_X^{p^+} \leq \psi_{\beta(x)}(u) \leq \|u\|_X^{p^-}$

(ii) $\|u\|_X \geq 1 \Rightarrow \|u\|_X^{p^-} \leq \psi_{\beta(x)}(u) \leq \|u\|_X^{p^+}$.

Proposition 2 (see [4, 7, 14, 17]) *Define the functional $\Psi_{\beta(x)} : X \to \mathbb{R}$ by*

$$\Psi_{\beta(x)}(u) = \int_\Omega \frac{1}{p(x)}(|\nabla\Delta u|^{p(x)} + a(x)|u|^{p(x)})dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)} d\sigma$$

for all $u \in X$. Then,

(i) *$\Psi_{\beta(x)} : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous, $\Psi_{\beta(x)} \in C^1(X, \mathbb{R})$, and its Gateaux derivative $\Psi'_{\beta(x)}$ is given by*

$$\Psi'_{\beta(x)}(u)(v) = \int_\Omega (|\nabla\Delta u|^{p(x)-2}\nabla\Delta u.\nabla\Delta v + a(x)|u|^{p(x)-2}uv)dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv d\sigma$$

for any $v \in X$.

(ii) *The mapping $\Psi'_{\beta(x)} : X \longrightarrow X^*$ is a strictly monotone, bounded homeomorphism, and is of type (S_+), i.e., if $u_n \rightharpoonup u$ in X and $\limsup_{n\longrightarrow\infty} \Psi'_{\beta(x)}(u_n)(u_n - u) \leq 0$, then $u_n \longrightarrow u$ in X.*

3 Main results

A function $u \in X$ is said to be a weak solution of the problem (1) if

$$\int_{\Omega} (|\nabla \Delta u|^{p(x)-2} \nabla \Delta u.\nabla \Delta v + a(x)\,|u|^{p(x)-2} uv)dx + \int_{\partial\Omega} \beta(x)\,|u|^{p(x)-2} uvd\sigma$$

$$-\lambda \int_{\Omega} (V_1(x)\,|u|^{q(x)-2} uv - V_2(x)\,|u|^{\alpha(x)-2} uv)dx = 0$$

for all $v \in X$.

In the sequel, define the energy functional $\Phi_\lambda : X \to \mathbb{R}$ denoted by

$$\Phi_\lambda(u) = \int_{\Omega} \frac{1}{p(x)}(|\nabla \Delta u|^{p(x)} + a(x)\,|u|^{p(x)})dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}\,|u|^{p(x)}\,d\sigma$$

$$-\lambda \int_{\Omega} (\frac{V_1(x)}{q(x)}\,|u|^{q(x)} - \frac{V_2(x)}{\alpha(x)}\,|u|^{\alpha(x)})dx$$

for $\lambda > 0$. It is known that Φ_λ is well defined from X to $\mathbb{R}$ and $\Phi_\lambda \in C^1(X, \mathbb{R})$, and its derivative at $u \in X$ is given by

$$\Phi'_\lambda(u)(v) = \int_{\Omega} (|\nabla \Delta u|^{p(x)-2} \nabla \Delta u.\nabla \Delta v + a(x)\,|u|^{p(x)-2} uv)dx + \int_{\partial\Omega} \beta(x)\,|u|^{p(x)-2} uvd\sigma$$

$$-\lambda \int_{\Omega} (V_1(x)\,|u|^{q(x)-2} uv - V_2(x)\,|u|^{\alpha(x)-2} uv)dx$$

for all $v \in X$. Moreover, the critical points of Φ_λ are the exact weak solutions of the problem (1).

Lemma 1 *There exists a* $\lambda^* > 0$ *such that*

$$\lambda^* = \inf_{u \in X, \|u\|_X > 1} \frac{\Psi_{\beta(x)}(u)}{\int_{\Omega} |u|^{p^-}\,dx}. \tag{3}$$

Proof By Theorem 1, X is continuously embedded into $L^{p^-}(\Omega)$. Then, there exists $C > 0$ such that

$$C\,\|u\|_{p^-} \le \|u\|_X \tag{4}$$

for any $u \in X$. In addition, by Proposition 1 (ii) we get

$$\psi_{\beta(x)}(u) \ge \|u\|_X^{p^-} \tag{5}$$

for all $u \in X$ with $\|u\|_X > 1$. From (4) and (5), we see that

$$\Psi_{\beta(x)}(u) \geq \frac{1}{p^+}\psi_{\beta(x)}(u) \geq \frac{C^{p^-}}{p^+}\int_\Omega |u|^{p^-}\,dx.$$

Hence, we find that there exists $\lambda^* > 0$ satisfying (3). This completes the proof. □

Lemma 2 *Let* $1 < \alpha^- \leq \alpha^+ < q^- \leq q^+ < p^- < p_3^*(x)$ *for any* $x \in \overline{\Omega}$. *For* $\lambda > 0$, *we obtain*

(i) Φ_λ *is bounded from below and coercive on* X.

(ii) Φ_λ *is sequentially weakly lower semi-continuous on* X.

Proof (i) Take $u \in X$ with $\|u\|_X > 1$. Then, we have

$$\begin{aligned}
\Phi_\lambda(u) &\geq \Psi_{\beta(x)}(u) - \lambda V_1^+\int_\Omega \frac{1}{q^-}|u|^{q(x)}\,dx + \lambda V_2^-\int_\Omega \frac{1}{\alpha^+}|u|^{\alpha(x)}\,dx \\
&\geq \Psi_{\beta(x)}(u) - \lambda\max\left\{V_1^+, V_2^-\right\}\int_\Omega \left(\frac{1}{q^-}|u|^{q(x)} - \frac{1}{\alpha^+}|u|^{\alpha(x)}\right)dx \\
&\geq \Psi_{\beta(x)}(u) - \frac{\lambda^*}{2}\int_\Omega |u|^{p^-}\,dx - c_\lambda|\Omega| \\
&\geq \frac{1}{2p^+}\|u\|_X^{p^-} - c_\lambda|\Omega|
\end{aligned}$$

for any $\lambda > 0$ by Lemma 1 and the inequality (2.3) in (Lemma 2.4, [20]). Hence, the functional Φ_λ is coercive, and is bounded from below.

(ii) Let $\{u_k\} \subset X$ such that $u_k \rightharpoonup u \in X$. By Proposition 2, $\Psi_{\beta(x)}$ is sequentially weakly lower semi-continuous on X. Then

$$\Psi_{\beta(x)}(u_k) \leq \liminf_{k\to\infty}\Psi_{\beta(x)}(u_k). \tag{6}$$

In addition, the space X is compactly embedded into $L^{q(.)}(\Omega)$ and $L^{\alpha(.)}(\Omega)$ by Theorem 1. So

$$u_k \to u \text{ in } L^{q(.)}(\Omega) \text{ and } L^{\alpha(.)}(\Omega). \tag{7}$$

Thus, from (6) and (7) we see that

$$\Phi_\lambda(u) \leq \liminf_{k\to\infty}\Phi_\lambda(u_k),$$

that is, Φ_λ is sequentially weakly lower semi-continuous on X. □

Lemma 3 *Let* $1 < \alpha^- \leq \alpha^+ < q^- \leq q^+ < p^-$. *Then, there exists* $\lambda_1 > 0$ *such that* $\inf_{u\in X}\Phi_\lambda(u) < 0$ *for all* $\lambda > \lambda_1$.

Proof Let $K \subset \Omega$ be a compact set such that $|K| > 0$. Then, there exists $t_0 > 1$ large enough such that

$$\frac{1}{\alpha^-} V_2^+ t_0^{\alpha^+} |\Omega \backslash K| - \left(\frac{1}{q^+} V_1^- t_0^{q^-} - \frac{1}{\alpha^-} V_2^+ t_0^{\alpha^+} \right) |K| < 0 \tag{8}$$

because of $t_0^{q^-} > t_0^{\alpha^+}$ for $\alpha^+ < q^-$. If we take $u_0 \in X$ such that $u(x) = t_0$ in K and $0 < u_0(x) \leq t_0$ in $\Omega \backslash K$, then we have

$$\begin{aligned}
\Phi_\lambda(u_0) \leq & \frac{1}{p^-} \int_{\Omega \backslash K} |\nabla \Delta u_0|^{p(x)} \, dx + \frac{1}{p^-} a^+ t_0^{p^+} |\Omega| + \frac{1}{p^-} \beta^+ t_0^{p^+} |\partial \Omega| \\
& - \lambda \left[\int_{\Omega \backslash K} \frac{V_1(x)}{q(x)} |u_0|^{q(x)} \, dx - \int_{\Omega \backslash K} \frac{V_2(x)}{\alpha(x)} |u_0|^{\alpha(x)} \, dx \right] \\
& - \lambda \left[\int_K \frac{V_1(x)}{q(x)} |t_0|^{q(x)} \, dx - \int_K \frac{V_2(x)}{\alpha(x)} |t_0|^{\alpha(x)} \, dx \right] \\
\leq & \frac{1}{p^-} \int_{\Omega \backslash K} |\nabla \Delta u_0|^{p(x)} \, dx + \frac{1}{p^-} \left(a^+ |\Omega| + \beta^+ |\partial \Omega| \right) t_0^{p^+} \\
& + \lambda \left[\frac{1}{\alpha^-} V_2^+ t_0^{\alpha^+} |\Omega \backslash K| - \left(\frac{1}{q^+} V_1^- t_0^{q^-} - \frac{1}{\alpha^-} V_2^+ t_0^{\alpha^+} \right) |K| \right]
\end{aligned} \tag{9}$$

If we combine (8) and (9), then we show that there exists $\lambda_1 > 0$ such that $\inf_{u \in X} \Phi_\lambda(u) < 0$ if $\lambda > \lambda_1$. The proof of the Lemma is completed. □

Theorem 3 *Let $1 < \alpha^- \leq \alpha^+ < q^- \leq q^+ < p^- < p_3^*(x)$ for any $x \in \overline{\Omega}$. Then, there exists $\lambda_1 > 0$ such that the problem (1) has at least one nontrivial weak solution $u(x)$ satisfying $\Phi_\lambda(u) < 0$.*

Proof Since Φ_λ is bounded from below, coercive and weakly lower semi-continuous on X by Lemma 2, then so it has a global minimizer $u \in X$ such that $\Phi'_\lambda(u) = 0$, which is a critical point of Φ_λ by Theorem 1.1 in [22] or Theorem 1.2 in [28]. In addition, using $\Phi_\lambda(0) = 0$ and $\Phi_\lambda(u) < 0$ for any $\lambda > \lambda_1$ by Lemma 3, then u is nontrivial weak solution, i.e. $u \neq 0$ when $\lambda > \lambda_1$. □

Now, to use Mountain Pass Theorem and find the weak solution of the problem (1) we need to prove the following two Lemmas.

Lemma 4 *Let $p^+, \alpha^+ < q^-$ and $q(x), \alpha(x) < p_3^*(x)$ for any $x \in \overline{\Omega}$. Then, the operator Φ_λ satisfies the (PS) condition for $V_2 > 0$.*

Proof Let $\{u_j\} \subset X$ be such that

$$\Phi_\lambda(u_j) \to \bar{c} \text{ and } \Phi'_\lambda(u_j) \to 0 \text{ in } X^* \text{ as } j \to \infty. \tag{10}$$

We prove that the sequence $\{u_j\}$ is bounded in X. Suppose that $\{u_j\}$ is not bounded in X. So we have $\|u_j\|_X \to \infty$ as $j \to \infty$, and $\|u_j\|_X > 1$ for all $j \in \mathbb{N}$. Hence

$$\begin{aligned} C + \|u_j\|_X &\geq \Phi_\lambda(u_j) - \frac{1}{q^-}\left\langle \Phi'_\lambda(u_j), u_j\right\rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\psi_{\beta(x)}\left(u_j\right) - \frac{\lambda}{q^-}\int\limits_\Omega V_1(x)\left|u_j\right|^{q(x)} dx \\ &\quad + \frac{\lambda}{\alpha^+}\int\limits_\Omega V_2(x)\left|u_j\right|^{\alpha(x)} dx \\ &\quad + \frac{\lambda}{q^-}\int\limits_\Omega \left(V_1(x)\left|u_j\right|^{q(x)} - V_2(x)\left|u_j\right|^{\alpha(x)}\right) dx \\ &= \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\psi_{\beta(x)}\left(u_j\right) + \lambda\left(\frac{1}{\alpha^+} - \frac{1}{q^-}\right)\int\limits_\Omega V_2(x)\left|u_j\right|^{\alpha(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\|u_j\|_X^{p^-}. \end{aligned}$$

If the last inequality is divided by $\|u_j\|_X^{p^-}$, and taking to the limit as $j \to \infty$, we get a contradiction due to $p^+ < q^-$. Hence, $\{u_j\}$ must be bounded in X. Because X is a reflexive Banach space, there exists a function $u \in X$ such that $u_j \rightharpoonup u$ (weakly) in X. By Theorem 1, the embeddings $X \hookrightarrow L^{q(.)}(\Omega)$ and $X \hookrightarrow L^{\alpha(.)}(\Omega)$ are compact. Therefore, we obtain

$$u_j \to u \text{ (strongly) in } L^{q(.)}(\Omega) \text{ and } L^{\alpha(.)}(\Omega)$$

as $j \to \infty$. On the other hand, if we use the Hölder inequality, we obtain

$$\begin{aligned} \left|\int\limits_\Omega V_1(x)\left|u_j\right|^{q(x)-2} u_j\left(u_j - u\right) dx\right| &\leq \|V_1\|_\infty \int\limits_\Omega \left|\left|u_j\right|^{q(x)-1}\left|u_j - u\right| dx\right| \\ &\leq 2\|V_1\|_\infty \left\|\left|u_j\right|^{q(x)-1}\right\|_{r(.)}\left\|u_j - u\right\|_{q(.)} \\ &= 2\|V_1\|_\infty \left\|u_j\right\|_{q(.)}\left\|u_j - u\right\|_{q(.)} \longrightarrow 0 \end{aligned}$$

as $j \to \infty$, where $\frac{1}{q(.)} + \frac{1}{r(.)} = 1$. Thus,

$$\lim_{j \longrightarrow \infty}\int\limits_\Omega V_1(x)\left|u_j\right|^{q(x)-2} u_j\left(u_j - u\right) dx = 0. \tag{11}$$

Similarly, we show that

$$\lim_{j\longrightarrow\infty}\int_{\Omega} V_2(x)\left|u_j\right|^{\alpha(x)-2}u_j\left(u_j-u\right)dx=0. \tag{12}$$

By (10) and the boundedness of $\left\{u_j-u\right\}$ in X, we have

$$\left|< \Phi_\lambda'(u_j)\left(u_j-u\right)>\right|\leq\left\|\Phi_\lambda'(u_j)\right\|_{op}\left\|u_j-u\right\|_X\to\infty$$

or

$$\Phi_\lambda'(u_j)\left(u_j-u\right)\to 0 \tag{13}$$

as $j\to\infty$. If we collect (11), (12) and (13), we achieve the following result

$$\lim_{j\longrightarrow\infty}\Psi_{\beta(x)}'(u_j)\left(u_j-u\right)=0.$$

So, we deduce that the sequence $u_j\to u$ (strongly) in X by Proposition 2. The proof is completed. □

Lemma 5 *Let $p^+,\alpha^+<q^-$ and $q(x)<p_3^*(x)$ for any $x\in\overline{\Omega}$. Then,*
(i) There exist $\gamma,\rho>0$ such that $\Phi_\lambda(u)\geq\gamma$ for any $u\in X$ with $\|u\|_X=\rho$.
(ii) There exist $u\in X$ such that $\Phi_\lambda(u)<0$.

Proof (i) Since $X\hookrightarrow L^{q(.)}(\Omega)$, then there exists a $C>0$

$$\|u\|_{q(.)}\leq C\|u\|_X \tag{14}$$

for any $u\in X$. Let $\|u\|_X=\rho<1$. Then, by (14) we write

$$\begin{aligned}\Phi_\lambda(u)&\geq\frac{1}{p^+}\|u\|_X^{p^+}-\frac{\lambda V_1^+}{q^-}\max\left\{\|u\|_{q(.)}^{q^-},\|u\|_{q(.)}^{q^+}\right\}\\&\geq\frac{1}{p^+}\|u\|_X^{p^+}-\frac{\lambda V_1^+}{q^-}\max\left\{C^{q^-}\|u\|_X^{q^-},C^{q^+}\|u\|_X^{q^+}\right\}\\&\geq\frac{1}{p^+}\|u\|_X^{p^+}-\frac{\lambda V_1^+C_*}{q^-}\|u\|_X^{q^-}\\&=\frac{1}{p^+}\rho^{p^+}-\frac{\lambda V_1^+C_*}{q^-}\rho^{q^-}\\&=\rho^{p^+}\left(\frac{1}{p^+}-\frac{\lambda V_1^+C_*}{q^-}\rho^{q^--p^+}\right),\end{aligned}$$

where $C_*=\max\left\{C^{q^-},C^{q^+}\right\}$. This inequality shows that if we take

$$\lambda_*=\frac{q^-}{2p^+V_1^+C_*\rho^{q^--p^+}},$$

then there exists $\gamma = \frac{\rho^{p^+}}{2p^+} > 0$ such that $\Phi_\lambda(u) \geq \gamma > 0$ for all $\lambda \in (0, \lambda_*)$.

(ii) Let $u \in X$ and $t > 1$. Thus, we have

$$\Phi_\lambda(tu) \leq \frac{t^{p^+}}{p^-}\psi_{\beta(x)}(u) - \frac{\lambda V_1^- t^{q^-}}{q^+}\int_\Omega |u|^{q(x)}\,dx + \frac{\lambda V_2^+ t^{\alpha+}}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\,dx$$

Therefore, $\Phi_\lambda(tu) \to -\infty$ as $t \to \infty$ due to $p^+, \alpha^+ < q^-$. □

Theorem 4 *Let $p^+, \alpha^+ < q^-$ and $q(x), \alpha(x) < p_3^*(x)$ for any $x \in \overline{\Omega}$. Then, the problem (1) has a nontrivial weak solution.*

Proof From Lemma 4 and Lemma 5, and $\Phi_\lambda(0) = 0$, we obtain that the functional Φ_λ satisfies the Mountain Pass Theorem (see [30]). So Φ_λ has a nontrivial critical point, i.e. the problem (1) has at least a nontrivial weak solution. □

For the multiplicity of solutions, we will use the Fountain Theorem. For this aim we have the following two Lemmas.

Lemma 6 (see [10]) *Let E be a reflexive and separable Banach space. Then, there exist $\{\zeta_j\} \subset E$ and $\left\{\zeta_j^*\right\} \subset E^*$ such that*

$$E = \overline{span\{\zeta_j \mid j = 1, 2, ...\}}, \quad E^* = \overline{span\left\{\zeta_j^* \mid j = 1, 2, ...\right\}}$$

and

$$\left\langle \zeta_j^*, \zeta_l \right\rangle = \begin{cases} 1, & j = l, \\ 0, & j \neq l, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between E and E^*.

For convenience, we write $E_j = span\{\zeta_j\}$, $Y_i = \bigoplus_{j=1}^{i} E_j$, $Z_i = \bigoplus_{j=i}^{\infty} E_j$.

Lemma 7 *Let $q(x) < p_3^*(x)$ for any $x \in \overline{\Omega}$. The set ϑ_j is defined by*

$$\vartheta_k = \sup\left\{\|u\|_{q(.)} : \|u\|_X = 1, u \in Z_k\right\},$$

then $\lim_{k\longrightarrow\infty} \vartheta_k = 0$.

Proof Since $q(x) < p_3^*(x)$ for all $x \in \overline{\Omega}$, then we have the continuous embedding $X \hookrightarrow L^{q(.)}(\Omega)$ and $\lim_{k\longrightarrow\infty} \vartheta_k = 0$ by the method in (Lemma 4.9, [10]). □

Theorem 5 *Let $p^+, \alpha^+ < q^-$ and $q(x), \alpha(x) < p_3^*(x)$ for any $x \in \overline{\Omega}$. Then, Φ_λ has a sequence of critical points $\{u_j\}$ such that $\Phi_\lambda(u_j) \longrightarrow +\infty$ as $j \to \infty$ and the problem (1) has infinite many pair of solutions for $V_2 > 0$.*

Proof By Lemma 4 the functional Φ_λ is even and satisfies (PS) condition. We show that there exists the real numbers η_k and γ_k such that $\eta_k > \gamma_k > 0$ and

$$(A_1)\ b_k := \inf\left\{\Phi_\lambda(u) \,|\, u \in Z_k, \|u\|_X = \gamma_k\right\} \longrightarrow \infty \text{ as } k \longrightarrow \infty$$

$$(A_2)\ a_k := \max\left\{\Phi_\lambda(u) \,|\, u \in Y_k, \|u\|_X = \eta_k\right\} \leq 0,$$

where k is large enough.

(A_1) For any $u \in Z_k$ such that $\|u\|_X = \gamma_k > 1$. Then, we get

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+}\|u\|_X^{p^-} - \frac{\lambda V_1^+}{q^-}\max\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\} \\
&\geq \begin{cases} \frac{1}{p^+}\|u\|_X^{p^-} - \lambda V_1^+, & \|u\|_{q(.)} \leq 1 \\ \frac{1}{p^+}\|u\|_X^{p^-} - \lambda V_1^+ \vartheta_k^{q^+}\|u\|_X^{q^+}, & \|u\|_{q(.)} \geq 1 \end{cases} \\
&\geq \frac{1}{p^+}\gamma_k^{p^-} - \lambda V_1^+ \vartheta_k^{q^+}\gamma_k^{q^+}.
\end{aligned}$$

If we take $\gamma_k = \left(\lambda V_1^+ q^+ \vartheta_k^{q^+}\right)^{\frac{1}{p^- - q^+}}$, then we get

$$\begin{aligned}
\lambda V_1^+ \vartheta_k^{q^+}\left(\lambda V_1^+ q^+ \vartheta_k^{q^+}\right)^{\frac{q^+}{p^- - q^+}} &= \lambda V_1^+ \vartheta_k^{q^+} \lambda^{\frac{q^+}{p^- - q^+}}\left(V_1^+\right)^{\frac{q^+}{p^- - q^+}}\left(\vartheta_k^{q^+}\right)^{\frac{q^+}{p^- - q^+}} \\
&= \lambda^{1+\frac{q^+}{p^- - q^+}}\left(V_1^+\right)^{1+\frac{q^+}{p^- - q^+}}\left(\vartheta_k^{q^+}\right)^{1+\frac{q^+}{p^- - q^+}}\left(q^+\right)^{\frac{p^-}{p^- - q^+}-1} \\
&= \frac{1}{q^+}\left(\lambda V_1^+ q^+ \vartheta_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+}\left(\lambda V_1^+ q^+ \vartheta_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} - \frac{1}{q^+}\left(\lambda V_1^+ q^+ \vartheta_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} \\
&= \left(\frac{1}{p^+} - \frac{1}{q^+}\right)\left(\lambda q^+ \alpha_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} \longrightarrow \infty
\end{aligned}$$

as $k \longrightarrow \infty$ due to $p^+ < q^+$, and $\vartheta_k \longrightarrow 0$. So (A_1) holds.

(A_2) Let $u \in Y_k$ be such that $\|u\|_X = \eta_k > \gamma_k > 1$. Then, we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{p^-}\|u\|_X^{p^+} - \frac{\lambda V_1^+}{q^+}\int_\Omega |u|^{q(x)}\,dx \\
&\leq \frac{1}{p^-}\|u\|_X^{p^+} - \frac{\lambda V_1^+}{q^+}\min\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\}.
\end{aligned}$$

Due to the fact that Y_k has finite dimension, the norms $\|.\|_X$ and $\|.\|_{q(.)}$ are equivalent. So,

$$\Phi_\lambda(u) \to -\infty \text{ as } \|u\|_X \to +\infty, u \in Y_k$$

by $p^+ < q^-$ and the Fountain Theorem (see [30, Theorem 3.6]). □

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1007/s11565-025-00593-1.

Acknowledgements The authors would like to thank the anonymous referees for their carefull reading, suggestions and comments for our paper. The authors extend their appreciation to Northern Border University, Saudi Arabia for supporting this work through project number (NBU-CRP-2025-1706).

Data Availability Data sharing not applicable to this article.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Aydın, I.: Dirichlet type problem with $p(.)$-triharmonic operator. Journal of Science and Arts. **4**(57), 1037–1042 (2021)
2. Aydin, I., Unal, C.: Existence and multiplicity of weak solutions for eigenvalue Robin problem with weighted p(.)-Laplacian. Ric. Mat. **72**, 511–528 (2023)
3. Aydın, I., Unal, C.: Three solutions to a Steklov problem involving the weighted $p(.)$-Laplacian. Rocky Mountain J. Math. **51**, 67–76 (2021)
4. Ayoujil, A., El Amrouss, A.R.: On the spectrum of a fourth order elliptic equation with variable exponent. Nonlinear Anal. **71**, 4916–4926 (2009)
5. Belakhdar, A., Belaouidel, H., Filali, M., Tsouli, N.: Positivity of the infimum eigenvalue for the $p(x)$-triharmonic operator with variable exponents. Mediterr. J. Math. **20**(63), 1–18 (2023)
6. Belakhdar, A., Belaouidel, H., Filali, M., Tsouli, N.: Existence of solutions and positivity of the infimum eigenvalue for elliptic equations driving by $p(x)$-triharmonic operator. J Elliptic Parabol Equ (2024). https://doi.org/10.1007/s41808-024-00303-6
7. Deng, S.G.: Positive solutions for Robin problem involving the $p(x)$-Laplacian. J. Math. Anal. Appl. **360**, 548–560 (2009)
8. Diening, L., Harjulehto, P., Hastö, P., Ruzicka, M.: Lebesgue and Sobolev spaces with variable exponents. Lecture Notes, vol. 2017. Springer-Verlag, Berlin (2011)
9. Fan, X.L., Wu, H.Q., Wang, F.Z.: Hartman-type results for $p(t)$-Laplacian systems. Nonlinear Anal. **52**, 585–594 (2003)
10. Fan, X.L., Zhang, Q.H.: Existence of solutions for $p(x)$ -Laplacian Dirichlet problem. Nonlinear Anal. **52**, 1843–1852 (2003)
11. Fan, X.: Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients. J. Math. Anal. Appl. **312**, 464–477 (2005)
12. Fan, X., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**, 424–446 (2001)
13. Fan, X.: Boundary trace embedding theorems for variable exponent Sobolev spaces. J. Math. Anal. Appl. **339**, 1395–1412 (2008)
14. Ge, B., Zhou, Q.M.: Multiple solutions for a Robin-type differential inclusion problem involving the $p(x)$-Laplacian. Math. Meth. Appl. Sci. **40**(18), 6229–6238 (2017)
15. Halsey, T.C.: Electrorheological fluids. Science **258**(5083), 761–766 (1992)
16. Han, X., Naghizadeh, Z., Zhang, B.: A study of Kirchhoff-type $p(x)$-triharmonic problem involving two parameters. Quaest. Math. **48**(1), 49–62 (2025)

17. Hsini, M., Irzi, N., Kefi, K.: Existence of solutions for a $p(x)$ -biharmonic problem under Neumann boundary conditions. Appl. Anal. **100**(10), 2188–2199 (2021)
18. Kefi, K., Rădulescu, V.D.: On a $p(x)$ -biharmonic problem with singular weights. Z. Angew. Math. Phys. 68(4) (2017)
19. Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math J **41**(4), 592–618 (1991)
20. Kong, L.: On a fourth order elliptic problem with a $p(x)$-biharmonic operator. Appl. Math. Lett. **27**, 21–25 (2014)
21. Kong, L.: Eigenvalues for a fourth order elliptic problem. Proc. Amer. Math. Soc. **143**, 249–258 (2015)
22. Mawhin, J., Willem, M.: Critical Point Theory and Hamiltonian Systems. Springer-Verlag, New York (1989)
23. Mihăilescu, M.: Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator. Nonlinear Anal. **67**, 1419–1425 (2007)
24. Mihăilescu, M., Rădulescu, V.: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. Lond. Ser. A **462**, 2625–2641 (2006)
25. Rahal, B.: Existence results of infinitely many solutions for $p(x)$-Kirshhoff type triharmonic operator with Navier boundary conditions. J. Math. Anal. Appl. **478**, 1133–1146 (2019)
26. Růžička, M.: Electrorheological fluids: modeling and mathematical theory. Lecture Notes in Mathematics 1748, Springer (2000)
27. Shokooh, S.: On a nonlinear differential equation involving the $p(.)$-triharmonic operator. J. Non Func. Analy. **2020**, 1–19 (2020)
28. Struwe, M.: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 2nd edn. Springer-Verlag, Berlin (1996)
29. Unal, C., Aydın, I.: Compact embeddings of weighted variable exponent Sobolev spaces and existence of solutions for weighted $p(.)$Laplacian. Complex Var. Elliptic Equ. **66**(10), 1755–1773 (2021)
30. Willem, M.: Minimax Theorems. Birkhauser, Boston (1996)
31. Zhao, X., Miao, Q.: Existence of Solutions for $p(x)$ -triharmonic problem with Navier boundary conditions. Journal of Nonlinear Mathematical Physics **32**(11), 1–30 (2025)
32. Zhikov, V.V.: Averaging of functionals of tha calculus of variations and elasticity theory. Math. USSR Izv. **9**, 33–66 (1987)